Vector Products

1°. Scalar Product

The *scalar product* of nonzero vectors $\vec{a}$ and $\vec{b}$ is a number equal to the product of the lengths of these vectors by the cosine of the angle between them. If at least one of the factors is a zero vector, the scalar product is considered to be zero.

The scalar product is denoted as $(\vec{a},\vec{b})$. Then, by definition,

$$(\vec{a},\vec{b}) = |\vec{a}||\vec{b}|\cos\varphi\,;\ 0 \le \varphi \le \pi\,,$$

where φ is the angle between the factor vectors.

Properties of the scalar product

1°/ $(\vec{a},\vec{b}) = 0$ with $\vec{a} \neq \vec{o}$ and $\vec{b} \neq \vec{o}$? if $\vec{a}$ and $\vec{b}$ are mutually orthogonal,

2°. $(\vec{a},\vec{b}) = (\vec{b},\vec{a})$ (commutativity),

3°. $(\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2, \vec{b}) = \lambda_1 (\vec{a}_1, \vec{b}) + \lambda_2 (\vec{a}_2, \vec{b})$ (linearity)

4°. $(\vec{a},\vec{a}) = |\vec{a}|^2 \geq 0 \ \forall \vec{a}$; $|\vec{a}| = \sqrt{(\vec{a},\vec{a})}$,
(conditions: $(\vec{a},\vec{a}) = 0$ and $\vec{a} = \vec{o}$ are equivalent),

5°. For $\vec{a} \neq \vec{o}$ and $\vec{b} \neq \vec{o}$ $\cos\varphi = \dfrac{(\vec{a},\vec{b})}{|\vec{a}||\vec{b}|}$.

2°. Vector product

A *vector product* of non-collinear vectors $\vec{a}$ and $\vec{b}$ a vector $\vec{c}$ such that

1º. $|\vec{c}|=|\vec{a}||\vec{b}|\sin\varphi$, where is the angle between the vectors $\vec{a},\vec{b}$; $0<\varphi<\pi$.

2º. The vector $\vec{c}$ is orthogonal to the vector $\vec{a}$ and the vector $\vec{b}$.

3º. The triple of vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ is right-oriented.

In the case where the factors are collinear, the vector product is considered equal to the zero vector.

The vector product is denoted as $[\vec{a},\vec{b}]$.

Properties of vector product

1°. $\left|[\vec{a},\vec{b}]\right|$ is equal to the area of the parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$.

2°. For nonzero vectors $\vec{a}$ and $\vec{b}$ to be collinear, it is necessary and sufficient that their vector product be equal to the zero vector.

3°. $[\vec{a},\vec{b}] = -[\vec{b},\vec{a}]$ (anticommutativity)

4°. $[\lambda\vec{a},\vec{b}] = \lambda[\vec{a},\vec{b}]$.

5°. $[\vec{a}+\vec{b},\vec{c}] = [\vec{a},\vec{c}] + [\vec{b},\vec{c}]$ (distributivity).

3°. Mixed product

The *mixed* (or *vector-scalar*) product of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, denoted as $(\vec{a}, \vec{b}, \vec{c})$, is the number $([\vec{a}, \vec{b}], \vec{c})$.

Properties of the mixed product

1°. The absolute value of the mixed product $(\vec{a}, \vec{b}, \vec{c})$ is equal to the volume of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. The sign of the mixed product is positive if the triple of $\vec{a}$, $\vec{b}$, $\vec{c}$ is right-oriented, and negative if it is left-oriented.

2°. $(\vec{a},\vec{b},\vec{c}) = (\vec{c},\vec{a},\vec{b}) = (\vec{b},\vec{c},\vec{a}) = -(\vec{b},\vec{a},\vec{c}) = -(\vec{c},\vec{b},\vec{a}) = -(\vec{a},\vec{c},\vec{b})$;

3°. $(\lambda\vec{a},\vec{b},\vec{c}) = \lambda(\vec{a},\vec{b},\vec{c})$;

4°. $(\vec{a}_1 + \vec{a}_2, \vec{b}, \vec{c}) = (\vec{a}_1,\vec{b},\vec{c}) + (\vec{a}_2,\vec{b},\vec{c})$,

The mixed product is equal to zero if there is at least one collinear pair among the factors.

