AFFINE TRANSFORMATIONS (Tasks)

We will use the following notations:

In the original coordinate system for a given point, with coordinate representation $\left\| \begin{matrix} x \\ y \end{matrix} \right\|$, the image will be a point with coordinate representation $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$. In a new coordinate system (if required), this point $\left\| \begin{matrix} x \\ y \end{matrix} \right\|$ will have coordinate representation $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$, and the coordinate representation of its image will be $\left\| \begin{matrix} x'^* \\ y'^* \end{matrix} \right\|$.

Task 10-2.01. For a straight line $3x-2y-2=0$, find its preimage under a plane transformation $\begin{cases} x^* = -2x+y-3, \\ y^* = -x+2y-3. \end{cases}$

Solution: 1) This transformation is affine, since $\det\begin{Vmatrix} -2 & 1 \\ -1 & 2 \end{Vmatrix} = -3 \neq 0$. It is known that under an affine transformation, a straight line is transformed into a straight line. Therefore, the problem consists of finding the relationship between the coordinates x and y, provided that $3x^* - 2y^* - 2 = 0$.

2) Substitute the transformation formula into the image equation $3x^* - 2y^* - 2 = 0$, and obtain the desired linear relationship

$$3(-2x+y-3) - 2(-x+2y-3) - 2 = 0 \quad \Rightarrow \quad 4x + y + 5 = 0 \ .$$

Solution is found

Task 10-2.02. For a line $9x-9y-7=0$, find its image under a plane transformation $\begin{cases} x^* = \phantom{3x+{}} 6y-5, \\ y^* = 3x+6y-5. \end{cases}$

Solution: 1) This transformation is affine, since $\det\begin{Vmatrix} 0 & 6 \\ 3 & 6 \end{Vmatrix} = -18 \neq 0$. Since under an affine transformation a line goes into a line, then. this problem consists of finding the relationship between the coordinates x^* and y^*, provided that $9x-9y-7=0$.

2) For this, we will need the formulas for the transformation inverse to the given one. It is known that for each affine transformation there is a unique inverse and it is also affine. In our case, we will consider the formulas for this transformation as a system of linear equations with unknowns x and y. Expressing x and y through x^* and x^*, we obtain

$$\begin{cases} x = -\dfrac{1}{3}x^* + \dfrac{1}{3}y^*, \\ y = \dfrac{1}{6}x^* \phantom{+\dfrac{1}{3}y^*} + \dfrac{5}{6}. \end{cases}$$

3) Now substituting the formulas for the inverse transformation into the equation of the preimage $9x-9y-7=0$, we obtain the equation of the desired image

$$9\left(-\frac{1}{3}x^* + \frac{1}{3}y^*\right) - 9\left(\frac{1}{6}x^* + \frac{5}{6}\right) - 7 = 0 \quad \Rightarrow \quad 9x^* - 6y^* + 58 = 0 \ .$$

Solution is found

Task 10-2.03. Some affine transformation of a plane the following holds:

$$\text{the image of a point } \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \text{ is the point } \begin{Vmatrix} 0 \\ 1 \end{Vmatrix},$$
$$\text{the image of a point } \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} \text{ is the point } \begin{Vmatrix} 1 \\ 1 \end{Vmatrix},$$
$$\text{the image of a point } \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} \text{ is the point } \begin{Vmatrix} 0 \\ 0 \end{Vmatrix}.$$

Which points of the plane remain fixed under this transformation?

Solution: 1) Let the given be given by the formulas $\begin{cases} x^* = \alpha_{11}x + \alpha_{12}y + \beta_1, \\ y^* = \alpha_{21}x + \alpha_{22}y + \beta_2. \end{cases}$. Let's find it.

2) From the first condition $\begin{cases} 0 = \alpha_{11}0 + \alpha_{12}0 + \beta_1, \\ 1 = \alpha_{21}0 + \alpha_{22}0 + \beta_2 \end{cases} \Rightarrow \begin{matrix} \beta_1 = 0 \\ \beta_2 = 1. \end{matrix}$

From the second condition $\begin{cases} 1 = \alpha_{11}0 + \alpha_{12}1 + 0, \\ 1 = \alpha_{21}0 + \alpha_{22}1 + 1 \end{cases} \Rightarrow \begin{matrix} \alpha_{12} = 1 \\ \alpha_{22} = 0. \end{matrix}$

Finally, from the third condition $\begin{cases} 0 = \alpha_{11}1 + 1 \cdot 1 \quad , \\ 0 = \alpha_{21}1 + 0 \cdot 1 + 1 \end{cases} \Rightarrow \begin{matrix} \alpha_{11} = -1, \\ \alpha_{22} = -1. \end{matrix}$

As a result: $\begin{cases} x^* = -x + y, \\ y^* = -x \quad\quad + 1. \end{cases}$ This transformation is obviously affine.

