Matrix objects

Definition.	A *matrix* of size $m \times n$ is an ordered rectangular table (or array) of numbers containing m rows and n columns.

The numbers that form a matrix, called its *elements* (or *components*), are characterized by both their value and the numbers of the rows and columns in which they are located. Let us agree to designate the matrix element located in the i-th row and j-th column as α_{ij}.

Definition	The numbers $m \times n$, m and n are called the *dimensions* of the matrix.

Matrices are denoted and written by listing their elements. For example, a matrix with elements

$$\alpha_{ij}\,;\; i=[1,m]\;;\; j=[1,n]$$

or in expanded form:

$$\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \dots & \alpha_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & \alpha_{mn} \end{Vmatrix},$$

If we need an unexpanded representation of a matrix, we will write it as $\|\alpha_{ij}\|$ or simply $\|A\|$.

Matrices are usually classified by the number of their rows and columns.

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Definition	If $m = n$, then the matrix is called *square*, of order n. A matrix of size $m \times 1$ is called an m-dimensional (or m-component) *column*. A matrix of size $1 \times n$ is called an n-dimensional (or n-component) *row*.

Note that, although two-index entries $\left\| \alpha_{1j} \right\|$ or $\left\| \alpha_{1j} \right\|$ should be used formally to denote rows or columns, it is customary to omit unchanging indices, as a result of which the row or column designations have the form $\left\| \alpha_j \right\|$ or $\left\| \beta_i \right\|$, respectively.

Some commonly used matrices with special element values have special names and notations.

Definition

A square matrix for which

$$\alpha_{ij} = \alpha_{ji} \quad \forall i, j = [1, n],$$

is called *symmetric*.

A matrix all of whose elements are zero is called the *zero* matrix. The zero matrix is denoted as $\|O\|$.

A square matrix of order n of the form

$$\begin{Vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{Vmatrix}$$

is called the *identity* matrix.

The identity matrix is usually denoted by $\|E\|$.

Matrix operations

Definition Two matrices $\| A \|$ and $\| B \|$ are considered *equal* (denoted by: $\| A \| = \| B \|$) if they are of the same size and if their corresponding components are equal, that is

$$\alpha_{ij} = \beta_{ij} \quad \forall i = [1, m] \text{ and } \forall j = [1, n].$$

Definition A matrix $\|C\|$ is called the *sum* of matrices $\|A\|$ and $\|B\|$ (denoted by: $\|C\| = \|A\| + \|B\|$) if matrices $\|A\|$, $\|B\|$, $\|C\|$ are of the same size and

$$\gamma_{ij} = \alpha_{ij} + \beta_{ij} \quad \forall i = [1, m], \ \forall j = [1, n],$$

where the numbers $\gamma_{ij} \quad \forall i = [1, m]$, $\forall j = [1, n]$ are the corresponding components of matrix $\|C\|$.

Definition A matrix $\|C\|$ is called the product of a number λ and a matrix $\|A\|$ (denoted by $\|C\| = \lambda\|A\|$), if the matrices $\|A\|$ and $\|C\|$ are of the same size and

$$\gamma_{ij} = \lambda\alpha_{ij} \quad \forall i = [1,m],\ \forall j = [1,n].$$

Note that a number can be multiplied by a matrix of any size.

Note: Any mathematical objects, for which the operations of *comparison*, *addition*, and *multiplication by a number* are suitably defined, can also be used as elements of a matrix.

Task 1.01 Answer the following questions with justification.

1) Are the matrices $\begin{Vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{Vmatrix}$ and $\begin{Vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{Vmatrix}$ equal?

2) What is the sum $\begin{Vmatrix} 1 & -2 & 3 \\ -3 & 2 & \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} \end{Vmatrix} + \begin{Vmatrix} -1 & 2 & -3 \\ 3 & -2 & \begin{Vmatrix} 2 & 2 \\ 2 & 2 \end{Vmatrix} \end{Vmatrix}$?

3) What is the expression $\begin{Vmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{Vmatrix} + \begin{Vmatrix} -\sin^2 x & \cos^2 x \\ \cos^2 x & -\sin^2 x \end{Vmatrix}$ equal to?

Решение

1) They are not equal, since they have elements with the same indices that are not equal to each other.

2) This sum is equal to a matrix object of the form

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & \begin{Vmatrix} 3 & 3 \\ 3 & 3 \end{Vmatrix} \end{Vmatrix} .$$

3) This expression is a matrix function of the form

$$\begin{Vmatrix} \cos 2x & 1 \\ 1 & \cos 2x \end{Vmatrix} .$$

Solution is found

Definition *Matrix transposition* is an operation that results in a new matrix, where the rows are the columns of the original matrix, written with the order of their sequence preserved (Fig. 1).

