

## Vector Products

### 1°. Scalar Product

The *scalar product* of nonzero vectors  $\vec{a}$  and  $\vec{b}$  is a number equal to the product of the lengths of these vectors by the cosine of the angle between them. If at least one of the factors is a zero vector, the scalar product is considered to be zero.

The scalar product is denoted as  $(\vec{a}, \vec{b})$ . Then, by definition,

$$(\vec{a}, \vec{b}) = |\vec{a}| |\vec{b}| \cos \varphi; \quad 0 \leq \varphi \leq \pi,$$

where  $\varphi$  is the angle between the factor vectors.

***Properties of the scalar product***

1°/  $(\vec{a}, \vec{b}) = 0$  with  $\vec{a} \neq \vec{o}$  and  $\vec{b} \neq \vec{o}$ ? if  $\vec{a}$  and  $\vec{b}$  are mutually orthogonal,

2°  $(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$  (commutativity),

3°  $(\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2, \vec{b}) = \lambda_1 (\vec{a}_1, \vec{b}) + \lambda_2 (\vec{a}_2, \vec{b})$  (linearity)

4°  $(\vec{a}, \vec{a}) = |\vec{a}|^2 \geq 0 \quad \forall \vec{a}; \quad |\vec{a}| = \sqrt{(\vec{a}, \vec{a})},$   
(conditions:  $(\vec{a}, \vec{a}) = 0$  and  $\vec{a} = \vec{o}$  are equivalent),

5° For  $\vec{a} \neq \vec{o}$  and  $\vec{b} \neq \vec{o}$   $\cos \varphi = \frac{(\vec{a}, \vec{b})}{|\vec{a}| |\vec{b}|}.$

## 2°. Vector product

A *vector product* of non-collinear vectors  $\vec{a}$  and  $\vec{b}$  a vector  $\vec{c}$  such that

- 1°.  $|\vec{c}| = |\vec{a}| |\vec{b}| \sin \varphi$ , where  $\varphi$  is the angle between the vectors  $\vec{a}, \vec{b}$ ;  $0 < \varphi < \pi$ .
- 2°. The vector  $\vec{c}$  is orthogonal to the vector  $\vec{a}$  and the vector  $\vec{b}$ .
- 3°. The triple of vectors  $\{\vec{a}, \vec{b}, \vec{c}\}$  is right-handed.

In the case where the factors are collinear, the vector product is considered equal to the zero vector.

The vector product is denoted as  $[\vec{a}, \vec{b}]$ .

***Properties of vector product***

- 1°.  $\left| [\vec{a}, \vec{b}] \right|$  is equal to the area of the parallelogram constructed on vectors  $\vec{a}$  and  $\vec{b}$ .
- 2°. For nonzero vectors  $\vec{a}$  and  $\vec{b}$  to be collinear, it is necessary and sufficient that their vector product be equal to the zero vector.
- 3°.  $[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$  (anticommutativity)
- 4°.  $[\lambda \vec{a}, \vec{b}] = \lambda [\vec{a}, \vec{b}]$ .
- 5°.  $[\vec{a} + \vec{b}, \vec{c}] = [\vec{a}, \vec{c}] + [\vec{b}, \vec{c}]$  (distributivity).

### 3°. Mixed product

The *mixed* (or *vector-scalar*) product of vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ , denoted as  $(\vec{a}, \vec{b}, \vec{c})$ , is the number  $([\vec{a}, \vec{b}], \vec{c})$ .

#### *Properties of the mixed product*

1°. The absolute value of the mixed product  $(\vec{a}, \vec{b}, \vec{c})$  is equal to the volume of the parallelepiped constructed on the vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ . The sign of the mixed product is positive if the triple of  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  is right-handed, and negative if it is left-handed.

$$2°. (\vec{a}, \vec{b}, \vec{c}) = (\vec{c}, \vec{a}, \vec{b}) = (\vec{b}, \vec{c}, \vec{a}) = -(\vec{b}, \vec{a}, \vec{c}) = -(\vec{c}, \vec{b}, \vec{a}) = -(\vec{a}, \vec{c}, \vec{b}) ;$$

$$3°. (\lambda \vec{a}, \vec{b}, \vec{c}) = \lambda (\vec{a}, \vec{b}, \vec{c}) ;$$

$$4°. (\vec{a}_1 + \vec{a}_2, \vec{b}, \vec{c}) = (\vec{a}_1, \vec{b}, \vec{c}) + (\vec{a}_2, \vec{b}, \vec{c}) ,$$

The mixed product is equal to zero if there is at least one collinear pair among the factors.

