

STRAIGHT LINE IN A PLANE

1°. Forms of defining a straight line in a plane

Let a coordinate system in a plane $\{O, \vec{g}_1, \vec{g}_2\}$ and a straight line L passing through a point $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be given. A nonzero vector $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ lying in L , is called the *direction vector* for L .

Then the following statement is true.

The set of position vectors of points in a straight line L can be represented as $\vec{r} = \vec{r}_0 + \tau \vec{a}$, where τ is an arbitrary real parameter.

Let's find the coordinate form for $\vec{r} = \vec{r}_0 + \tau \vec{a}$ in the chosen coordinate system.

We have: from $\vec{r} = \vec{r}_0 + \tau \vec{a}$ it follows $\left\| \vec{r} \right\|_g = \left\| \vec{r}_0 + \tau \vec{a} \right\|_g$.

Using the statements of the theorems on actions with vectors in coordinates, we obtain

$$\left\| \vec{r} \right\|_g = \left\| \vec{r}_0 \right\|_g + \left\| \tau \vec{a} \right\|_g \quad \text{and} \quad \left\| \vec{r} \right\|_g = \left\| \vec{r}_0 \right\|_g + \tau \left\| \vec{a} \right\|_g.$$

Let in the chosen coordinate system $\left\| \vec{r} \right\|_g = \left\| \begin{matrix} x \\ y \end{matrix} \right\|$, $\left\| \vec{r}_0 \right\|_g = \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\|$ and $\left\| \vec{a} \right\|_g = \left\| \begin{matrix} a_x \\ a_y \end{matrix} \right\|$, then the equation of the line in *matrix* form will be $\left\| \begin{matrix} x \\ y \end{matrix} \right\| = \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| + \tau \left\| \begin{matrix} a_x \\ a_y \end{matrix} \right\|$.

Using operations with *matrices*, we reduce the resulting equation to the form

$$\left\| \begin{matrix} x \\ y \end{matrix} \right\| = \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| + \left\| \begin{matrix} \tau a_x \\ \tau a_y \end{matrix} \right\| \quad \Rightarrow \quad \left\| \begin{matrix} x \\ y \end{matrix} \right\| = \left\| \begin{matrix} x_0 + \tau a_x \\ y_0 + \tau a_y \end{matrix} \right\|.$$

Finally, by the definition of equality of matrices, we obtain $\begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y \end{cases} \quad \forall \tau \in \mathbf{R}.$

In coordinate representation it follows from $\begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y \end{cases}$

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y}; \quad a_x a_y \neq 0$$

$$y = y_0; \quad \forall x, \quad \text{if } a_y = 0$$

$$x = x_0; \quad \forall y, \quad \text{if } a_x = 0,$$

This notation is called the *symmetrical* form of the equation of a straight line in a plane.

We obtain that the following statements are true

- Any line in any Cartesian coordinate system can be defined by an equation of the form $Ax + By + C = 0$, $|A| + |B| > 0$.
- Each equation of the form $Ax + By + C = 0$, $|A| + |B| > 0$ in any Cartesian coordinate system is an equation of some line.
- In order for two equations of the form $A_1x + B_1y + C_1 = 0$, $|A_1| + |B_1| > 0$ and $A_2x + B_2y + C_2 = 0$, $|A_2| + |B_2| > 0$ to be equations of the same line, it is necessary and sufficient that there exists a number $\lambda \neq 0$ such that

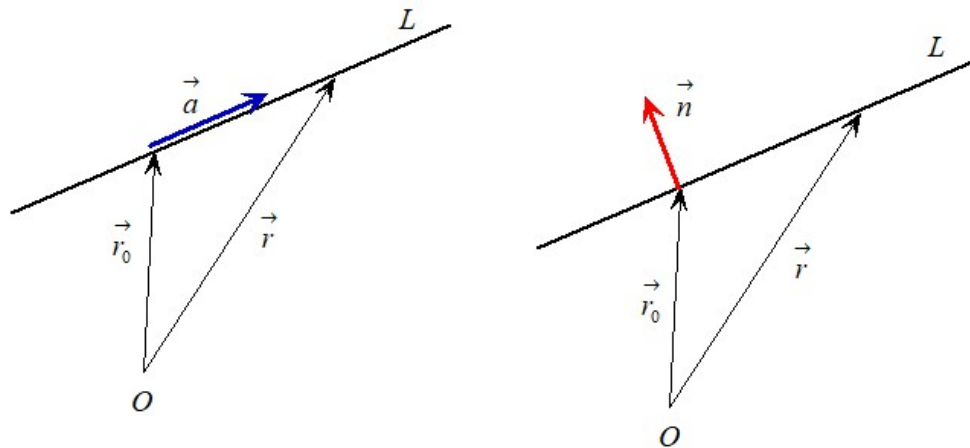
$$A_1 = \lambda A_2 ; B_1 = \lambda B_2 ; C_1 = \lambda C_2 .$$

Task 5.01 *What set of points in the coordinate system $\{O, \vec{g}_1, \vec{g}_2\}$ in the plane does the equation $Ax + By + C = 0$ describe?*

Solution

- 1°. If $|A| + |B| > 0$ and $C \in \mathbf{R}$, then this set is some *line*.
- 2°. If $A = 0, B = 0$ and $C \neq 0$, then the set is the *empty set*.
- 3°. If $A = 0, B = 0$ and $C = 0$, then the set is the *entire coordinate plane*.

