

STRAIGHT LINE AND PLANE IN SPACE

Forms of defining a plane in space

Let a coordinate system in space $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ and a plane S passing through a point $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be given, with non-collinear vectors $\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}$ lying on S . Vectors $\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}$ are called *direction vectors* for S . Then the following statement is true:

- The set of position vectors of points of plane S can be represented as

$$\vec{r} = \vec{r}_0 + \varphi \vec{p} + \theta \vec{q},$$

where φ and θ are arbitrary real numbers.

Since this equation is equivalent to the condition of coplanarity of vectors $\vec{r}-\vec{r}_0$, \vec{p} and \vec{q} , it can be written as

$$\boxed{(\vec{r}-\vec{r}_0, \vec{p}, \vec{q}) = 0.}$$

Or, in coordinate $\det \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix} = 0 .$

- Any plane in any Cartesian coordinate system can be defined by an equation of the form $Ax + By + Cz + D = 0$, $|A| + |B| + |C| > 0$.
- Each equation of the form $Ax + By + Cz + D = 0$, $|A| + |B| + |C| > 0$ in any Cartesian coordinate system defines a certain plane.

The equation of a plane S passing through a point with a position vector $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ perpendicular to a nonzero vector $\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ has the form

$$\boxed{(\vec{n}, \vec{r} - \vec{r}_0) = 0, \quad \text{or} \quad (\vec{n}, \vec{r}) = d, \quad \text{where} \quad d = (\vec{n}, \vec{r}_0).}$$

The vector \vec{n} is called the *normal vector* of the plane S .

- If a plane S is defined in an *orthonormal* coordinate system $\{\vec{O}, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ by the equation $Ax + By + Cz + D = 0$, $|A| + |B| + |C| > 0$, then the vector $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is orthogonal to this plane S .

The equation of a plane passing through three pairwise non-coinciding and non-collinear points $\vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$; $\vec{r}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$; $\vec{r}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ has the form $(\vec{r} - \vec{r}_1, \vec{r}_2 - \vec{r}_1, \vec{r}_3 - \vec{r}_1) = 0$.

Or, in coordinate representation

$$\det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0..$$

Forms of defining a straight line in space

A straight line L in space, having a non-zero direction vector $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and passing through a point with position vector $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, is defined by an equation of the form

$$\vec{r} = \vec{r}_0 + \tau \vec{a} .$$

If the parameter τ is excluded from the scalar notation of this equation $\begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y \\ z = z_0 + \tau a_z \end{cases}$, then we obtain a *standard* system of the form

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z} .$$

A straight line L in space can be defined as the intersection line of planes of the form $(\vec{n}_1, \vec{r}) = d_1$ and $(\vec{n}_2, \vec{r}) = d_2$. Here \vec{n}_1 and \vec{n}_2 are *non-collinear* normal vectors of these planes, and d_1 and d_2 are some numbers. The vector description of the straight line L will be

$$\begin{cases} (\vec{n}_1, \vec{r}) = d_1 \\ (\vec{n}_2, \vec{r}) = d_2 \end{cases}$$

If the position vector \vec{r}_0 of some point of a straight line L is known, then its description will be $\begin{cases} (\vec{n}_1, \vec{r} - \vec{r}_0) = 0, \\ (\vec{n}_2, \vec{r} - \vec{r}_0) = 0. \end{cases}$ The coordinate form of the description of L in these cases will be

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

The condition of collinearity of vectors \vec{a} and $\vec{r}-\vec{r}_0$ when specifying a straight line L can be written as $[\vec{a}, \vec{r}-\vec{r}_0] = \vec{o}$ or