#### 4° **Double vector product**

The *double vector product* of vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  is called the vector  $[\vec{a} [\vec{b} \vec{c}]]$ .

#### *Property of the double vector product*

$$[\vec{a}, [\vec{b}, \vec{c}]] = \vec{b}(\vec{a}, \vec{c}) - \vec{c}(\vec{a}, \vec{b}),$$

Task 4.01 What angle do vectors  $\vec{a}$  and  $\vec{b}$  form if it is known that  $\vec{a}+2\vec{b}$  and  $5\vec{a}-4\vec{b}$  are orthogonal?

Solution

If vectors  $\vec{a}+2\vec{b}$  and  $5\vec{a}-4\vec{b}$  are orthogonal, then their scalar product is zero. Taking into account the commutativity of the scalar product and the conditions  $|\vec{a}|=|\vec{b}|=1$  we have

$$\begin{aligned} 0 &= (\vec{a}+2\vec{b}, 5\vec{a}-4\vec{b}) = 5(\vec{a}, \vec{a}) - 4(\vec{a}, \vec{b}) + 10(\vec{b}, \vec{a}) - 8(\vec{b}, \vec{b}) = \\ &= 5|\vec{a}|^2 + 6(\vec{a}, \vec{b}) - 8|\vec{b}|^2 = 6(\vec{a}, \vec{b}) - 3. \end{aligned}$$

$$\text{Since } (\vec{a}, \vec{b}) = \frac{1}{2} \text{ and } \cos \varphi = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}.$$

Solution is found.

Task .4.02

*Show that the vector product of a pair of vectors does not change if a vector collinear to the first factor is added to the second factor.*

Solution

Let  $[\vec{a}, \vec{b}]$  and  $\vec{c} = \vec{b} + \lambda \vec{a}$  be given. For  $[\vec{a}, \vec{c}]$  we have

$$[\vec{a}, \vec{c}] = [\vec{a}, \vec{b} + \lambda \vec{a}] = [\vec{a}, \vec{b}] + \lambda [\vec{a}, \vec{a}] = [\vec{a}, \vec{b}],$$

since  $[\vec{a}, \vec{a}] = \vec{o}$ .

Solution is found

Note that we have also shown that it is impossible to uniquely indicate the second factor for a vector product and one of its factors.

Task 4.03

Find a vector  $\vec{x}$  lying in the plane of vectors  $\vec{a}$  and  $\vec{b}$  if

$$\begin{cases} (\vec{a}, \vec{x}) = \alpha, \\ (\vec{b}, \vec{x}) = \beta, \end{cases}$$

and vectors  $\vec{a}$  and  $\vec{b}$  are non-collinear.

Solution

Vectors  $\vec{a}$  and  $\vec{b}$  form a basis in their plane. Therefore, vector  $\vec{x}$  can be (and uniquely) expanded in this basis

$$\vec{x} = \xi \vec{a} + \eta \vec{b} .$$

We can find the expansion coefficients from the system of equations

$$\begin{cases} (\vec{a}, \vec{a})\xi + (\vec{a}, \vec{b})\eta = \alpha, \\ (\vec{b}, \vec{a})\xi + (\vec{b}, \vec{b})\eta = \beta. \end{cases}$$

Solution is found.

Task 4.04

Find vector  $\vec{x}$  if

$$\begin{cases} (\vec{a}, \vec{x}) = \alpha, \\ (\vec{b}, \vec{x}) = \beta, \\ (\vec{c}, \vec{x}) = \gamma, \end{cases}$$

and the vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are not coplanar.

Solution

The vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are linearly independent, so the vectors  $[\vec{a}, \vec{b}]$ ,  $[\vec{b}, \vec{c}]$  and  $[\vec{c}, \vec{a}]$  are also linearly independent. Therefore, they form a basis in space and vector  $\vec{x}$  can be (and uniquely) expanded in this basis

$$\vec{x} = \xi[\vec{a}, \vec{b}] + \eta[\vec{b}, \vec{c}] + \kappa[\vec{c}, \vec{a}] .$$

We can find the expansion coefficients from the system of equations

$$\begin{cases} (\vec{a}, \vec{a}, \vec{b})\xi + (\vec{a}, \vec{b}, \vec{c})\eta + (\vec{a}, \vec{c}, \vec{a})\kappa = \alpha, \\ (\vec{b}, \vec{a}, \vec{b})\xi + (\vec{b}, \vec{b}, \vec{c})\eta + (\vec{b}, \vec{c}, \vec{a})\kappa = \beta, \\ (\vec{c}, \vec{a}, \vec{b})\xi + (\vec{c}, \vec{b}, \vec{c})\eta + (\vec{c}, \vec{c}, \vec{a})\kappa = \gamma, \end{cases}$$

which, by the properties of the mixed product, is equivalent to the system

$$\begin{cases} (\vec{a}, \vec{b}, \vec{c})\eta = \alpha, \\ (\vec{b}, \vec{c}, \vec{a})\kappa = \beta, \\ (\vec{c}, \vec{a}, \vec{b})\xi = \gamma. \end{cases}$$

Solution is found.

