Directed segments

Definition	A line segment whose endpoints are points A and B is called a *directed segment*, if it is specified which of these two points is the beginning and which is the endpoint of the segment. A directed segment whose beginning and endpoint coincide is called a *zero directed segment*.

Operations with directed segments

Definition Two non-zero directed segments $\overline{AB}$ and $\overline{CD}$ are called *equal* if their beginnings and their ends can be combined by parallel translation of one of these segments.

Note that by virtue of this definition, *parallel translation* of directed segments does not change them.

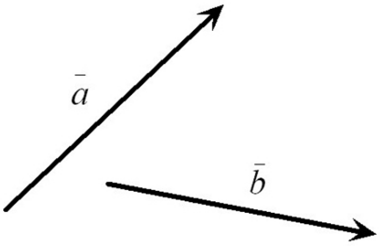
Let two directed segments $\overline{a}$ and $\overline{b}$ be given.

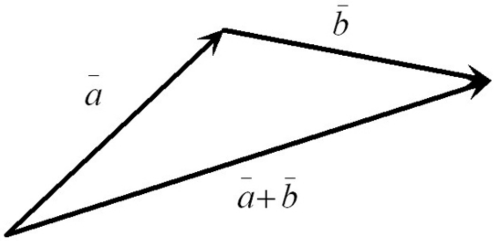
Definition

Let us match the beginning of the segment $\overline{b}$ with the end of $\overline{a}$ (that is, construct a directed segment $\overline{b'}$ equal to $\overline{b}$, the beginning of which coincides with the end of the segment $\overline{a}$),

then the directed segment $\overline{c}$, the beginning of which coincides with the beginning of $\overline{a}$ and the end with the end of $\overline{b'}$, is called the *sum* of the directed segments $\overline{a}$ and $\overline{b}$.

This definition is sometimes called the *triangle rule.*

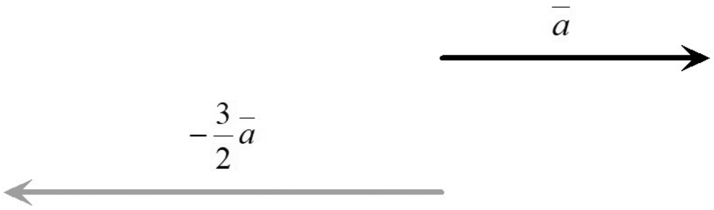




Definition

The *product* $\lambda\overline{a}$ of a directed segment $\overline{a}$ by a number λ is understood as:

it is the zero directed segment, when $\lambda = 0$
when $\lambda \neq 0$, it is a directed segment for which
the length is $|\lambda||\overline{a}|$ and
the direction of $\lambda\overline{a}$ coincides with the direction of $\overline{a}$, if $\lambda > 0$;
the direction of $\lambda\overline{a}$ is opposite to the direction of $\overline{a}$, if $\lambda < 0$.



Definition of a set of vectors

Definition The set of *all* directed segments for which the following operations are introduced:

- comparisons ;
- additions
- multiplications by a real number

is called a *set of vectors*.

A specific element of this set will be called a vector and denoted by a symbol with an upper arrow, for example, $\vec{a}$.

The zero vector is denoted by the symbol $\vec{o}$.

Theorem **The operations of addition and multiplication by a real number on a set of vectors have the properties:**

1°. Commutativity $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

2°. Associativity

$$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}\,;$$

$$\lambda(\mu\vec{a})=(\lambda\mu)\vec{a}.$$

3°. Distributivity

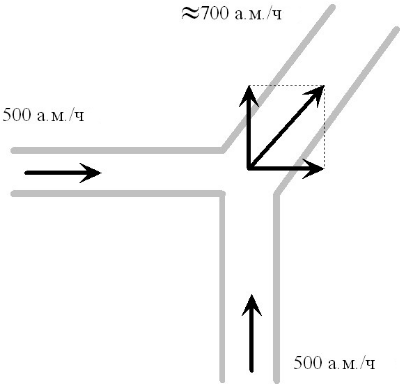
$$\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}\,;$$

$$(\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a}$$

for any vectors $\vec{a}$, $\vec{b}$ **and** $\vec{c}$ **and any real numbers** λ **and** λ.

