

Lines and surfaces on the plane and in space (Theory)

Let a coordinate system in the plane $\{O, \vec{g}_1, \vec{g}_2\}$ and a numerical set Ω be given that is an interval (possibly infinite).

We will say that a line L in the plane is defined *parametrically* by a vector function $\vec{r} = \vec{F}(\tau)$ (or in coordinate form

$$\left\| \begin{array}{c} \vec{r} \\ \parallel_g \end{array} \right\| = \left\| \begin{array}{c} F_x(\tau) \\ F_y(\tau) \end{array} \right\|,$$

where $F_x(\tau), F_y(\tau)$ are continuous, scalar functions of argument τ , defined for $\tau \in \Omega$), if

- 1) for any $\tau \in \Omega$ point $\vec{r} = \vec{F}(\tau)$ lies in L ;
- 2) for any point \vec{r}_0 lying on L , there exists $\tau_0 \in \Omega$ such that the equality holds $\vec{r}_0 = \vec{F}(\tau_0)$.

Sometimes a line in a plane is defined as an equation $G(x, y) = 0$, which is obtained by eliminating the parameter τ from the system of equations $\begin{cases} x = F_x(\tau) \\ y = F_y(\tau) \end{cases}, \tau \in \Omega$.

1°. A straight line, for example, is defined by a vector function $\vec{r} = \vec{r}_0 + \tau \vec{a}$, where \vec{a} is the direction vector, and \vec{r}_0 is one of the points of this line. The scalar form of defining a line in this case has the form

$$\begin{cases} x = x_0 + \tau a_x, \\ y = y_0 + \tau a_y, \end{cases} \tau \in (-\infty, +\infty),$$

that is, $\begin{cases} F_x(\tau) = x_0 + \tau a_x, \\ F_y(\tau) = y_0 + \tau a_y, \end{cases} \tau \in (-\infty, +\infty),$

or $Ax + By + C = 0, |A| + |B| > 0$, where $G(x, y) = Ax + By + C$.

2°. In a Cartesian coordinate system with an *orthonormal* basis, a circle of radius R with center at a point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in parametric form can be defined as

$$\begin{cases} x = x_0 + R \cos \tau, \\ y = y_0 + R \sin \tau, \end{cases} \quad \tau \in [0, 2\pi),$$

that is,

$$\begin{cases} F_x(\tau) = x_0 + R \cos \tau, \\ F_y(\tau) = y_0 + R \sin \tau, \end{cases} \quad \tau \in [0, 2\pi),$$

or by the equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2,$$

where $G(x, y) = (x - x_0)^2 + (y - y_0)^2 - R^2$.

A line L is called *algebraic* if its equation in a Cartesian coordinate system has the form $\sum_{k=0}^m \alpha_k x^{p_k} y^{q_k} = 0$, where p_k and q_k are non-negative integers, and the numbers α_k are not equal to zero simultaneously.

The number $N = \max_{k=[0,m]} \{p_k + q_k\}$ is called the *order of the algebraic equation*, where the maximum is found over all k , for which $\alpha_k \neq 0$. The *smallest* of the orders of the algebraic equations defining a given algebraic line is called the *order of the algebraic line*.

Name	Equation	Order
<i>Straight line</i>	$x + 3y + 2 = 0$	$(N = 1)$
<i>Square parabola</i>	$y - x^2 = 0$	$(N = 2)$
<i>Hyperbola</i>	$xy - 1 = 0$	$(N = 2)$
<i>"Cartesian leaf"</i>	$x^3 + y^3 - xy = 0$	$(N = 3)$

Theorem **The order of an algebraic line does not depend on the choice of coordinate system.**

Proof.

Let an algebraic line L have an equation $G(x, y) = 0$ and order N in the coordinate system $\{O, \vec{g}_1, \vec{g}_2\}$. Let us move to the coordinate system $\{O, \vec{g}'_1, \vec{g}'_2\}$. The transition formulas have the form

$$\begin{cases} x = \sigma_{11}x' + \sigma_{12}y' + \beta_1, \\ y = \sigma_{21}x' + \sigma_{22}y' + \beta_2, \end{cases}$$

the equation of the line L in the “new” coordinate system will be

$$G(\sigma_{11}x' + \sigma_{12}y' + \beta_1, \sigma_{21}x' + \sigma_{22}y' + \beta_2) = 0.$$

It follows from this that $N \geq N'$, that is, when moving to the “new” coordinate system, the order of the algebraic curve cannot increase.

Using similar reasoning for the reverse transition from the coordinate system $\{O, \vec{g}'_1, \vec{g}'_2\}$ to the system $\{O, \vec{g}_1, \vec{g}_2\}$, we obtain $N \leq N'$ and finally $N = N'$.

Theorem is proven.

Figures in the plane can be defined using inequality-type constraints.

1°. In an *orthonormal* coordinate system, a set of conditions $\begin{cases} x \geq 0, \\ y \geq 0, \\ x + y - 5 \leq 0 \end{cases}$ defines a right isosceles triangle whose legs lie in the coordinate axes and have lengths of 5.