The matrix resulting from the transposition of the matrix $\|A\|$ is denoted by $\|A\|^{\mathrm{T}}$.

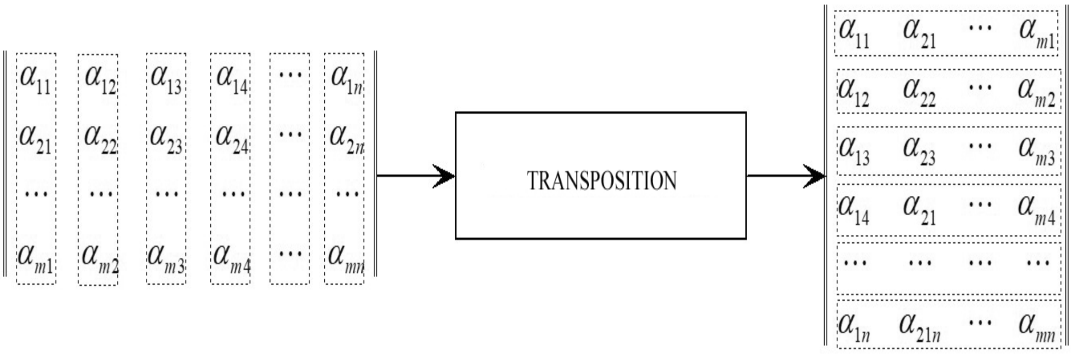

Fig. 1

For the elements of the transposed matrix $\|A\|^{\mathrm{T}}$, the equality is true

$$\alpha^{\mathrm{T}}_{ij} = \alpha_{ji} \quad \forall i = [1, m] \text{ , } \forall j = [1, n] \text{ .}$$

The transposition operation, for example, does not change a symmetric matrix, but transfers a row of size $1 \times m$ to a column of size $m \times 1$ and vice versa.

Task 1.02 Answer the following questions with justification.

1) Is the operation feasible $\left\|\begin{matrix}1 & 2 & 3\\ 3 & 2 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 3\\ 2 & 2\\ 3 & 1\end{matrix}\right\|$?

2) Is the operation feasible $\left\|\begin{matrix}1 & 2 & 3\\ 3 & 2 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 3\\ 2 & 2\\ 3 & 1\end{matrix}\right\|^{\mathrm{T}}$?

3) Is the operation feasible $\left\|\begin{matrix}1 & 2 & 3\\ 3 & \vec{x} & 1\end{matrix}\right\|^{\mathrm{T}} + \left\|\begin{matrix}1 & 3\\ 2 & 2\\ 3 & 1\end{matrix}\right\|$?

4) Is the operation feasible $\left\|\begin{matrix}1 & 2 & 3\\ 3 & \sin x & 1\end{matrix}\right\|^{\mathrm{T}} + \left\|\begin{matrix}1 & 3\\ 2 & 2\\ 3 & 1\end{matrix}\right\|$?

Solution

1) Not feasible, since the sizes of the addends are not equal.

2) Feasible, since the matrices being added have the same sizes.

3) Not feasible, since the operation $\vec{x} + 2$ is not feasible.

4) Feasible in the case where "2" denotes a function identically equal to two.

Solution is found

Determinants (determinants) of square matrices of the 2nd and 3rd orders

A special numerical characteristic is introduced for square matrices only. It is called a determinant and denoted as $\det\|A\|$.

A description of the properties of determinants of square matrices of the n-th order will be given later. Here we will consider the cases of $n = 2$ and $n = 3$.

Definition. The determinant of a square matrix of the 2nd order $\begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix}$ is the number

$$\det\begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}.$$

Definition. The determinant of a square matrix of the 3rd order $\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix}$ is the number

$$\det\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix} =$$

$$= \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{13}\alpha_{21}\alpha_{32} + +\alpha_{12}\alpha_{23}\alpha_{31} -$$
$$- \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33}.$$

The following theorems are valid for determinants of square matrices:

Theorem **The determinant of a matrix of the 3rd order can be expressed in terms of determinants of the 2nd order by the formula:**

$$\det\begin{Vmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{Vmatrix}=$$

$$=\alpha_{11}\det\begin{Vmatrix}\alpha_{22} & \alpha_{23}\\ \alpha_{32} & \alpha_{33}\end{Vmatrix}-\alpha_{12}\det\begin{Vmatrix}\alpha_{21} & \alpha_{23}\\ \alpha_{31} & \alpha_{33}\end{Vmatrix}+\alpha_{13}\det\begin{Vmatrix}\alpha_{21} & \alpha_{22}\\ \alpha_{31} & \alpha_{32}\end{Vmatrix}.$$

This formula is called the expansion of the determinant along the first row.

