

## Second-order lines on a plane

Let an *orthonormal* coordinate system be given on a plane  $\{O, \vec{e}_1, \vec{e}_2\}$ .

**Definition** A line  $L$  is called an *algebraic second-order line* if its equation in a given coordinate system has the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where the numbers  $A$ ,  $B$  and  $C$  are not equal to zero simultaneously, and  $x$  and  $y$  are the coordinates of the radius vector of a point belonging to  $L$ .

Since the coefficients of this equation depend on the choice of coordinate system, when studying the properties of second-order lines, it is advisable to try to find another orthonormal coordinate system  $\{O', \vec{e}'_1, \vec{e}'_2\}$  in which the equation of the line is *simpler*.

The solution to this problem is given by

**Theorem** For any second-order line, there exists an *orthonormal* coordinate system in which the equation of this line (for  $a \geq b > 0, p > 0$ ) has one of the following nine (called *canonical*) forms:

$$T \setminus, a \setminus, b \setminus, l \setminus, e \ 1$$

<i>Line type</i>	<i>Elliptic</i> $\Delta > 0$	<i>Hyperbolic</i> $\Delta < 0$	<i>Parabolic</i> $\Delta = 0$
<i>Line class</i>			
<i>Empty sets</i>	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = -1$		$y'^2 = -a^2 \quad \forall x'$
<i>Isolated points</i>	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 0$		
<i>Coincident lines</i>			$y'^2 = 0 \quad \forall x'$
<i>Non-coincident lines</i>		$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 0$	$y'^2 = a^2 \quad \forall x'$
<i>Curves</i>	<i>Ellipse</i> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$	<i>Hyperbola</i> $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$	<i>Parabola</i> $y'^2 = 2px'$

where  $\Delta = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$ .

The equations of the *tangents* to an ellipse, hyperbola and parabola passing through a point  $(x_0, y_0)$  belonging to a line in the canonical coordinate system have, respectively, of the form

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \quad \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1 \quad \text{and} \quad yy_0 = p(x + x_0).$$

The following problems illustrate methods for studying the properties of second-order lines.

Task 8.1. Find all points on an ellipse  $\frac{x^2}{32} + \frac{y^2}{8} = 1$  for which the tangent is parallel to the line  $x - 2y + 4 = 0$ .

Solution: 1) The equation of the tangent to a given ellipse passing through a point on the ellipse with coordinates  $(x_0, y_0)$  is of the form:  $\frac{x_0x}{32} + \frac{y_0y}{8} = 1$ .

2) The condition of parallelism or coincidence on the plane of two lines

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2 = 0$$

has the form:  $\exists \lambda \neq 0$  such that  $A_1 = \lambda A_2$  and  $B_1 = \lambda B_2$ . In this problem, this condition gives the relations

$$\begin{aligned} \frac{x_0}{32} = \lambda \cdot 1 \quad \text{and} \quad \frac{y_0}{8} = \lambda \cdot (-2) &\Rightarrow \\ \Rightarrow \quad x_0 = 32\lambda \quad \text{and} \quad y_0 = -16\lambda. & \end{aligned}$$

3) The point with coordinates  $x_0$  and  $y_0$  belongs to the ellipse, therefore

$$\frac{x_0^2}{32} + \frac{y_0^2}{8} = 1 \Rightarrow \frac{(32\lambda)^2}{32} + \frac{(-16\lambda)^2}{8} = 1 \Rightarrow 64\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{8}.$$

From which we obtain two points:  $(4, -2)$  and  $(-4, 2)$ , which are the solution to the problem.

Solution is found

Task 8.2. *On a hyperbola  $\frac{x^2}{4} - y^2 = 1$ , find all points whose distance to one of the asymptotes is equal to three times the distance to the other.*

Solution: 1) The equation of the tangent to a given hyperbola passing through a point with coordinates  $x_0$  and  $y_0$  has the form:  $\frac{x_0 x}{4} - y_0 y = 1$ . In this case, the equations of the asymptotes of this hyperbola will be  $y = \pm \frac{x}{2}$ .