4° Double vector product

The *double vector product* of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is called the vector $[\vec{a}\,[\vec{b}\;\vec{c}\,]]$.

Property of the double vector product

$$[\vec{a},[\vec{b},\vec{c}\,]] = \vec{b}(\vec{a},\vec{c}\,) - \vec{c}(\vec{a},\vec{b}\,)\,,$$

Task 4.01 *What angle do vectors $\vec{a}$ and $\vec{b}$ form if it is known that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are orthogonal?*

Solution

If vectors $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are orthogonal, then their scalar product is zero. Taking into account the commutativity of the scalar product and the conditions $\left|\vec{a}\right|=\left|\vec{b}\right|=1$ we have

$$0=(\vec{a}+2\vec{b},5\vec{a}-4\vec{b})=5(\vec{a},\vec{a})-4(\vec{a},\vec{b})+10(\vec{b},\vec{a})-8(\vec{b},\vec{b})=$$
$$=5\left|\vec{a}\right|^2+6(\vec{a},\vec{b})-8\left|\vec{b}\right|^2=6(\vec{a},\vec{b})-3.$$

Since $(\vec{a},\vec{b})=\frac{1}{2}$ and $\cos\varphi=\frac{1}{2} \Rightarrow \varphi=\frac{\pi}{3}$.

Solution is found.

Task .4.02 *Show that the vector product of a pair of vectors does not change if a vector collinear to the first factor is added to the second factor.*

Solution

Let $[\vec{a},\vec{b}]$ and $\vec{c}=\vec{b}+\lambda\vec{a}$ be given. For $[\vec{a},\vec{c}]$ we have

$$[\vec{a},\vec{c}]=[\vec{a},\vec{b}+\lambda\vec{a}]=[\vec{a},\vec{b}]+\lambda[\vec{a},\vec{a}]=[\vec{a},\vec{b}],$$

since $[\vec{a},\vec{a}]=\vec{o}$.

Solution is found

Note that we have also shown that it is impossible to uniquely indicate the second factor for a vector product and one of its factors.

Task 4.03

Find a vector $\vec{x}$ *lying in the plane of vectors* $\vec{a}$ *and* $\vec{b}$ *if*

$$\begin{cases} (\vec{a},\vec{x}) = \alpha, \\ (\vec{b},\vec{x}) = \beta, \end{cases}$$

and vectors $\vec{a}$ *and* $\vec{b}$ *are non-collinear.*

Solution

Vectors $\vec{a}$ and $\vec{b}$ form a basis in their plane. Therefore, vector $\vec{x}$ can be (and uniquely) expanded in this basis

$$\vec{x} = \xi\vec{a} + \eta\vec{b} \quad .$$

We can find the expansion coefficients from the system of equations

$$\begin{cases} (\vec{a},\vec{a})\xi + (\vec{a},\vec{b})\eta = \alpha, \\ (\vec{b},\vec{a})\xi + (\vec{b},\vec{b})\eta = \beta. \end{cases}$$

Solution is found.

Task 4.04

Find vector $\vec{x}$ *if*

$$\begin{cases} (\vec{a},\vec{x}) = \alpha, \\ (\vec{b},\vec{x}) = \beta, \\ (\vec{c},\vec{x}) = \gamma, \end{cases}$$

and the vectors $\vec{a}$*,* $\vec{b}$ *and* $\vec{c}$ *are not coplanar.*

Solution

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent, so the vectors $[\vec{a},\vec{b}]$, $[\vec{b},\vec{c}]$ and $[\vec{c},\vec{a}]$ are also linearly independent. Therefore, they form a basis in space and vector $\vec{x}$ can be (and uniquely) expanded in this basis

$$\vec{x} = \xi[\vec{a},\vec{b}] + \eta[\vec{b},\vec{c}] + \kappa[\vec{c},\vec{a}] \quad .$$