Determinant can be expanded along any row (or column), provided that the sign of each term with a factor α_{ij} is equal to $(-1)^{i+j}$.

Sometimes it is more convenient to calculate the value of the determinant of a matrix of the 3rd order in a different way ('Triangle Method', see Fig. 2)

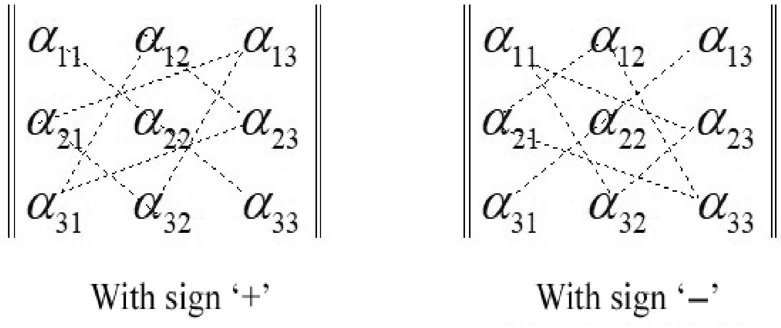

Fig. 2

Consequence **When *transposing* square matrices of the 2nd or 3rd order, their determinants do not change.**

Task 1.03 Calculate the determinants of the matrices.

1) $\begin{Vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{Vmatrix}$.

2) $\begin{Vmatrix} 5 & -4 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & -4 \end{Vmatrix}$.

3) $\begin{Vmatrix} 123457 & 123456 \\ 123456 & 123455 \end{Vmatrix}$.

4) $\begin{Vmatrix} 50 & -44 & 55 \\ -50 & 71 & -55 \\ 100 & 28 & 110 \end{Vmatrix}$.

Solution

1) -1.

2) -20. It is advisable to use the expansion of the determinant along the second row, or along the third column.

3) Let $a = 123456$, then the determinant is equal to

$$\begin{Vmatrix} a+1 & a \\ a & a-1 \end{Vmatrix} = \left(a^2 - 1\right) - a^2 = -1 .$$

4) This determinant is equal to zero, since the third column is equal to the first, multiplied by $\frac{11}{10}$.

Solution is found

In terms of determinants of second-order matrices, the condition for the unique solvability of a system of two linear equations with two unknowns can be formulated quite conveniently.

Theorem (Cramer). **In order for a system of linear equations**

$$\begin{cases} \alpha_{11}\xi_1 + \alpha_{12}\xi_2 = \beta_1, \\ \alpha_{21}\xi_1 + \alpha_{22}\xi_2 = \beta_2 \end{cases}$$

to have a unique solution, it is necessary and sufficient that

$$\det \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} \neq 0.$$

Task 1.04. Find all solutions of a system of linear equations $\begin{cases} \lambda\xi_1 + 4\xi_2 = \lambda \\ \xi_1 + \lambda\xi_2 = \lambda - 1 \end{cases}$ for any values of the parameter $\lambda \in \mathbf{R}$.

Solution: 1. Cramer's theorem states: in order for a system of linear equations $\begin{cases} \alpha_{11}\xi_1 + \alpha_{12}\xi_2 = \beta_1 \\ \alpha_{21}\xi_1 + \alpha_{22}\xi_2 = \beta_2 \end{cases}$ to have a unique solution $\{\xi_1^*;\xi_2^*\}$, it is necessary and sufficient that $\Delta \neq 0$, while $\xi_1^* = \dfrac{\Delta_1}{\Delta}$ and $\xi_2^* = \dfrac{\Delta_2}{\Delta}$, where

$$\Delta = \det\begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix}; \quad \Delta_1 = \det\begin{Vmatrix} \beta_1 & \alpha_{12} \\ \beta_2 & \alpha_{22} \end{Vmatrix}; \quad \Delta_2 = \det\begin{Vmatrix} \alpha_{11} & \beta_1 \\ \alpha_{21} & \beta_2 \end{Vmatrix}.$$

In the case when $\Delta = 0$, a special study is required..