Test 4.05

Find all vectors  $\vec{x}$  satisfying the relation

$$[\vec{a}, \vec{x}] + [\vec{x}, \vec{b}] = [\vec{a}, \vec{b}],$$

if vectors  $\vec{a}$  and  $\vec{b}$  are non-collinear.

Решение

We multiply both sides of this equation scalarly by  $\vec{b}$ , we get

$$([\vec{a}, \vec{x}], \vec{b}) + ([\vec{x}, \vec{b}], \vec{b}) = ([\vec{a}, \vec{b}], \vec{b}) \quad \text{or} \quad (\vec{a}, \vec{x}, \vec{b}) + (\vec{x}, \vec{b}, \vec{b}) = (\vec{a}, \vec{b}, \vec{b}).$$

According to the properties of the mixed product  $(\vec{x}, \vec{b}, \vec{b}) = (\vec{a}, \vec{b}, \vec{b}) = 0$ , that is  $(\vec{a}, \vec{x}, \vec{b}) = 0$ .This means that vectors  $\vec{a}$ ,  $\vec{x}$  and  $\vec{b}$  are coplanar and linearly dependentIn this case, the vector  $\vec{x}$  can be represented as a linear combination of vectors  $\vec{a}$  and  $\vec{b}$ .Therefore,  $\vec{x} = \alpha \vec{a} + \beta \vec{b}$ .Now we find at what values of  $\alpha$  and  $\beta$  the vector  $\vec{x} = \alpha \vec{a} + \beta \vec{b}$  will satisfy the original relation. Substituting, we get

$$\begin{aligned} [\vec{a}, \vec{x}] + [\vec{x}, \vec{b}] &= [\vec{a}, \alpha \vec{a} + \beta \vec{b}] + [\alpha \vec{a} + \beta \vec{b}, \vec{b}] = \\ &= \alpha [\vec{a}, \vec{a}] + \beta [\vec{a}, \vec{b}] + \alpha [\vec{a}, \vec{b}] + \beta [\vec{b}, \vec{b}] = (\alpha + \beta) [\vec{a}, \vec{b}] = [\vec{a}, \vec{b}], \end{aligned}$$

that is, it is necessary that  $\alpha + \beta = 1$ . Therefore  $\vec{x} = \alpha \vec{a} + (1 - \alpha) \vec{b}$ ,  $\forall \alpha$ .

Solution is found

Task 4.06

Find vector  $\vec{x}$  from a system of equations

$$\begin{cases} [\vec{a}, \vec{x}] = \vec{b}, \\ (\vec{c}, \vec{x}) = \alpha, \end{cases}$$

subject to  $(\vec{c}, \vec{a}) \neq 0$

Solution

We multiply both sides of the first equation vectorially from the left by  $\vec{c}$ . Then we use the property of double vector product. We get

$$[\vec{c}, [\vec{a}, \vec{x}]] = \vec{a}(\vec{c}, \vec{x}) - \vec{x}(\vec{c}, \vec{a}) = [\vec{c}, \vec{b}]$$

$$\alpha \vec{a} - \vec{x}(\vec{c}, \vec{a}) = [\vec{c}, \vec{b}],$$

since due to the second equation of the system there will be  $(\vec{c}, \vec{x}) = \alpha$ .

Where we finally get

$$\vec{x} = \frac{\alpha \vec{a} - [\vec{c}, \vec{b}]}{(\vec{c}, \vec{a})}.$$

Solution is found.