2) Recall that the distance from a point with coordinates  $x_0$  and  $y_0$  to a line of the form  $Ax + By + C = 0$ , in an orthonormal coordinate system is given by the formula

$$L = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Therefore, the distances to the first and second asymptotes in our problem will be

$$L_1 = \frac{\left| -\frac{x_0}{2} + y_0 \right|}{\sqrt{\left(\frac{1}{2}\right)^2 + 1^2}} = \frac{|-x_0 + 2y_0|}{\sqrt{5}} \quad \text{and} \quad L_2 = \frac{\left| \frac{x_0}{2} + y_0 \right|}{\sqrt{\left(\frac{1}{2}\right)^2 + 1^2}} = \frac{|x_0 + 2y_0|}{\sqrt{5}}.$$

Since the condition does not specify which of the asymptotes the point is closer to, we will have to consider two cases:  $L_1 = 3L_2$  and  $3L_1 = L_2$ .

3) Let's consider the first case. Since the point with coordinates  $x_0$  and  $y_0$  belongs to the hyperbola, it is necessary to solve the system of equations

$$\begin{cases} |-x_0 + 2y_0| = 3|x_0 + 2y_0|, \\ \frac{x_0^2}{4} - y_0^2 = 1. \end{cases}$$

The expansion of the absolute value symbols leads to two variants:

$$\begin{cases} -x_0 + 2y_0 = 3(x_0 + 2y_0), \\ \frac{x_0^2}{4} - y_0^2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} x_0 - 2y_0 = 3(x_0 + 2y_0), \\ \frac{x_0^2}{4} - y_0^2 = 1 \end{cases}$$

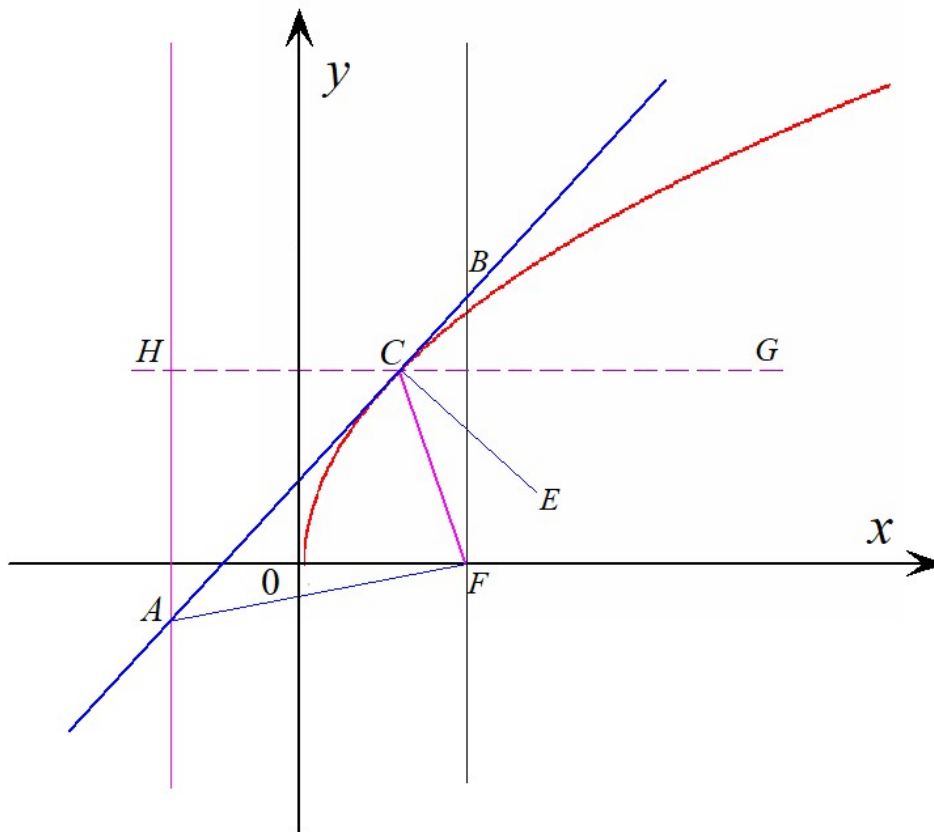
In the first variant, from the first equation we have  $-x_0 = y_0$ , which, by virtue of the second equation, yields  $-\frac{3y_0^2}{4} = 1$ , that is, there are no solutions here.