2. In our case

$$\Delta = \det\begin{Vmatrix} \lambda & 4 \\ 1 & \lambda \end{Vmatrix} = \lambda^2 - 4\,, \qquad \Delta_1 = \det\begin{Vmatrix} \lambda & 4 \\ \lambda - 1 & \lambda \end{Vmatrix} = \lambda^2 - 4\lambda + 4$$

and

$$\Delta_2 = \det\begin{Vmatrix} \lambda & \lambda \\ 1 & \lambda - 1 \end{Vmatrix} = \lambda^2 - 2\lambda\,.$$

Therefore, when $\lambda \in (-\infty,-2) \cup (-2,2) \cup (2,+\infty)$ by Cramer's theorem the system has a unique solution

$$\xi_1^* = \frac{\lambda - 2}{\lambda + 2}\,; \quad \xi_2^* = \frac{\lambda}{\lambda + 2}\,.$$

3. Finally, when $\lambda = -2$ the system has the form $\begin{cases} -2\xi_1 + 4\xi_2 = -2, \\ \xi_1 - 2\xi_2 = -3. \end{cases}$ There are no solutions here.

If then $\lambda = 2$, the system will be $\begin{cases} 2\xi_1 + 4\xi_2 = 2, \\ \xi_1 + 2\xi_2 = 1. \end{cases}$ It has an infinite number of solutions, described by the formula

$$\begin{cases} \xi_1^* = 1 - 2\tau \\ \xi_2^* = \tau \end{cases}; \; \tau \in (-\infty,+\infty)\,.$$

Matrix product

Definition A matrix $\|C\|$ of size $m \times n$ with elements

$$\gamma_{ji} \quad \forall i = [1, n],\ \forall j = [1, m]$$

is called the *product* of a matrix $\|A\|$ of size $m \times l$ with elements

$$\alpha_{jk}\ \forall j = [1, m],\ \forall k = [1, l]$$

by a matrix $\|B\|$ of size $l \times n$ with elements $\beta_{ki}\ \forall k = [1, l]$, $\forall i = [1, n]$,
where

$$\gamma_{ji} = \sum_{k=1}^{l} \alpha_{jk} \beta_{ki} \quad \forall i = [1, n],\ \forall j = [1, m].$$

The result of *matrix multiplication* is the matrix of size $m \times n$ for any natural l, which is denoted as $\| C \| = \| A \| \| B \|$. The rule for calculating the components of a product from the components of the factors of a matrix product is illustrated by the Fig. 3.

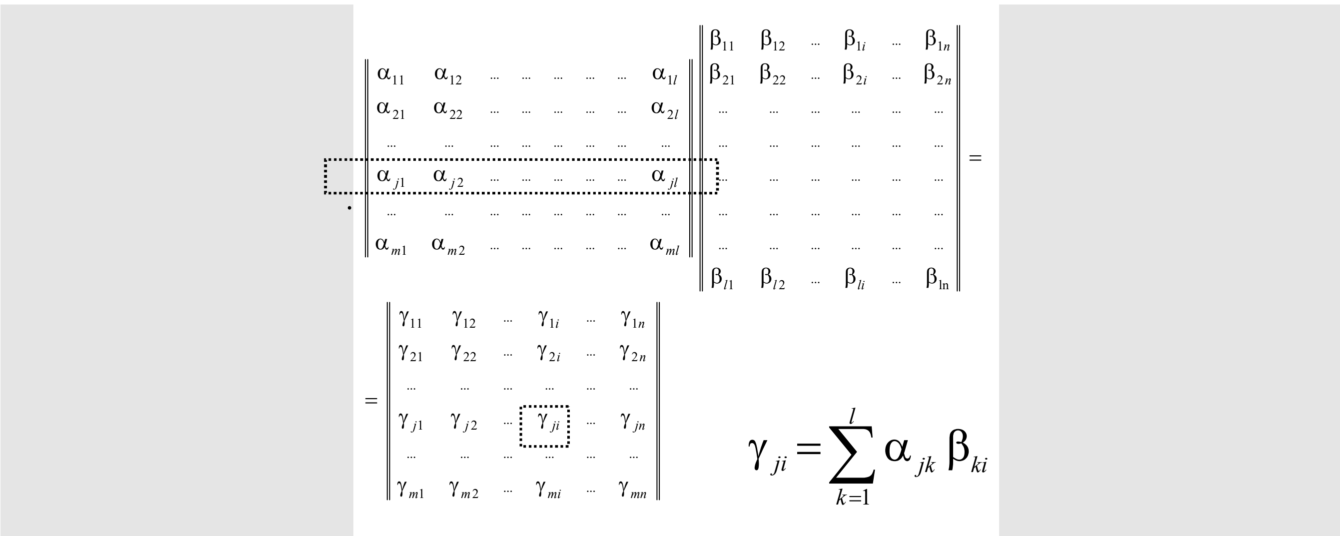

Fig.3

From the definition of matrix product it follows directly that for matrices of suitable sizes:

1) matrix multiplication is *non-commutative*, i.e. in the general case $\|A\|\|B\| \neq \|B\|\|A\|$,

2) matrix multiplication is *associative*

$$\|A\|(\|B\|\|C\|) = (\|A\|\|B\|)\|C\|,$$

3) matrix multiplication has the property of *distributivity*

$$\|A\|(\|B\| + \|C\|) = \|A\|\|B\| + \|A\|\|C\|.$$

Task 1.05 For two matrices, find out in what order they can be multiplied and find the size of the possible product.

1) $\left\|\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 0 \end{matrix}\right\|$ and $\left\|\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{matrix}\right\|$.

2) $\left\|\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{matrix}\right\|$ and $\left\|\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{matrix}\right\|$.

3) $\left\|\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\right\|$ and $\left\|\begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix}\right\|$.

Solution

1) Multiplication is not possible in any order.

2) The second matrix can be multiplied by the first, you get a 4x3 matrix.

3) Multiplication is possible in both orders. If a column is multiplied by a row, you get a 5x5 matrix. If you multiply a row by a column, you get a 1x1 matrix (it is a number!).

Solution is found

Definition A matrix $\|A\|^{-1}$ called the inverse of a square matrix $\|A\|$, if the equalities are satisfied.

$$\|A\|^{-1}\|A\| = \|A\|\|A\|^{-1} = \|E\|.$$

The inverse matrix does not exist for an arbitrary square matrix. For the existence of a matrix inverse to $\|A\|$, it is necessary and sufficient that the condition is satisfied $\det\| A \|\neq 0$.

Definition A matrix $\|A\|$, for which $\det\|A\| = 0$, is called *singular*, and a matrix, for which $\det\|A\| \neq 0$, is called *non-singular*.

Lemma **If the inverse matrix exists, then it is unique.**

There is a useful formula for a non-singular matrix $\|A\| = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$:

$$\|A\|^{-1} = \frac{1}{\det\|A\|}\begin{Vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{Vmatrix} . \qquad (*)$$

Task 1.06 Write the condition and solution in matrix form for a system of linear equations:

$$\begin{cases} 3x_1 - 5x_2 = -1, \\ 4x_1 + 2x_2 = 16. \end{cases}$$

Solution

The original system in expanded matrix form is

$$\begin{Vmatrix} 3 & -5 \\ 4 & 2 \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} = \begin{Vmatrix} -1 \\ 16 \end{Vmatrix}$$

The solution will be

$$\begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} = \begin{Vmatrix} 3 & -5 \\ 4 & 2 \end{Vmatrix}^{-1} \begin{Vmatrix} -1 \\ 16 \end{Vmatrix}$$

or, according to formula (*),

$$\begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} = \frac{1}{26} \begin{Vmatrix} 2 & 5 \\ -4 & 3 \end{Vmatrix} \begin{Vmatrix} -1 \\ 16 \end{Vmatrix} == \frac{1}{26} \begin{Vmatrix} 78 \\ 52 \end{Vmatrix} = \begin{Vmatrix} 3 \\ 2 \end{Vmatrix} .$$

Answer: $x_1 = 3$ and $x_2 = 2$.

Solution is found

Theorem. **The relation $(\|A\|\|B\|)^{\mathrm{T}} = \|B\|^{\mathrm{T}}\|A\|^{\mathrm{T}}$ is valid.**

Theorem **For non-singular square matrices of the same size $\|A\|$ and $\|B\|$ the relation**

$$(\|A\|\|B\|)^{-1} = \|B\|^{-1}\|A\|^{-1}$$

is true.

Problem 1.07 *Check the identity* $(\|A\|^{-1})^{\mathrm{T}} = (\|A\|^{\mathrm{T}})^{-1}$.

Definition A non-singular square matrix $\|Q\|$, for which $\|Q\|^{-1} = \|Q\|^{\mathrm{T}}$, is called *orthogonal*.

Problem 1.08 *Check that the matrix* $\left\| \begin{matrix} \cos x & -\sin x \\ \sin x & \cos x \end{matrix} \right\|$ *is orthogonal.*