Solution is found



The equation of a straight line L passing through a given point $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, perpendicular to a non-zero vector $\vec{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$, has the form $(\vec{n}, \vec{r} - \vec{r}_0) = 0$.

Or $(\vec{n}, \vec{r}) = d$, where $d = (\vec{n}, \vec{r}_0)$.

Vector \vec{n} is called the *normal* vector for L

Task 5.02 Write the equation $(\vec{n}, \vec{r}) = d$ of a straight line L as $(\vec{n}, \vec{r} - \vec{r}_0) = 0$.

Solution

Since the vector \vec{n} is contained in both forms of the equation, we only need to find the vector \vec{r}_0 .

As \vec{r}_0 we can take the position vector of any point for the line L . Therefore, we take as \vec{r}_0 the vector of the orthogonal projection of the origin onto the straight line L , which exists for any straight line. In this case, the equality $\vec{r}_0 = \lambda \vec{n}$ is obviously satisfied, where λ is some number.

Since the point \vec{r}_0 belongs to the straight line L , we have

$$(\vec{n}, \lambda \vec{n}) = d \quad \Rightarrow \quad \lambda = \frac{d}{|\vec{n}|^2}.$$

Solution is found

By comparing different forms of the straight line:

$$\frac{x-x_0}{a_x} = \frac{y-y_0}{a_y}, (\vec{n}, \vec{r}) = d, (\vec{n}, \vec{r}-\vec{r}_0) = 0 \text{ и } Ax+By+C=0, |A|+|B|>0,$$

we can establish the geometric meaning of the coefficients A, B, C .

In any Cartesian coordinate system $A = a_y, B = -a_x$. In a Cartesian coordinate system with an orthonormal basis $A = n_x, B = n_y$.

The equation of a straight line passing through two non-coinciding points $\vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

in vector form will look like $\vec{r} = (1 - \tau)\vec{r}_1 + \tau\vec{r}_2$.

In coordinate representation (after eliminating τ):

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, \quad \text{if } (x_2 - x_1)(y_2 - y_1) \neq 0$$

$$y = y_1; \quad \forall x, \quad \text{if } y_2 = y_1$$

$$x = x_1; \quad \forall y, \quad \text{if } x_2 = x_1.$$

A linear inequality $Ax + By + C \geq 0$, $|A| + |B| > 0$ defines a part of the coordinate plane whose points satisfy the inequality. In this case the straight line $Ax + By + C = 0$, $|A| + |B| > 0$ is a fragment of the boundary of this part of the plane.

Task 5.03

Construct a coordinate description of a right triangle ABC whose right angle vertex C is at the origin of the orthonormal coordinate system $\{Oxy\}$, and the other two vertices have position vectors $\vec{r}_A = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\vec{r}_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively.

Solution

The desired description is the following system of non-strict linear inequalities

$$\begin{cases} x \geq 0, \\ y \geq 0, \\ x + 3y \leq 3. \end{cases}$$

Solution is found

Task 5.04 A coordinate system $\{O, \vec{g}_1, \vec{g}_2\}$ in a plane and a line L with the equation $(\vec{n}, \vec{r} - \vec{r}_0) = 0$ are given. Find the distance from this line to a point M whose position vector is $\vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

Solution

1°. Let $\vec{MK} = \lambda \vec{n}$, then $\vec{r} = \vec{r}_1 + \lambda \vec{n}$.

2°. The point K belongs to this line, so the equality is true

$$(\vec{n}, \vec{r}_1 + \lambda \vec{n} - \vec{r}_0) = 0.$$

Whence $\lambda = -\frac{(\vec{n}, \vec{r}_1 - \vec{r}_0)}{|\vec{n}|^2}$.

3°. Substituting λ into the formula for \vec{MK} , we

obtain $|\vec{MK}| = \left| \left(\vec{r}_1 - \vec{r}_0, \frac{\vec{n}}{|\vec{n}|} \right) \right|$.

4°. Let the coordinate system be orthonormal. Then for the line

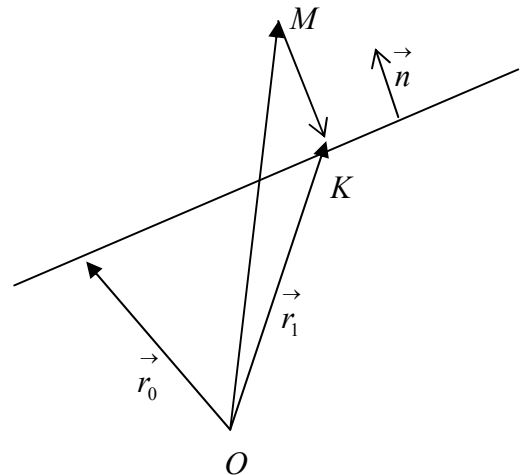
$$Ax + By + C = 0, \quad |A| + |B| > 0$$

the vector $\vec{n} = \begin{pmatrix} A \\ B \end{pmatrix}$ is normal. Therefore

$$|\vec{MK}| = \frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{A^2 + B^2}}.$$

Considering that the point \vec{r}_0 lies on the line L , we have $Ax_0 + By_0 + C = 0$. Therefore the answer can be written as

$$|\vec{MK}| = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$



Solution is found.

