Operators and functionals. Mappings and transformations of the plane

Definition We will say that an *operator* $\hat{A}$ is given that acts on a set Ω with values in the set Ω if a rule is specified according to which *each* element of the set Ω is assigned a *unique* element from the set Θ.

Symbolically, the result of the operator $\hat{A}$ is denoted as follows: $y = \hat{A}x, \quad x \in \Omega, \quad y \in \Theta$. In this case, the element y is called the *image* of the element x, the element x is called the *preimage* of the element y.

Definition Let Θ be the range of some operator and Θ is a *numerica*l set. In this case we say that a *functional* is defined on the set Ω.

Functionals are usually denoted in the same way as functions: for example, $y = \Phi(x), \; x \in \Omega$.

Definition

An *operator* $\hat{A}$ *mapping a plane* (or simply a *mapping of a plane*) P *onto a plane* Q, is called a rule according to which *each* point of the plane P is assigned a *unique* point of the plane Q.

A mapping of a plane is usually denoted as follows: $\hat{A}: P \to Q$.

Definition

A mapping $\hat{A}: P \to Q$ is called *one-to-one* if each point of the plane Q has a *pre-image* and, moreover, a unique one.

A mapping $\hat{A}$ of a plane P into itself is called a *transformation* of the plane P.

Definition The successive execution of the transformations

$$M^{*}=\hat{A}M \quad \text{and} \quad M^{**}=\hat{B}M^{*}$$

is called the *product* (or *composition*) of these transformations.

The product of operators is written as $M^{**}=\hat{B}\hat{A}\ M$. Note that in the general case this product is *not commutative*, but *associative* and *distributive*.

Definition The inverse transformation of a one-to-one transformation $\hat{A}:P\to P$ is an operator such that $\hat{A}^{-1}:P\to P$ for each point P in the plane we have

$$\hat{A}^{-1}(\hat{A}M)=\hat{A}(\hat{A}^{-1}M)=M.$$

Definition	The set of points on P that is mapped into *itself* for $\hat{A}$ is called the *invariant set* of the transformation $\hat{A}$.

For example, for a fixed line, every point is fixed. While for an invariant line, the image of a point does not necessarily coincide with the preimage, but it belongs to this line.

Linear operators on a plane

Let each point M on a plane with a Cartesian coordinate system $\{O, \vec{g}_1, \vec{g}_2\}$ be assigned a one-to-one correspondence with a point M^*.

And let the coordinate representations of the radius vectors of these points be $\left\| \vec{r}_M \right\|_g = \left\| \begin{matrix} x \\ y \end{matrix} \right\|$ and $\left\| \vec{r}_{M^*} \right\|_g = \left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$, then the coordinates x^* and y^* will be some functions of x and y,

$$\begin{cases} x^* = F_x(x,y) \\ y^* = F_y(x,y) \end{cases},$$

Then the equality $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\| = \left\| \begin{matrix} F_x(x,y) \\ F_y(x,y) \end{matrix} \right\|$ can be considered as a representation of the operator $\vec{r}_{M^*} = \hat{A}\vec{r}_M$ in the coordinate system $\{O, \vec{g}_1, \vec{g}_2\}$.

Below we will consider particular, but important for applications, types of functions $F_x(x,y)$ and $F_y(x,y)$.

Definition	An operator $\vec{r}_{M^*} = \hat{A}\vec{r}_M$ is called a *linear operator* if in each Cartesian coordinate system $\{O, \vec{g}_1, \vec{g}_2\}$ it is defined by the formulas $$\begin{cases} x^* = \alpha_{11}x + \alpha_{12}y + \beta_1, \\ y^* = \alpha_{21}x + \alpha_{22}y + \beta_2. \end{cases}$$

Using matrix operations, a linear operator can be written in the form $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\| = \| \hat{A} \|_g \left\| \begin{matrix} x \\ y \end{matrix} \right\| + \left\| \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \right\|$, where the matrix $\| \hat{A} \|_g = \left\| \begin{matrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{matrix} \right\|$ is called the *matrix of the linear operator* $\hat{A}$ (*coordinate representation* $\hat{A}$) in $\{O, \vec{g}_1, \vec{g}_2\}$.