2°. In an *orthonormal* coordinate system, an inequality of the type $x^2 + y^2 - 4 \leq 0$ defines a circle of radius 2 with center at the origin.

Lines in space

Let a Cartesian coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ be given.

We will say that a line L in space is defined parametrically by a vector function $\vec{r} = \vec{F}(\tau)$ (or in coordinate form

$$\begin{cases} x \\ y \\ z \end{cases} = \begin{cases} F_x(\tau) \\ F_y(\tau) \\ F_z(\tau) \end{cases},$$

where $F_x(\tau), F_y(\tau), F_z(\tau)$ are continuous, scalar functions of τ , defined for $\tau \in \Omega$, if

- 1) for any $\tau \in \Omega$ point $\vec{r} = \vec{F}(\tau)$ lies in L ,
- 2) for any point \vec{r}_0 lying in L , there exists $\tau_0 \in \Omega$, such that the equality is satisfied $\vec{r}_0 = \vec{F}(\tau_0)$.

Sometimes a line in space is defined by a system of equations

$$\begin{cases} G(x, y, z) = 0, \\ H(x, y, z) = 0, \end{cases}$$

which is obtained by excluding the parameter τ from the relations

$$\begin{cases} x = F_x(\tau), \\ y = F_y(\tau), \\ z = F_z(\tau), \end{cases} \quad \tau \in \Omega,$$

or by an equivalent equation, for example, of the form

$$G^2(x, y, z) + H^2(x, y, z) = 0.$$

1°. In a Cartesian coordinate system, a second-order algebraic line $x^2 + y^2 = 0 \quad \forall z$ is a *straight line*.

2°. In an *orthonormal* coordinate system, a helical line of radius R with a pitch $2\pi a$ can be specified in the following parametric form:

$$\begin{cases} x = R \cos \tau, \\ y = R \sin \tau, \tau \in (-\infty, +\infty), \\ z = a\tau \end{cases} \quad \text{or} \quad \begin{cases} x = R \cos \frac{z}{a}, \\ y = R \sin \frac{z}{a}. \end{cases}$$

Surfaces in space

Let there be a Cartesian coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ and Ω is a set of ordered pairs of numbers φ, θ , defined by the conditions: $\alpha \leq \varphi \leq \beta, \gamma \leq \theta \leq \delta$.

We will say that in space a surface S is defined parametrically by a vector function

$\vec{r} = \vec{F}(\varphi, \theta)$ (or in coordinate form

$$\left\| \vec{r} \right\|_g = \left\| \begin{matrix} F_x(\varphi, \theta) \\ F_y(\varphi, \theta) \\ F_z(\varphi, \theta) \end{matrix} \right\|$$

where $F_x(\varphi, \theta), F_y(\varphi, \theta), F_z(\varphi, \theta)$ are continuous scalar functions of two arguments φ, θ , defined for $\varphi, \theta \in \Omega$), if

- 1) for any ordered pair of numbers $\varphi, \theta \in \Omega$ the point $\vec{r} = \vec{F}(\varphi, \theta)$ lies in S ,
- 2) for any \vec{r}_0 point lying in S , there exists an ordered pair of numbers $\varphi_0, \theta_0 \in \Omega$, such that the equality $\vec{r}_0 = \vec{F}(\varphi_0, \theta_0)$ holds.

Иногда поверхность в пространстве задается в виде уравнения $G(x, y, z) = 0$, которое получается исключением φ и θ из системы уравнений Sometimes a surface in space is defined in the form of an equation $G(x, y, z) = 0$, which is obtained by excluding φ and θ from the system of equations

$$\begin{cases} x = F_x(\varphi, \theta), \\ y = F_y(\varphi, \theta), \\ z = F_z(\varphi, \theta). \end{cases} \quad \varphi, \theta \in \Omega.$$

In an *orthonormal* coordinate system, a *sphere* of radius R with center at a point $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ can be *parametrically* defined as

$$\begin{cases} x = x_0 + R \cos \varphi \sin \theta, & 0 \leq \varphi < 2\pi, \\ y = y_0 + R \sin \varphi \sin \theta, & 0 \leq \theta \leq \pi, \\ z = z_0 + R \cos \theta, \end{cases}$$

and its equation in coordinates

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

A surface S is called *algebraic* if its equation in a Cartesian coordinate system has the form $\sum_{k=0}^m \alpha_k x^{p_k} y^{q_k} z^{r_k} = 0$, where p_k, q_k and r_k are non-negative integers, and the numbers α_k are not equal to zero simultaneously.

The number $N = \max_{k=[0,m]} \{p_k + q_k + r_k\}$ is called the *order of the algebraic equation*, where the maximum is found over all k for which $\alpha_k \neq 0$. The *smallest* of the orders of the algebraic equations defining a given algebraic surface is called the *order of the algebraic surface*.

Name	Equation	Order
<i>Right circular cylinder</i>	$x^2 + y^2 - 1 = 0$	$(N = 2)$
<i>Sphere</i>	$x^2 + y^2 + z^2 - R^2 = 0$	$(N = 2)$

Theorem The order of an algebraic surface does not depend on the choice of coordinate system.