Task 5.05 *In a general Cartesian coordinate system, a line $9x - 5y - 8 = 0$ and points $\vec{r}_A = \begin{pmatrix} -8 \\ -9 \end{pmatrix}$ and $\vec{r}_B = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$ are given. Determine analytically whether these points lie on the same or different sides of the line.*

Solution:

- 1) Find the value for the linear function $9x - 5y - 8$ at point A :
we get $9(-8) - 5(-9) - 8 = -35 < 0$.
- 2) Find the value for the linear function $9x - 5y - 8$ at point B :
we get $9(-2) - 5(-6) - 8 = 4 > 0$.

This means that points A and B lie on different sides of the line $9x - 5y - 8 = 0$.

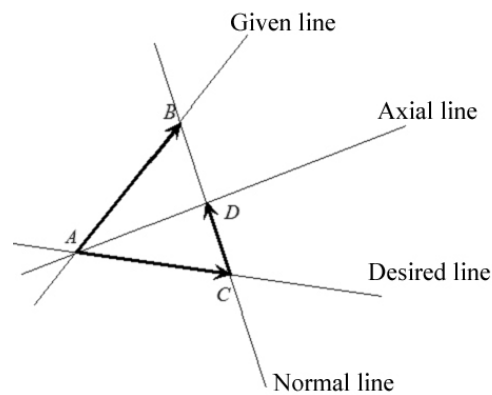
Solution is found

Task 5.06 *On a plane with an orthonormal coordinate system, find the "desired" line, which is symmetrical to the "given" line with an equation $11x - 2y + 31 = 0$, relative to the "axial" line with the equation $4x - 3y + 9 = 0$.*

Solution:

1°. Let the *desired* line pass through two non-coinciding points with coordinates $(x_0; y_0)$ and

$(x_1; y_1)$. Then its equation will be $\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$.



2°. As the first point, we can choose point A – the intersection point of the *given* and *axial* lines. The coordinates of this point are obviously determined by the system of equations:

$$\begin{cases} 11x_0 - 2y_0 = -31, \\ 4x_0 - 3y_0 = -9, \end{cases}$$

which has a solution $\begin{cases} x_0 = -3, \\ y_0 = -1. \end{cases}$

3°. On the *given* line, we choose some point B that does not coincide with point A . For example, a point with coordinates $(-1; 10)$. Let us draw a line through B *normal* (that is, perpendicular) to the *axial* line. Let point C , belonging to the *normal* line, be symmetrical to point B relative to the *axial* line,

In this case, if we designate the intersection point of the *axial* and *normal* lines as D , we will have $|BD| = |DC|$.

4°. Now let us find the coordinates of point C . First, let us note that, due to the choice of points B and C , the normal vector of the *axial* line \vec{CD} will be the direction vector of the *normal* line. And, since the coordinate system is rectangular, the equation of the *normal* line will be $3x + 4y + K = 0$, where K is some constant.

We will find the value of K , taking into account that point B belongs to the *normal* line, that is,

$$3 \cdot (-1) + 4 \cdot 10 + K = 0 \quad \Rightarrow \quad K = -37.$$

Therefore, the *normal* line has an equation $3x + 4y - 37 = 0$ and point C belongs to this line.

5°. On the other hand, let point C have coordinates $(x_1; y_1)$. Then, since $|BD| = |DC|$, using the formula for the *distance from a point to a line*, we write this equality as

$$\frac{|4 \cdot (-1) - 3 \cdot 10 + 9|}{\sqrt{4^2 + 3^2}} = \frac{|4x_1 - 3y_1 + 9|}{\sqrt{4^2 + 3^2}} \quad \text{or} \quad \frac{|-25|}{5} = \frac{|4x_1 - 3y_1 + 9|}{5}.$$

Knowing that points B and C lie on different sides of the *axial* line, we open modules with opposite signs for their interiors, which gives $4x_1 - 3y_1 - 16 = 0$.

6°. Thus, for the coordinates of point C we have two conditions, which we write down in the form of a system $\begin{cases} 3x_1 + 4y_1 = 37, \\ 4x_1 - 3y_1 = 16, \end{cases}$ from which we obtain that $\begin{cases} x_1 = 7, \\ y_1 = 4. \end{cases}$

7°. Finally, substituting the values of the coordinates of points A and C into the formula from point 1°, we obtain the equation of the desired line:

$$\frac{x - (-3)}{7 - (-3)} = \frac{y - (-1)}{4 - (-1)} \quad \text{or, after simplification,} \quad x - 2y + 1 = 0.$$

Solution is found