Notes on defining vectors

Not all objects that require a numerical value and direction to be described are vectors. For example, flows of liquid, gases, electric charges, or cars on the street.



Such objects, for example, can be summed with each other, but not according to the parallelogram rule (see figure).

Linear dependence and independence of vectors

Definition	Two vectors parallel to the same line are called *collinear*. Three vectors parallel to the same plane are called *coplanar*.

The zero vector is considered collinear to any other vector. The zero vector is considered coplanar to any pair of vectors.

Definition	An expression of the form $\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \ldots + \lambda_n \vec{a}_n$, where $\lambda_i ; i = [1, n]$ are some numbers, is called a *linear combination* of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. A linear combination is called *trivial* if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Otherwise it is called *non-trivial*.

Summation convention

In cases where explicitly writing the sum of a number of terms is impractical or impossible, but it is known how the value of each term depends on its number, then a special form of writing the summation operation is allowed:

$$F(k) + F(k+1) + \ldots + F(n) = \sum_{i=k}^{n} F(i),$$

$F(i)$ (read: "sum $F(i)$ over i from k to n"), where i is the summation index, k is the minimum value of the summation index, n is the maximum value of the summation index, and, finally, $F(i)$ is the general form of the term.

Definition

Vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are called *linearly dependent* if there exists a non-trivial linear combination of them equal to the zero vector,

that is, such that $\sum\limits_{i=1}^{n} \lambda_i \vec{a}_i = \vec{o}$.

Definition

Vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are called linearly independent if the condition $\sum\limits_{i=1}^{n} \lambda_i \vec{a}_i = \vec{o}$ implies the triviality of the linear combination $\sum\limits_{i=1}^{n} \lambda_i \vec{a}_i$, that is, that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

Lemma **For vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ to be linearly dependent it is necessary and sufficient that one of them be a linear combination of the others.**

The following statements are true.

Theorem **One vector is linearly dependent if and only if it is zero.**

Theorem **Two vectors are linearly dependent if and only if they are collinear.**

Theorem **Three vectors are linearly dependent if and only if they are coplanar.**

Theorem **If among the vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ there is a subset of linearly dependent ones, then all vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ are linearly dependent.**

Corollary **If among the vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ there is at least one zero one, then the vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ are linearly dependent.**

Task *Based on the definition of linear dependence of vectors, prove that*

a) vectors lying on two adjacent sides of a rectangle and a vector lying on one of its diagonals are linearly dependent;

b) vectors lying on two adjacent sides of a rectangle are linearly independent.

Solution

a) Consider a rectangle $ABCD$ (Fig.02.02.02.) Note that $\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$. Therefore

$$1\cdot\overrightarrow{AB}+(-1)\overrightarrow{AD}+1\cdot\overrightarrow{BD}=\overrightarrow{o}.$$

That is, there is a nontrivial linear combination of vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{BD}$ equal to the zero vector. Consequently, the vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{BD}$ are linearly dependent.

b) Equate some linear combination of the studied set of vectors to the zero vector

$$\lambda_1\overrightarrow{AD}+\lambda_2\overrightarrow{AB}=\overrightarrow{o}$$

The orthogonal projection of this equality onto a straight line AD has the form

$$\overrightarrow{\text{Pr}}_{AD}(\lambda_1\overrightarrow{AD}+\lambda_2\overrightarrow{AB})=\overrightarrow{\text{Pr}}_{AD}\overrightarrow{o}$$

or $\lambda_1\overrightarrow{AD}=\overrightarrow{o}$. Whence $\lambda_1=0$, $\overrightarrow{AD}\neq\overrightarrow{o}$.

But then and $\lambda_2=0$. That is, the linear combination of vectors $\overrightarrow{AB}$; $\overrightarrow{AD}$ is trivial and, therefore, these vectors are linearly independent.

Solution is found.

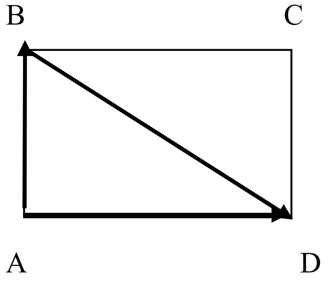


Figure 02.02.02.