In the second variant, from the first equation we obtain  $x_0 = -4y_0$ . Then from the second equation we have  $3y_0^2 = 1$ , or  $y_0 = \pm \frac{1}{\sqrt{3}} \Rightarrow x_0 = \pm \frac{4}{\sqrt{3}}$ . This yields two solutions to the problem  $\left(-\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$  and  $\left(\frac{4}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ .

4) The reasoning in the case  $3L_1 = L_2$  is similar. They yield two solutions to the problem  $\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$  and  $\left(-\frac{4}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ , which can also be obtained from the first pair of solutions and the symmetry of the hyperbola branches relative to the canonical coordinate axes.

Solution is found

Task 8.3. Find out whether the statement is true: *the tangent to the parabola intersects the directrix and the focal chord perpendicular to the parabola axis at points equidistant from the focus. Justify your answer.*



Explanations for the figure:

the red line is the parabola,  
 the blue line is the tangent,  
 line  $AH$  is the directrix,  
 line  $FB$  is the focal chord,  
 $C$  is the point of tangency of the parabola,  
 $A$  is the point of intersection of the tangent and the directrix,  
 $B$  is the point of intersection of the tangent and the focal chord,  
 прямая  $HG$  параллельна оси  $Ox$  the line  $HG$  is parallel to the axis  $Ox$ ,  
 the line  $CE$  is perpendicular to the tangent.



Let us give two versions of proof of the validity of this statement: standard geometric (using a drawing) and analytical, not using visual images.

### **Geometric proof**

Angles  $GCE$  and  $ECF$  are equal by the optical property of a parabola and are equal to  $\alpha$ .

Angles  $ECF$  and  $CBF$  are equal  $\alpha$  as angles with mutually perpendicular sides

Angles  $DAC$  and  $CBF$  are equal  $\alpha$  as intersecting at parallel lines.

Angles  $ACF$  and  $BCG$  are equal to  $\beta = \frac{\pi}{2} - \alpha$ .

Angle  $HCA$  is also equal to  $\beta = \frac{\pi}{2} - \alpha$ .

Then, by virtue of  $HC = CF$  (since the eccentricity for a parabola is 1), triangles  $AHC$  and  $ACF$  are equal, which means angles  $DAC$  and  $CAF$  are equal.

But then angles  $CAF$  and  $CBF$  are equal  $\alpha$ , that is, triangle  $ABF$  is isosceles and  $AF = BF$ .

**Analytical proof**

Let the point of tangency have coordinates  $x_0$  and  $y_0$ . Then we have that  $y_0^2 = 2px_0$ , and the equation of the tangent will be  $y_0y = p(x + x_0)$ . In addition, we take into account that the equation of the directrix is  $x = -\frac{p}{2}$ , and the equation of the focal chord is  $x = \frac{p}{2}$ . Also note that  $FB$  is equal to the modulus of the ordinate of point  $B$ , and  $FA$  is the distance between points  $F$  and  $A$ .