# Vector Products in Coordinates

## Expression of Scalar Product in Coordinates

Let a basis  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$  and two vectors  $\vec{a}$  and  $\vec{b}$  with coordinate expansions in this basis:

$$\vec{a} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3, \quad \vec{b} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3.$$

Then by properties 3) and 4) of the scalar product

$$\begin{aligned} (\vec{a}, \vec{b}) &= (\xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3, \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3) = \\ &= \xi_1 \eta_1 (\vec{g}_1, \vec{g}_1) + \xi_1 \eta_2 (\vec{g}_1, \vec{g}_2) + \xi_1 \eta_3 (\vec{g}_1, \vec{g}_3) + \\ &+ \xi_2 \eta_1 (\vec{g}_2, \vec{g}_1) + \xi_2 \eta_2 (\vec{g}_2, \vec{g}_2) + \xi_2 \eta_3 (\vec{g}_2, \vec{g}_3) + \\ &+ \xi_3 \eta_1 (\vec{g}_3, \vec{g}_1) + \xi_3 \eta_2 (\vec{g}_3, \vec{g}_2) + \xi_3 \eta_3 (\vec{g}_3, \vec{g}_3) = \\ &= \sum_{j=1}^3 \left( \xi_j \eta_1 (\vec{g}_j, \vec{g}_1) + \xi_j \eta_2 (\vec{g}_j, \vec{g}_2) + \xi_j \eta_3 (\vec{g}_j, \vec{g}_3) \right) = \\ &= \sum_{j=1}^3 \sum_{i=1}^3 \xi_j \eta_i (\vec{g}_j, \vec{g}_i). \end{aligned}$$

In the case of an *orthonormal* basis, this formula is simplified, since for pairwise scalar products of basis vectors the equality  $(\vec{e}_j, \vec{e}_i) = \delta_{ji} =$

$$\begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases}$$

where the number  $\delta_{ji}$  is called the *Kronecker delta*.

From where for the scalar product of vectors in an orthonormal basis we obtain the formula

$$(\vec{a}, \vec{b}) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$

from which the following relations follow:  $|\vec{a}| = \sqrt{(\vec{a}, \vec{a})} = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ , and for the case  $\vec{a} \neq \vec{o}$  And  $\vec{b} \neq \vec{o}$ ,

$$\cos \varphi = \frac{(\vec{a}, \vec{b})}{|\vec{a}| |\vec{b}|} = \frac{\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \cdot \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}}.$$

Note that this equality in combination with the condition  $|\cos \varphi| \leq 1$  leads to the correct  $\forall \xi_i, \eta_j \quad i, j = 1 \dots 3$  *Cauchy - Bunyakovsky inequality*:

$$|\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3| \leq \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \cdot \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}.$$

**Task 4.07** Find the distance between two points in an Cartesian coordinate system with orthonormal basis if the coordinates of these points are known.

**Solution.** Let the following be given: orthonormal coordinate system  $\{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$  and two points  $M_1$  and  $M_2$ , whose radii-vectors have coordinate representations of the form

$$\left\| \vec{OM}_1 \right\|_e = \left\| \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \right\| \quad \text{and} \quad \left\| \vec{OM}_2 \right\|_e = \left\| \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \end{array} \right\|.$$

Then, using the formula for the length of the vector

$$M_1 \vec{M}_2 = (\xi_1 - \eta_1) \vec{e}_1 + (\xi_2 - \eta_2) \vec{e}_2 + (\xi_3 - \eta_3) \vec{e}_3,$$

**Solution is found.** we obtain in the orthonormal coordinate system

$$\left| M_1 \vec{M}_2 \right| = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2}.$$

## Expression of vector product in coordinates

Let *right* basis  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$  be given (i.e. such that vectors  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$  form *right-handed* triple) and let in this basis vectors  $\vec{a}$  and  $\vec{b}$  have coordinate decompositions

$$\vec{a} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3 \quad \text{and} \quad \vec{b} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3.$$

By properties 2) and 3) of the vector product we have

$$\begin{aligned} [\vec{a}, \vec{b}] &= [\xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3, \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3] = \\ &= \xi_1 \eta_1 [\vec{g}_1, \vec{g}_1] + \xi_1 \eta_2 [\vec{g}_1, \vec{g}_2] + \xi_1 \eta_3 [\vec{g}_1, \vec{g}_3] + \\ &+ \xi_2 \eta_1 [\vec{g}_2, \vec{g}_1] + \xi_2 \eta_2 [\vec{g}_2, \vec{g}_2] + \xi_2 \eta_3 [\vec{g}_2, \vec{g}_3] + \\ &+ \xi_3 \eta_1 [\vec{g}_3, \vec{g}_1] + \xi_3 \eta_2 [\vec{g}_3, \vec{g}_2] + \xi_3 \eta_3 [\vec{g}_3, \vec{g}_3] = \\ &= \sum_{j=1}^3 \sum_{i=1}^3 \xi_j \eta_i [\vec{g}_j, \vec{g}_i]. \end{aligned}$$