3) Let $\begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix}$ be a fixed point of the obtained transformation. Then there must be

$\begin{cases} x_0 = -x_0 + y_0, \\ y_0 = -x_0 \quad\quad + 1. \end{cases} \Rightarrow \begin{matrix} x_0 = \dfrac{1}{3}, \\ y_0 = \dfrac{2}{3} \end{matrix}$. It is a *unique* fixed point.

Solution is found

Task 10-2.04. Find an affine transformation of the plane, under which all points of the line $x + y - 1 = 0$ are fixed, and the point $\left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\|$ has as its image a point with coordinates $\left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\|$.

Solution: 1) Let the desired transformation be given by the formulas

$$\begin{cases} x^* = \alpha_{11}x + \alpha_{12}y + \beta_1, \\ y^* = \alpha_{21}x + \alpha_{22}y + \beta_2. \end{cases}$$

We immediately obtain from the conditions $\left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\| \to \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\|$ that $\left\| \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \right\| = \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\|$.

2) Now we use the condition of fixedness of each point of the line $x + y - 1 = 0$.

Let the point $\left\| \begin{matrix} x_0 \\ 1 - x_0 \end{matrix} \right\|$ $\forall x_0 \in \mathbf{R}$ be an arbitrary point of this line. Then the condition of its fixedness will be:

$$\begin{cases} x_0 = \alpha_{11}x_0 + \alpha_{12}(1 - x_0) + 1, \\ 1 - x_0 = \alpha_{21}x_0 + \alpha_{22}(1 - x_0) + 1. \end{cases}$$

3) We regroup these equalities to the form

$$\begin{cases}(\alpha_{11}-\alpha_{12}-1)x_0+(\alpha_{12}+1)=0,\\ (\alpha_{21}-\alpha_{22}+1)x_0+\alpha_{22}=0.\end{cases}$$

From where, taking into account $\forall x_0 \in \mathbf{R}$, from

$$\begin{cases}\alpha_{11}-\alpha_{12}-1=0,\\ \alpha_{12}+1=0,\\ \alpha_{21}-\alpha_{22}+1=0,\\ \alpha_{22}=0,\end{cases}$$ we obtain the answer to the task: $$\begin{cases}x^*=-y+1,\\ y^*=-x+1.\end{cases}$$

Solution is found

Task 10-2.05. Find all invariant lines of an affine transformation of the plane

$$\begin{cases} x^* = -28x + 18y + 65, \\ y^* = -45x + 29y + 107. \end{cases}$$

Solution: 1) Let the preimage line have an equation $Ax + By + C = 0, \quad A^2 + B^2 > 0$, its image has an equation $Ax^* + By^* + C = 0$ or

$$A(-28x + 18y + 65) + B(-45x + 29y + 107) + C = 0 \ .$$

And this means that $(-28A - 45B)x + (18A + 29B)y + (65A + 107B + C) = 0$.

2) Since the equation of the line is determined up to a non-zero factor, the *invariance condition* will have the following form:

$$\begin{cases} A^* = kA = -28A - 45B, \\ B^* = kB = 18A + 29B, \\ C^* = kC = 65A + 107B + C. \end{cases}$$

3) Note that the first two equations form a linear homogeneous system:

$$\begin{cases} (-28-k)A-45B=0, \\ 18A+(29-k)B=0. \end{cases}$$

Since A and B cannot be equal to 0 simultaneously, it is necessary to satisfy the condition (following from the theory of systems of linear equations):

$$\det\begin{Vmatrix} -28-k & -45 \\ 18 & 29-k \end{Vmatrix}=0,$$

which gives

$$(k+28)(k-29)+18\cdot 45=0 \quad \Rightarrow \quad k^2-k-2=0.$$

From where either $k=2$, or $k=-1$ and can be taken

for $k=2$ $A=3, B=-2, C=-7,$ and for $k=-1$ $A=5, B=-3, C=24.$

Consequently, the desired invariant lines will be:

$$3x-2y-7=0 \quad \text{and} \quad 5x-3y+24=0.$$

Solution is found

Task 10-2.06. Find an affine transformation of the plane such that:

1) each of the lines $x-2y-3=0$ and $-x+y+1=0$ is invariant,

2) the point $M=\begin{Vmatrix}4\\-5\end{Vmatrix}$ has as its image the point $M^*=\begin{Vmatrix}-39\\-32\end{Vmatrix}$.

Solution:

1). Let us move to a *new* coordinate system in which the *origin* O' is the point of intersection of the given lines, and the *new basis* vectors are the direction vectors of these same lines.

The coordinates of O' are coordinates of the intersection point of the lines, determined from the system of equations:

$$\begin{cases} x_0-2y_0=3, \\ -x_0+y_0=-1. \end{cases}$$

Whence $x_0=-1$ and $y_0=-2$. This means that $\begin{Vmatrix}\vec{OO'}\end{Vmatrix}=\begin{Vmatrix}-1\\-2\end{Vmatrix}$.