The focal point  $F$  has coordinates  $\left(\frac{p}{2}, 0\right)$ . Then the coordinates of point  $B$  are defined as follows:

$$\begin{cases} x_B = \frac{p}{2} \\ y_0 y_B = p(x_B + x_0) \end{cases} \Rightarrow y_B = \frac{p}{y_0} \left( \frac{p}{2} + x_0 \right) \Rightarrow y_B = \frac{p^2 + px_0}{2y_0}.$$

On the other hand, the coordinates of point  $A$  are found from the system

$$\begin{cases} x_A = -\frac{p}{2} \\ y_0 y_A = p(x_A + x_0) \end{cases} \Rightarrow y_A = \frac{p}{y_0} \left( -\frac{p}{2} + x_0 \right) \Rightarrow y_A = \frac{-p^2 + 2px_0}{2y_0}.$$

Since the coordinate system is orthonormal, then taking into account  $y_0^2 = 2px_0$  we obtain

$$\begin{aligned} FA &= \sqrt{\left(x_A - \frac{p}{2}\right)^2 + y_A^2} = \sqrt{\left(-\frac{p}{2} - \frac{p}{2}\right)^2 + \left(\frac{-p^2 + 2px_0}{2y_0}\right)^2} = \\ &= \sqrt{\frac{4p^2 y_0^2 + p^4 - 4p^3 x_0 + 4p^2 x_0^2}{4y_0^2}} = \sqrt{\frac{8p^3 x_0 + p^4 - 4p^3 x_0 + p^2 x_0^2}{4y_0^2}} = \left| \frac{p^2 + px_0}{2y_0} \right| = FB. \end{aligned}$$

## Practical classification of second-order lines

I. The analysis is simplified if the equation of the second-order line has a special form.

Task 8.4. Find out the type of the second-order line (without reducing its equation to canonical form)  $(x - 4y + 3)^2 + (2x + 3y - 1) = 0$ .

Solution. Let us denote  $\begin{cases} x' = x - 4y + 3, \\ y' = 2x + 3y - 1, \end{cases}$  then we obtain  $x'^2 + y' = 0$ . In this case

$\det \begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} = 11 \neq 0$ . It will mean that these relations can be *transition formulas*. Since the *affine classification* of lines is preserved when replacing the coordinate system, the line in the condition is a *parabola*.

Solution is found

### General case of a second-order line

$$\text{Let } Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \quad A^2 + B^2 + C^2 \neq 0.$$

1) First, we note that without loss of generality we can assume that the conditions  $B \geq 0$  and  $A \geq C$  are satisfied. Indeed, if  $B < 0$ , then we can change the signs of all the coefficients in the equation of the line.

If  $A < C$ , then, by moving to a new orthonormal coordinate system for which  $\vec{e}'_1 = \vec{e}_2$ ;  $\vec{e}'_2 = \vec{e}_1$ ;  $\vec{OO}' = \vec{o}$ , we obtain the desired relation, since with such a transition, equalities  $x = y'$ ;  $y = x'$  hold according to the rules for constructing transition formulas. We also note that with this replacement,  $\Delta$  does not change, since

$$\det \begin{vmatrix} C & B \\ B & A \end{vmatrix} = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \Delta.$$

2) The rotation around the origin of coordinates  $\begin{cases} x = x' \cos \alpha - y' \sin \alpha, \\ y = x' \sin \alpha + y' \cos \alpha, \end{cases}$  which leads to the fact that the term  $2Bxy$  disappears ( $B' = 0$ ), is satisfied under the condition:

$$\text{tg } 2\alpha = \frac{2B}{A-C}; \quad \alpha = \frac{1}{2} \text{arctg } \frac{2B}{A-C} \quad \text{or, equivalent to it,} \quad \text{tg}^2 \alpha + \frac{A-C}{B} \text{tg } \alpha - 1 = 0, \quad B \neq 0,$$

$$\text{This gives} \quad A' = \frac{A+C}{2} + \frac{1}{2} \sqrt{4B^2 + (A-C)^2} \quad \text{and} \quad C' = \frac{A+C}{2} - \frac{1}{2} \sqrt{4B^2 + (A-C)^2}.$$

Task 8.5: *In a rectangular coordinate system, construct a line of the second order*

$$3x^2 + 12xy + 12y^2 - 2x + 46y + 67 = 0 .$$

Solution:

1°. Since  $B \geq 0$ , but  $A \leq C$ , then we make a change of variables:  $\begin{cases} x = y', \\ y = x'. \end{cases}$  (A)

We get the equation  $12x'^2 + 12x'y' + 3y'^2 + 46x' - 2y' + 67 = 0$ .