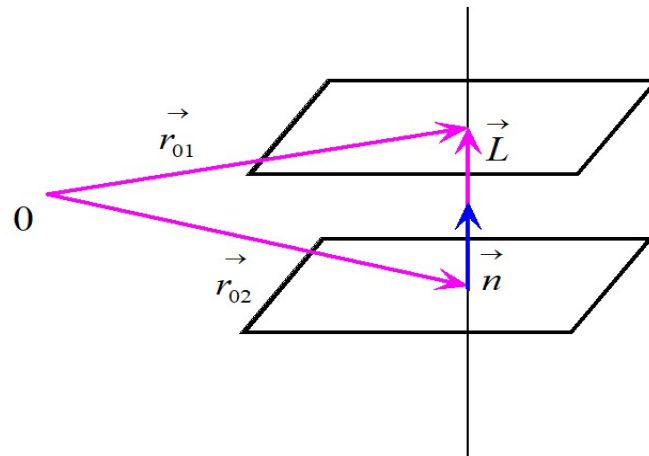
$$[\vec{a}, \vec{r}] = \vec{b},$$

where $\vec{b} = [\vec{a}, \vec{r}_0]$.

In an Cartesian coordinate system $\{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ with the *orthonormal basis*, this method of describing a straight line in space L has the form

$$\det \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix} = \vec{b} \quad \text{or} \quad \begin{cases} a_y z - a_z y = b_x, \\ a_z x - a_x z = b_y, \\ a_x y - a_y x = b_z. \end{cases}$$

Note that in the latter system only two equations of the three are independent.



Task 6.1. Find the distance between planes $(\vec{n}, \vec{r}) = d_1$ and $(\vec{n}, \vec{r}) = d_2$.

Solution: 1) Let \vec{r}_{01} and \vec{r}_{02} be the position vectors of the intersection points of the common perpendicular with the planes. Then we have $\vec{L} = \vec{r}_{01} - \vec{r}_{02}$ and $\vec{L} = \lambda \vec{n}$. The desired distance is obviously equaled to $S = |\vec{L}|$.

2) We can find λ from the condition $\vec{r}_{01} - \vec{r}_{02} = \lambda \vec{n}$. Multiplying both parts scalarly by \vec{n} , we obtain

$$(\vec{r}_{01}, \vec{n}) - (\vec{r}_{02}, \vec{n}) = \lambda (\vec{n}, \vec{n}) \Rightarrow d_1 - d_2 = \lambda |\vec{n}|^2 \Rightarrow \lambda = \frac{d_1 - d_2}{|\vec{n}|^2}.$$

$$3) \text{ Whence } S = |\vec{L}| = |\lambda \vec{n}| = |\lambda| |\vec{n}| = \frac{|d_1 - d_2|}{|\vec{n}|}$$

Solution is found

Task 6.2. Write an equation of a plane passing through a point $A(5,-4,3)$, perpendicular to a line passing through points $B(-1,-2,1)$ and $C(-6,-4,3)$. The coordinate system is Cartesian with an orthonormal basis.

Solution: 1) Let the points A, B, C have position vectors \vec{r}_0, \vec{r}_1 and \vec{r}_2 , respectively, and an arbitrary point of the plane has position vector \vec{r} .

2) Note that the vectors $\vec{r}_2 - \vec{r}_1$ and $\vec{r} - \vec{r}_0$ in this case are orthogonal for any \vec{r} . Therefore, the desired equation will be $(\vec{r}_2 - \vec{r}_1, \vec{r} - \vec{r}_0) = 0$.

3) Let the vector \vec{r} has coordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then the equation of the plane has

the form

$$(-6+1)(x-5) + (-4+2)(y+4) + (3-1)(z-3) = 0 \quad \Rightarrow \quad 5x + 2y - 2z = 11$$

Solution is found

Task 6.3. Find a plane passing through the line $\begin{cases} x = 1 - 5\tau \\ y = 2 - \tau \\ z = 1 + 4\tau \end{cases}$, parallel to the line

$$\begin{cases} x = -3 + 3\tau \\ y = -2 - 5\tau \\ z = 2 - 4\tau \end{cases}. \text{ The Cartesian coordinate system is arbitrary.}$$

Solution: 1) Let the given lines have the following parametric form in vector form

$$\vec{r} = \vec{r}_{01} + \tau \vec{a}_1 \quad \text{and} \quad \vec{r} = \vec{r}_{02} + \tau \vec{a}_2.$$

That is, we know the point \vec{r}_{01} , through which the sought plane is guaranteed to pass, since the first line also passes through this point.