Definition	An operator $\vec{r}_{M^*} = \hat{A}\vec{r}_M$ is called a linear *homogeneous* operator if it satisfies Definition 5.3.1 and, in addition, $\beta_1 = \beta_2 = 0.$ IIf $\lvert \beta_1 \rvert + \lvert \beta_2 \rvert > 0$, then the operator $\hat{A}$ is called *non-homogeneous*.

Example The following are linear homogeneous operators:

- operator $\hat{A}$, the action of which is reduced to multiplying the coordinates of the radius vector of the pre-image by fixed positive numbers, called the "*operator of compression to the axes*", or simply "compression to the axes", having a matrix $\left\| \hat{A} \right\|_g = \left\| \begin{matrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{matrix} \right\|$, where the numbers κ_1 and κ_2 are the *compression coefficients*;

- operator of orthogonal projection of the radius vectors of points of the plane onto some given axis passing through the origin;

- homothety with the coefficient κ and with the center at the origin.

Theorem **For a linear homogeneous operator $\hat{A}$ the following relations are valid:**

1°. $\hat{A}(\vec{r}_1 + \vec{r}_2) = \hat{A}\vec{r}_1 + \hat{A}\vec{r}_2 \quad \forall \vec{r}_1, \vec{r}_2 .$

2°. $\hat{A}(\lambda \vec{r}) = \lambda \hat{A}\vec{r} \quad \forall \vec{r}, \lambda .$

An important corollary follows from these theorems.

Corollary **The columns of the matrix of a linear homogeneous operator $\hat{A}$ in the basis $\{\vec{g}_1, \vec{g}_2\}$ are the coordinate representations of the vectors $\hat{A}\vec{g}_1$ and $\hat{A}\vec{g}_2$.**

Each linear homogeneous operator of plane transformation in a specific basis corresponds to a uniquely determined square matrix of the second order, and each square matrix of the second order defines a linear homogeneous operator in this basis.

Let us check the first statement.

Obviously, in the basis $\{\vec{g}_1, \vec{g}_2\}$, the coordinate representations (coordinate columns) of the basis vectors themselves have the form $\begin{Vmatrix} 1 \\ 0 \end{Vmatrix}$ and $\begin{Vmatrix} 0 \\ 1 \end{Vmatrix}$. Then, from the formulas $\begin{cases} x^* = \alpha_{11}x + \alpha_{12}y, \\ y^* = \alpha_{21}x + \alpha_{22}y, \end{cases}$ describing the action of the linear homogeneous operator $\hat{A}$, it follows that the images of the basis elements $\{\vec{g}_1^*, \vec{g}_2^*\}$ will have coordinate representations $\begin{Vmatrix} \alpha_{11} \\ \alpha_{21} \end{Vmatrix}$ and $\begin{Vmatrix} \alpha_{12} \\ \alpha_{22} \end{Vmatrix}$, which are the columns of the matrix $\| \hat{A} \|_g = \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix}$.

We also note that if a pair of vectors $\{\vec{g}_1^*, \vec{g}_2^*\}$ (in the case of their linear independence) is taken as a new basis on the plane, then the transition matrix from the original basis to the new one, by virtue of its definition, will coincide with the transformation $\hat{A}$, that is, the equality will be true

$$\| S \| = \| \hat{A} \|_g = \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix}.$$

Task 12-1.01 *Based on the rules of operation with matrices, show that the following statements are true for linear homogeneous operators on the plane:*

1°. *The product matrix of linear homogeneous operators is equal to the product of the matrices of the multipliers*: $\| \hat{A}\hat{B} \|_g = \| \hat{A} \|_g \| \hat{B} \|_g$.