2°. We rotate the coordinate system by an angle  $\alpha$  counterclockwise. Moreover,  $0 \leq \alpha \leq \frac{\pi}{4}$ . The formulas for this change  $\begin{cases} x' = x'' \cos \alpha - y'' \sin \alpha, \\ y' = x'' \sin \alpha + y'' \cos \alpha. \end{cases}$

We find the value  $\alpha$  from the condition:  $\operatorname{tg} 2\alpha = \frac{2B}{A-C} = \frac{2 \cdot 6}{12-3} = \frac{4}{3}$ .

Using the formulas:  $\frac{1}{\cos^2 2\alpha} = \operatorname{tg}^2 2\alpha + 1$  and  $\begin{cases} 1 + \cos 2\phi = 2 \cos^2 \phi \\ 1 - \cos 2\phi = 2 \sin^2 \phi \end{cases}$ ,

we find that  $\cos 2\alpha = \frac{3}{5}$  and  $\begin{cases} \cos \alpha = \frac{2}{\sqrt{5}} \\ \sin \alpha = \frac{1}{\sqrt{5}} \end{cases}$ . That is,  $\begin{cases} x' = \frac{2}{\sqrt{5}} x'' - \frac{1}{\sqrt{5}} y'' \\ y' = \frac{1}{\sqrt{5}} x'' + \frac{2}{\sqrt{5}} y'' \end{cases}$  **(B)**

Since

$$A'' = \frac{A' + C'}{2} + \frac{1}{2}\sqrt{4B'^2 + (A' - C')^2} = \frac{12 + 3}{2} + \frac{1}{2}\sqrt{4 \cdot 6^2 + (12 - 3)^2} = \frac{15}{2} + \frac{\sqrt{144 + 81}}{2} = 15$$

$$\text{and } C'' = \frac{A' + C'}{2} - \frac{1}{2}\sqrt{4B'^2 + (A' - C')^2} = 0,$$

then we arrive at an equation of the form:

$$15x''^2 + \frac{90}{\sqrt{5}}x'' - \frac{50}{\sqrt{5}}y'' + 67 = 0 \quad \text{or} \quad x''^2 + 2\frac{3}{\sqrt{5}}x'' - 2\frac{5}{3\sqrt{5}}y'' + \frac{67}{15} = 0.$$



3°. We select the full square:  $x''^2 + 2\frac{3}{\sqrt{5}}x'' + \frac{9}{5} - 2\frac{5}{3\sqrt{5}}y'' + \frac{67}{15} - \frac{9}{5} = 0$  or

$$\left(x'' + \frac{3}{\sqrt{5}}\right)^2 - 2\frac{5}{3\sqrt{5}}y'' + \frac{8}{3} = 0.$$

From which we obtain: 
$$\left(x'' + \frac{3}{\sqrt{5}}\right)^2 = 2\frac{\sqrt{5}}{3}\left(y'' - \frac{4}{\sqrt{5}}\right).$$

We obtain the canonical form of the equation of our line by renaming the axes and shift-

ing the origin of coordinates, that is, by replacing: 
$$\begin{cases} x''' = y'' - \frac{4}{\sqrt{5}}, \\ y''' = x'' + \frac{3}{\sqrt{5}}. \end{cases} \quad (\text{C})$$

This form will look like  $y'''^2 = 2px'''$ , that is, it is a parabola, for which the focal parameter  $p = \frac{\sqrt{5}}{3}$ .