2) Vectors \vec{a}_1 and \vec{a}_2 are colinear with the desired plane by condition. Let the vector \vec{r} of an arbitrary point of this plane has coordinates $\begin{vmatrix} x \\ y \\ z \end{vmatrix}$. This means that the triple of vectors $\vec{r}-\vec{r}_{01}$, \vec{a}_1 and \vec{a}_2 is coplanar with the desired plane, and its equation can be written as $(\vec{r}-\vec{r}_{01}, \vec{a}_1, \vec{a}_2) = 0$.

3) In an arbitrary coordinate system, this equation will be

$$\det \begin{vmatrix} x-1 & y-2 & z-1 \\ -5 & -1 & 4 \\ 3 & -5 & -4 \end{vmatrix} \cdot (g_1, g_2, g_3) = 0,$$

where the basis vectors $\{g_1; g_2; g_3\}$ are linearly independent and, therefore, their mixed product is nonzero. Then the equation of the desired plane will be

$$\det \begin{vmatrix} x-1 & y-2 & z-1 \\ -5 & -1 & 4 \\ 3 & -5 & -4 \end{vmatrix} = 0 \Rightarrow 6x - 2y + 7z = 9.$$

Solution is found

Task 6.4. Find the intersection point of a line $\begin{cases} x = -3 + 4\tau \\ y = 1 - 4\tau \\ z = -5 + \tau \end{cases}$ and a plane $x + 4z = -7$.

The Cartesian coordinate system is arbitrary.

Solution: 1) Let the given line and plane have the following form in vector form $\vec{r} = \vec{r}_0 + \tau \vec{a}$ and $(\vec{n}, \vec{r}) = d$. Let us denote the desired intersection point as \vec{R} , and we can assume that $\vec{R} = \vec{r}_0 + \lambda \vec{a}$.

2) Since \vec{R} belongs to both the line and the plane, the value of the parameter λ can be found from the system of equations

$$\begin{cases} \vec{R} = \vec{r}_0 + \lambda \vec{a} \\ (\vec{n}, \vec{R}) = d \end{cases} \Rightarrow (\vec{n}, \vec{r}_0 + \lambda \vec{a}) = d \Rightarrow \lambda = \frac{d - (\vec{n}, \vec{r}_0)}{(\vec{n}, \vec{a})}.$$

Where from $\vec{R} = \vec{r}_0 + \frac{d - (\vec{n}, \vec{r}_0)}{(\vec{n}, \vec{a})} \vec{a}$.

3) It is more convenient to find the values of the coordinates of the desired point $\vec{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ not using the obtained formula, but directly from the system of linear equations

$$\begin{cases} X = -3 + 4\lambda, \\ Y = 1 - 4\lambda, \\ Z = -5 + \lambda, \\ X + 4Z + 7 = 0. \end{cases}$$

If we substitute X and Z from the first and third equations into the fourth,

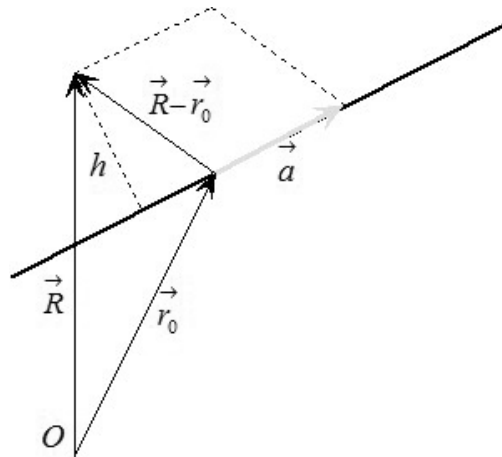
then we immediately get that $\lambda = 2$ and, therefore, $\vec{R} = \begin{pmatrix} 5 \\ -7 \\ -3 \end{pmatrix}$.