2°. *If* $\hat{A}^{-1}$ *there is an operator inverse to a linear homogeneous operator* $\hat{A}$, *then*

$$\| \hat{A}^{-1} \|_g = \| \hat{A} \|_g^{-1} .$$

Let us now find out how the matrix of a linear homogeneous operator changes when changing the basis. We have

Theorem **Let some homogeneous linear operator (transformation) have a matrix** $\|\hat{A}\|_g$ **in the coordinate system** $\{O, \vec{g}_1, \vec{g}_2\}$. **Then this operator will have a matrix** $\|\hat{A}\|_{g'}$ **in the coordinate system** $\{O, \vec{g}'_1, \vec{g}'_2\}$ **and**

$$\| \hat{A} \|_{g'} = \| S \|^{-1} \| \hat{A} \|_g \| S \| ,$$

where $\|S\|$ **is the transition matrix.**

Corollary **The quantity** $\det\|\hat{A}\|_g$ **does not depend on the choice of basis.**

Affine transformations and their properties

Linear operators that transform a plane into itself (i.e. linear operators of the form $\hat{A}: P \to P$) and have an *inverse operator* play an important role from a practical point of view and therefore are allocated to a special class.

Definition A linear operator

$$\begin{Vmatrix} x^* \\ y^* \end{Vmatrix} = \| \hat{A} \|_g \begin{Vmatrix} x \\ y \end{Vmatrix} + \begin{Vmatrix} \beta_1 \\ \beta_2 \end{Vmatrix},$$

mapping a plane P onto itself, with a matrix $\| \hat{A} \|_g = \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix}$ for which in any basis $\det \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} \neq 0$, is called an *affine transformation* of the plane.

Theorem (affinity criterion) **If for a linear transformation of a plane $\det\|\hat{A}\|_g \neq 0$ in some Cartesian coordinate system, then this condition will be satisfied in any other Cartesian coordinate system.**

Theorem **Every affine transformation has a unique inverse, which is also affine.**

Theorem **Under an affine transformation, every basis is transformed into a basis, and for any two bases there is a unique affine transformation that takes the first basis to the second.**

Theorem **Under an affine transformation, the image of a straight line is a straight line.**

Theorem **Under an affine transformation, the image of parallel lines is parallel lines, the common point of intersecting preimage lines is mapped to the intersection point of their images.**

Theorem **Under an affine transformation, the division of a segment in a given ratio is preserved.**

Theorem **Under an affine transformation, the ratio of the lengths of the images of two segments lying on parallel lines is equal to the ratio of the lengths of their preimages.**

Theorem **Under an affine transformation, any Cartesian coordinate system is mapped to a Cartesian coordinate system, and the coordinates of the image of each point of the plane in the new coordinate system will coincide with the coordinates of the preimage in the original.**

The following will be true:

Theorem

1°. Under an affine transformation, the quantities S^* – the area of the image of a parallelogram and S – the area of the pre-image of a parallelogram are related by the relation

$$S^* = \left| \det \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} \right| \cdot S .$$

2°. Under an affine transformation, the orientation of the images of a pair of non-collinear vectors coincides with the orientation of the pre-images if

$$\det \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} > 0 ,$$

and changes to the opposite if

$$\det \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} < 0 .$$

Theorem **For any second-order line specified in the formulation of the canonical classification theorem:**

- **its type and form cannot change under an affine transformation;**

- **there is an affine transformation that takes it to any other second-order line of the same type and form.**

Theorem **For any affine transformation, there is a pair of mutually orthogonal directions that are transformed by the given affine transformation into mutually orthogonal ones.**

Orthogonal transformations of the plane

Definition

An *orthogonal transformation* of the plane P is a linear operator $\hat{Q}$ of the form

$$\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\| = \| \hat{Q} \|_e \left\| \begin{matrix} x \\ y \end{matrix} \right\| + \left\| \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \right\| ,$$

whose *matrix* $\| \hat{Q} \|_e = \left\| \begin{matrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{matrix} \right\|$ *is orthogonal* in any orthonormal coordinate system.

Theorem

In any orthonormal Cartesian coordinate system an orthogonal transformation of the plane preserves:

1) **the scalar product of vectors;**
2) **the lengths of vectors and the distances between points of the plane;**
3) **the angles between lines.**

Theorem

Each affine transformation can be represented as a product of an orthogonal transformation and two contractions along mutually orthogonal directions.