We can find the expansion coefficients from the system of equations

$$\begin{cases} (\vec{a},\vec{a},\vec{b},)\xi + (\vec{a},\vec{b},\vec{c},)\eta + (\vec{a},\vec{c},\vec{a},)\kappa = \alpha \ , \\ (\vec{b},\vec{a},\vec{b},)\xi + (\vec{b},\vec{b},\vec{c},)\eta + (\vec{b},\vec{c},\vec{a},)\kappa = \beta \ , \\ (\vec{c},\vec{a},\vec{b},)\xi + (\vec{c},\vec{b},\vec{c},)\eta + (\vec{c},\vec{c},\vec{a},)\kappa = \gamma \ , \end{cases}$$

which, by the properties of the mixed product, is equivalent to the system

$$\begin{cases} (\vec{a},\vec{b},\vec{c},)\eta = \alpha \ , \\ (\vec{b},\vec{c},\vec{a},)\kappa = \beta \ , \\ (\vec{c},\vec{a},\vec{b},)\xi = \gamma \ . \end{cases}$$

Solution is found.

Test 4.05 *Find all vectors* $\vec{x}$ *satisfying the relation*

$$[\vec{a},\vec{x}]+[\vec{x},\vec{b}]=[\vec{a},\vec{b}],$$

if vectors $\vec{a}$ *and* $\vec{b}$ *are non-collinear.*

Решение

We multiply both sides of this equation scalarly by $\vec{b}$, we get

$$([\vec{a},\vec{x}],\vec{b})+([\vec{x},\vec{b}],\vec{b})=([\vec{a},\vec{b}],\vec{b}) \quad \text{or} \quad (\vec{a},\vec{x},\vec{b})+(\vec{x},\vec{b},\vec{b})=(\vec{a},\vec{b},\vec{b}).$$

According to the properties of the mixed product $(\vec{x},\vec{b},\vec{b})=(\vec{a},\vec{b},\vec{b})=0$, that is $(\vec{a},\vec{x},\vec{b})=0$. This means that vectors $\vec{a}$, $\vec{x}$ and $\vec{b}$ are coplanar and linearly dependent

In this case, the vector $\vec{x}$ can be represented as a linear combination of vectors $\vec{a}$ and $\vec{b}$. Therefore, $\vec{x}=\alpha\vec{a}+\beta\vec{b}$.

Now we find at what values of α and β the vector $\vec{x}=\alpha\vec{a}+\beta\vec{b}$ will satisfy the original relation. Substituting, we get

$$[\vec{a},\vec{x}]+[\vec{x},\vec{b}]=[\vec{a},\alpha\vec{a}+\beta\vec{b}]+[\alpha\vec{a}+\beta\vec{b},\vec{b}]=$$

$$=\alpha[\vec{a},\vec{a}]+\beta[\vec{a},\vec{b}]+\alpha[\vec{a},\vec{b}]+\beta[\vec{b},\vec{b}]=(\alpha+\beta)[\vec{a},\vec{b}]=[\vec{a},\vec{b}],$$

that is, it is necessary that $\alpha+\beta=1$. Therefore $\vec{x}=\alpha\vec{a}+(1-\alpha)\vec{b}, \quad \forall\alpha$.

Solution is found

Task 4.06

Find vector $\vec{x}$ *from a system of equations*

$$\begin{cases} [\vec{a},\vec{x}] = \vec{b}, \\ (\vec{c},\vec{x}) = \alpha, \end{cases}.$$

subject to $(\vec{c},\vec{a}) \neq 0$

Solution

We multiply both sides of the first equation vectorially from the left by $\vec{c}$. Then we use the property of double vector product. We get

$$[\vec{c},[\vec{a},\vec{x}]] = \vec{a}(\vec{c},\vec{x}) - \vec{x}(\vec{c},\vec{a}) = [\vec{c},\vec{b}]$$

$$\alpha\,\vec{a} - \vec{x}(\vec{c},\vec{a}) = [\vec{c},\vec{b}],$$

since due to the second equation of the system there will be $(\vec{c},\vec{x}) = \alpha$.

Where we finally get

$$\vec{x} = \frac{\alpha\,\vec{a} - [\vec{c},\vec{b}]}{(\vec{c},\vec{a})}.$$

Solution is found.