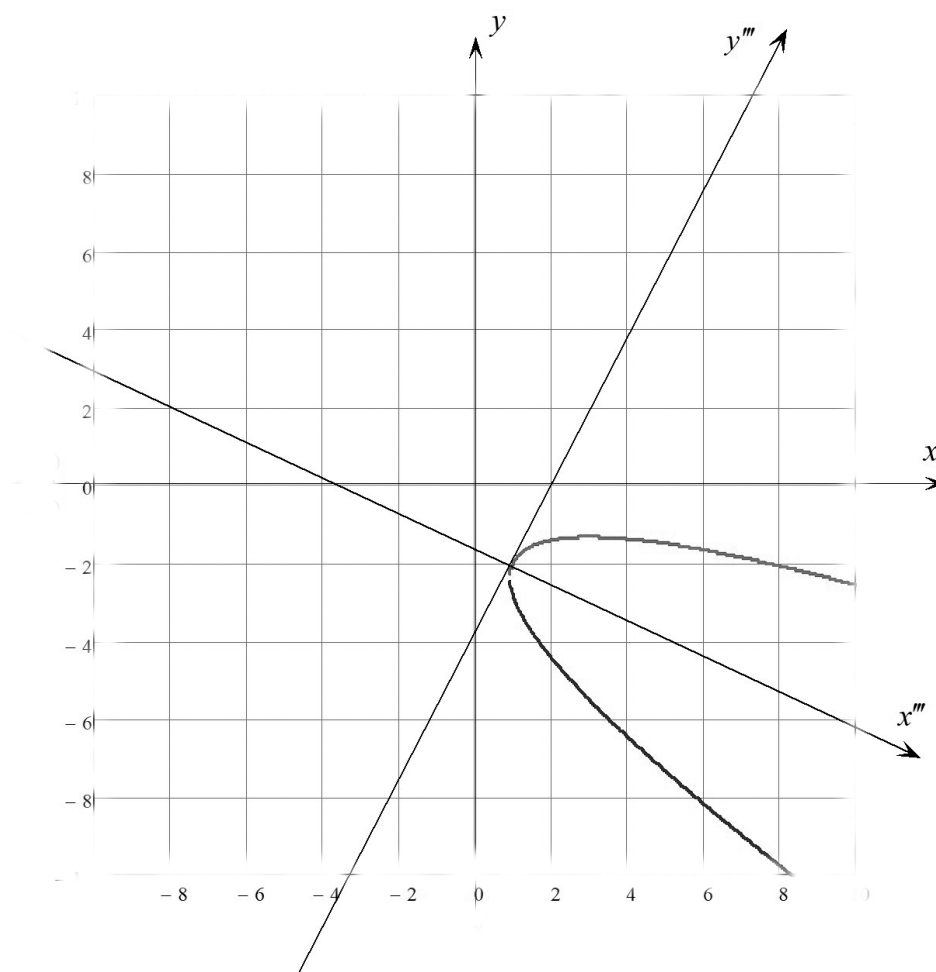
4°. We find the final transition formulas by substituting (C) into (B), and we substitute the

result of this substitution into (A). We finally obtain

$$\begin{cases} x = \frac{2}{\sqrt{5}}x''' + \frac{1}{\sqrt{5}}y''' + 1, \\ y = -\frac{1}{\sqrt{5}}x''' + \frac{2}{\sqrt{5}}y''' - 2. \end{cases}$$

This means that the final transition matrix  $\|S\| = \begin{vmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{vmatrix}$ , and the final origin (the vertex of the parabola) will be at the point  $\begin{vmatrix} 1 \\ -2 \end{vmatrix}$ .

5°. Considering that the columns of the matrix  $\|S\|$  are the coordinate columns of the basis vectors  $\vec{e}_1''', \vec{e}_2'''$ , and the vector  $\vec{e}_1'''$  is the direction vector of the parabola axis, we construct a sketch:



A graphical solution to the problem is obtained by removing ('erasing') the axes of the canonical coordinate system  $\{O''', x''', y'''\}$  from the drawing.