As the *direction vectors* of the new axes, we can obviously take $\begin{Vmatrix}\vec{g}'_1\end{Vmatrix}=\begin{Vmatrix}2\\1\end{Vmatrix}$ and $\begin{Vmatrix}\vec{g}'_2\end{Vmatrix}=\begin{Vmatrix}1\\1\end{Vmatrix}$. Then, according to the theorem on *transition formulas*, we obtain that the expression of the old coordinates through the new ones has the form: $\begin{cases} x=2x'+y'-1, \\ y=x'+y'-2. \end{cases}$

Whence we find the formulas for the *inverse* transition: $\begin{cases} x'=x-y-1, \\ y'=-x+2y+3. \end{cases}$

2). Based on the conditions of the problem, we can assert that under the sought-for affine transformation:

1) , The point O' will be fixed,
2) and the coordinate axes of the new coordinate system will be invariant lines.

Then in the new coordinate system the formulas of the sought-for affine transformation will have the following, very simple form: $\begin{cases} x^{*}{}' = \lambda x', \\ y^{*}{}' = \mu y', \end{cases}$ where λ and μ are some constants

3). We will find the values of λ and μ from the condition $M \to M^{*}$ specified in the condition of the problem. Indeed, using the formulas of the inverse transition, we obtain that in the new coordinate system $M(8;-11)$, and $M^{*}(-8;-22)$, Whence it is obvious that $\lambda = -1$ and $\mu = 2$.

4). Finally, substituting the inverse transition formulas into the equalities $\begin{cases} x^{*}{}' = -x', \\ y^{*}{}' = 2y', \end{cases}$, we obtain the equations of the relationship between the coordinates of the image and the pre-image in the *original* coordinate system (that is, the formulas defining the desired affine transformation):

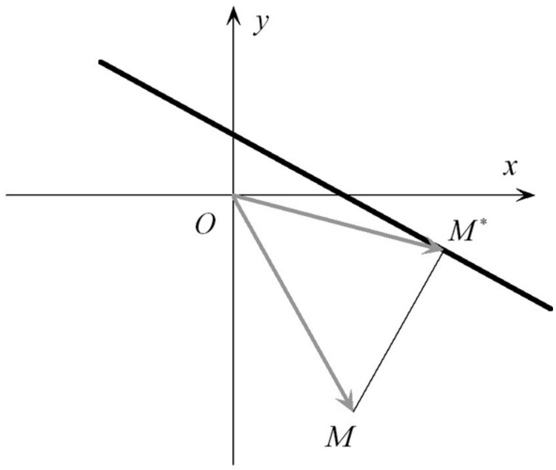
$$\begin{cases} x^{*} = -4x + 6y + 7, \\ y^{*} = -3x + 5y + 5. \end{cases}$$

Solution is found

Task 10-2.07. In an orthonormal coordinate system, find the matrix of the operator that orthogonally projects the position vectors of the points of the coordinate plane onto the line $x + 3y - 2 = 0$.

Sulution:

1). Let the point-preimage M have the position vector $\vec{r}_0 = \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\|$, and the point-image M^* of the point M have its position vector $\vec{r}_0^{\,*} = \left\| \begin{matrix} x_0^* \\ y_0^* \end{matrix} \right\|$.



It follows from the figure that there is a point M^* of intersection of the line $x + 3y - 2 = 0$ and the perpendicular to it passing through M .

2) Since the normal vector of a straight line $x+3y-2=0$ is the direction vector of this perpendicular, the equation of the latter will have the form

$$\left\| \begin{matrix} x \\ y \end{matrix} \right\| = \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| + \tau \left\| \begin{matrix} 1 \\ 3 \end{matrix} \right\| .$$

From which it follows that the coordinates of the position vector of a point M^* will satisfy the system of equations

$$\begin{cases} x_0^* = x_0 + \tau, \\ y_0^* = y_0 + 3\tau, \\ x_0^* + 3y_0^* - 2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} x_0^* = \dfrac{9}{10}x_0 - \dfrac{3}{10}y_0 + \dfrac{1}{5}, \\ y_0^* = -\dfrac{3}{10}x_0 + \dfrac{1}{10}y_0 + \dfrac{3}{5}. \end{cases}$$

3) Using the rules of operations with matrices, we finally obtain that

$$\left\| \begin{matrix} x_0^* \\ y_0^* \end{matrix} \right\| = \left\| \begin{matrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{matrix} \right\| \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| + \left\| \begin{matrix} \frac{1}{5} \\ \frac{3}{5} \end{matrix} \right\| ,$$

that is

$$\| \hat{A} \|_e = \left\| \begin{matrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{matrix} \right\| .$$

Solution is found

Note that the operator of orthogonal projection of points of a plane onto a fixed line is linear, but not affine, since there is no one-to-one relationship between the images and preimages. Therefore, $\det \| \hat{A} \|_e = 0$.