Solution is found

Task 6.5 Find the distance from a point with a position vector \vec{R} to a line
 $\vec{r} = \vec{r}_0 + \tau \vec{a}$.

Solution: The distance h in space from a certain point with a position vector \vec{R} to a line $\vec{r} = \vec{r}_0 + \tau \vec{a}$ can be found using the following property. The area S of a parallelogram constructed on a pair of vectors is equal to the length of the vector product of these vectors. As a result, we obtain

$$h = \frac{S}{|\vec{a}|} = \frac{|[\vec{R} - \vec{r}_0, \vec{a}]|}{|\vec{a}|}.$$

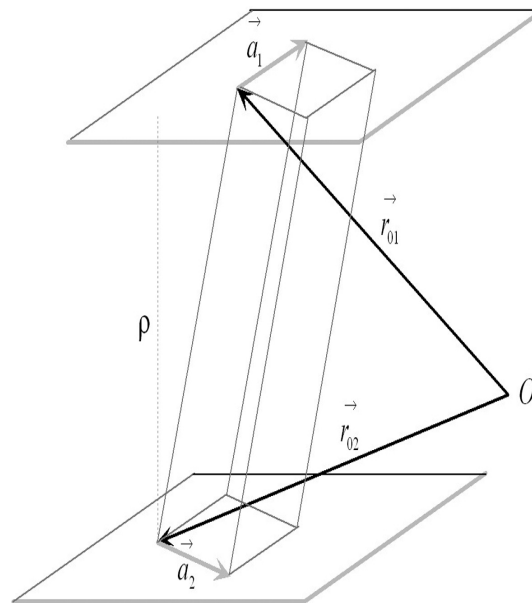


Solution is found

Task 6.6 Find the distance between the lines $\vec{r} = \vec{r}_{01} + \tau \vec{a}_1$ and $\vec{r} = \vec{r}_{02} + \tau \vec{a}_2$.

Solution: 1°. If the vectors \vec{a}_1 and \vec{a}_2 are collinear, then the solution is similar to the solution of the task 6.5.

2°. Let the vectors \vec{a}_1 and \vec{a}_2 be non-collinear, then we construct a pair of planes parallel to these vectors, one of which contains the point \vec{r}_{01} , and the other the point \vec{r}_{02} .



The volume of the parallelepiped constructed on the vectors \vec{a}_1 , \vec{a}_2 and $\vec{r}_{02} - \vec{r}_{01}$, is equal, on the one hand, to the product of the area of the parallelogram located at the base, by the desired value ρ and

$\left| (\vec{r}_{02} - \vec{r}_{01}, \vec{a}_1, \vec{a}_2) \right|$ – on the other. Whence we find that

$$\rho = \frac{|(\vec{r}_{02} - \vec{r}_{01}, \vec{a}_1, \vec{a}_2)|}{|[\vec{a}_1, \vec{a}_2]|}.$$

Solution is found

Task 6.7 A plane $(\vec{n}, \vec{r}) = d$ and a line $[\vec{a}, \vec{r}] = \vec{b}$ are given. Find the position vector of their intersection point if $(\vec{n}, \vec{a}) \neq 0$.

Solution: 1°. Multiplying both sides of the vector equation of the line from the left by \vec{n} , we obtain $[\vec{n}, [\vec{a}, \vec{r}]] = [\vec{n}, \vec{b}]$. Substituting the desired vector \vec{R} into this relation and applying the "bac-cab" formula, we arrive at the equality

$$\vec{a}(\vec{n}, \vec{R}) - \vec{R}(\vec{n}, \vec{a}) = [\vec{n}, \vec{b}].$$

Since the point \vec{R} belongs to the given plane, the equality $(\vec{n}, \vec{R}) = d$ is true.

Then, under the constraint $(\vec{n}, \vec{a}) \neq 0$, we obtain

$$\vec{R} = \frac{d \vec{a} - [\vec{n}, \vec{b}]}{(\vec{n}, \vec{a})}.$$

Solution is found