If we introduce the notation

$$\vec{f}_1 = [\vec{g}_2, \vec{g}_3], \quad \vec{f}_2 = [\vec{g}_3, \vec{g}_1], \quad \vec{f}_3 = [\vec{g}_1, \vec{g}_2],$$

then we get an easy-to-remember formula

$$\begin{aligned} [\vec{a}, \vec{b}] &= (\xi_2 \eta_3 - \xi_3 \eta_2) \vec{f}_1 - (\xi_1 \eta_3 - \xi_3 \eta_1) \vec{f}_2 + (\xi_1 \eta_2 - \xi_2 \eta_1) \vec{f}_3 = \\ &= \vec{f}_1 \det \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} - \vec{f}_2 \det \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix} + \vec{f}_3 \det \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} = \\ &= \det \begin{vmatrix} \vec{f}_1 & \vec{f}_2 & \vec{f}_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix}. \end{aligned}$$

## The case of an orthonormal basis

Let the original basis  $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$  *orthonormal*, forming a *right* triple of vectors, then by the definition of the vector product

$$\vec{f}_1 = \vec{e}_1, \quad \vec{f}_2 = \vec{e}_2, \quad \vec{f}_3 = \vec{e}_3.$$

Then the formula for the vector product of vectors in the *right orthonormal* basis will be noticeably simplified:

$$[\vec{a}, \vec{b}] = \det \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix}.$$

The case of an orthonormal basis

The above formulas have useful consequences.

**Corollary** For vectors  $\vec{a}$  and  $\vec{b}$  to be collinear, it is necessary and sufficient that in any basis

$$\det \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} = \det \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix} = \det \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} = 0,$$

or (in the case of  $\vec{b} \neq \vec{o}$ )  $\frac{\xi_1}{\eta_1} = \frac{\xi_2}{\eta_2} = \frac{\xi_3}{\eta_3}$ .

**Corollary** In an orthonormal basis, the area of a parallelogram constructed on the vectors  $\vec{a}$  and  $\vec{b}$ , is

$$S = \sqrt{\det^2 \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} + \det^2 \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix} + \det^2 \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}},$$

and for the case on the plane  $S = \left| \det \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} \right|$ .

## Expression of a mixed product in coordinates

Let a *right* basis be given  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$  and let in this basis the vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$  have coordinate decompositions

$$\vec{a} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3, \quad \vec{b} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3$$

and accordingly  $\vec{c} = \kappa_1 \vec{g}_1 + \kappa_2 \vec{g}_2 + \kappa_3 \vec{g}_3$ .

It was shown earlier that the vector product in coordinates can be represented as

$$[\vec{a}, \vec{b}] = \vec{f}_1 \det \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} - \vec{f}_2 \det \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix} + \vec{f}_3 \det \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix},$$

where

$$\vec{f}_1 = [\vec{g}_2, \vec{g}_3], \quad \vec{f}_2 = [\vec{g}_3, \vec{g}_1], \quad \vec{f}_3 = [\vec{g}_1, \vec{g}_2].$$

Using the definition of a mixed product, it is easy to verify that the last equalities imply the relations

$$(\vec{g}_k, \vec{f}_j) = \begin{cases} (\vec{g}_1, \vec{g}_2, \vec{g}_3) & \text{for } k = j, \\ 0 & \text{for } k \neq j \end{cases} \quad \forall k, j = 1, 2, 3.$$

Then we get for the mixed product will be

$$\begin{aligned} (\vec{a}, \vec{b}, \vec{c}) &= ([\vec{a}, \vec{b}], \vec{c}) = \left( \kappa_1 \det \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} - \right. \\ &\quad \left. -\kappa_2 \det \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix} + \kappa_3 \det \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} \right) (\vec{g}_1, \vec{g}_2, \vec{g}_3) = \\ &= \det \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix} (\vec{g}_1, \vec{g}_2, \vec{g}_3). \end{aligned}$$

Indeed, the expression in large parentheses is the expansion of the determinant of the 3rd order by its last line.

**Notes:** 1) In the case of a right orthonormal basis we have  $(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$ . Therefore, in such a basis

$$(\vec{a}, \vec{b}, \vec{c}) = \det \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{vmatrix}.$$

2) For a triple of vectors  $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$  is true

**Theorem** Triple of vectors  $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$  linearly independent.

**Corollary** The triple of vectors  $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$  forms a basis (called *reciprocal to the basis*  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$  ).