

AFFINE TRANSFORMATIONS (Tasks)

We will use the following notations:

In the original coordinate system for a given point, with coordinate representation $\begin{pmatrix} x \\ y \end{pmatrix}$, the image will be a point with coordinate representation $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$. In a new coordinate system (if required), this point $\begin{pmatrix} x \\ y \end{pmatrix}$ will have coordinate representation $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$, and the coordinate representation of its image will be $\begin{pmatrix} x'^* \\ y'^* \end{pmatrix}$.

Task 10-2.01. For a straight line $3x - 2y - 2 = 0$, find its preimage under a plane transformation $\begin{cases} x^* = -2x + y - 3, \\ y^* = -x + 2y - 3. \end{cases}$

Solution: 1) This transformation is affine, since $\det \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} = -3 \neq 0$. It is known that under an affine transformation, a straight line is transformed into a straight line. Therefore, the problem consists of finding the relationship between the coordinates x and y , provided that $3x^* - 2y^* - 2 = 0$.

2) Substitute the transformation formula into the image equation $3x^* - 2y^* - 2 = 0$, and obtain the desired linear relationship

$$3(-2x + y - 3) - 2(-x + 2y - 3) - 2 = 0 \quad \Rightarrow \quad 4x + y + 5 = 0 .$$

Solution is found

Task 10-2.02. For a line $9x - 9y - 7 = 0$, find its image under a plane transformation

$$\begin{cases} x^* = 6y - 5, \\ y^* = 3x + 6y - 5. \end{cases}$$

Solution:

1) This transformation is affine, since $\det \begin{vmatrix} 0 & 6 \\ 3 & 6 \end{vmatrix} = -18 \neq 0$. Since under an affine transformation a line goes into a line, then this problem consists of finding the relationship between the coordinates x^* and y^* , provided that $9x - 9y - 7 = 0$.

2) For this, we will need the formulas for the transformation inverse to the given one. It is known that for each affine transformation there is a unique inverse and it is also affine. In our case, we will consider the formulas for this transformation as a system of linear equations with unknowns x and y .

Expressing x and y through x^* and y^* , we obtain

$$\begin{cases} x = -\frac{1}{3}x^* + \frac{1}{3}y^*, \\ y = \frac{1}{6}x^* + \frac{5}{6}. \end{cases}$$

3) Now substituting the formulas for the inverse transformation into the equation of the preimage $9x - 9y - 7 = 0$, we obtain the equation of the desired image

$$9\left(-\frac{1}{3}x^* + \frac{1}{3}y^*\right) - 9\left(\frac{1}{6}x^* + \frac{5}{6}\right) - 7 = 0 \quad \Rightarrow \quad 9x^* - 6y^* + 58 = 0.$$

Solution is found

Task 10-2.03. Some affine transformation of a plane the following holds:

the image of a point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the point $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

the image of a point $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

the image of a point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Which points of the plane remain fixed under this transformation?

Solution: 1) Let the given be given by the formulas $\begin{cases} x^* = \alpha_{11}x + \alpha_{12}y + \beta_1, \\ y^* = \alpha_{21}x + \alpha_{22}y + \beta_2. \end{cases}$ Let's find it.

$$2) \text{ From the first condition } \begin{cases} 0 = \alpha_{11} \cdot 0 + \alpha_{12} \cdot 0 + \beta_1, \\ 1 = \alpha_{21} \cdot 0 + \alpha_{22} \cdot 0 + \beta_2 \end{cases} \Rightarrow \begin{cases} \beta_1 = 0 \\ \beta_2 = 1. \end{cases}$$

$$\text{From the second condition } \begin{cases} 1 = \alpha_{11} \cdot 0 + \alpha_{12} \cdot 1 + 0, \\ 1 = \alpha_{21} \cdot 0 + \alpha_{22} \cdot 1 + 1 \end{cases} \Rightarrow \begin{cases} \alpha_{12} = 1 \\ \alpha_{22} = 0. \end{cases}$$

$$\text{Finally, from the third condition } \begin{cases} 0 = \alpha_{11} \cdot 1 + 1 \cdot 1, \\ 0 = \alpha_{21} \cdot 1 + 0 \cdot 1 + 1 \end{cases} \Rightarrow \begin{cases} \alpha_{11} = -1, \\ \alpha_{21} = -1. \end{cases}$$

As a result: $\begin{cases} x^* = -x + y, \\ y^* = -x + 1. \end{cases}$ This transformation is obviously affine.

3) Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be a fixed point of the obtained transformation. Then there must be

$$\begin{cases} x_0 = -x_0 + y_0, \\ y_0 = -x_0 + 1. \end{cases} \Rightarrow \begin{cases} x_0 = \frac{1}{3}, \\ y_0 = \frac{2}{3}. \end{cases} \text{ It is a } \textit{unique} \text{ fixed point.}$$

Solution is found

Task 10-2.04. Find an affine transformation of the plane, under which all points of the line $x + y - 1 = 0$ are fixed, and the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has as its image a point with coordinates $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution: 1) Let the desired transformation be given by the formulas

$$\begin{cases} x^* = \alpha_{11}x + \alpha_{12}y + \beta_1, \\ y^* = \alpha_{21}x + \alpha_{22}y + \beta_2. \end{cases}$$

We immediately obtain from the conditions $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ that $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2) Now we use the condition of fixedness of each point of the line $x + y - 1 = 0$.

Let the point $\begin{pmatrix} x_0 \\ 1 - x_0 \end{pmatrix} \forall x_0 \in \mathbf{R}$ be an arbitrary point of this line. Then the condition of its fixedness will be:

$$\begin{cases} x_0 = \alpha_{11}x_0 + \alpha_{12}(1 - x_0) + 1, \\ 1 - x_0 = \alpha_{21}x_0 + \alpha_{22}(1 - x_0) + 1. \end{cases}$$

3) We regroup these equalities to the form

$$\begin{cases} (\alpha_{11} - \alpha_{12} - 1)x_0 + (\alpha_{12} + 1) = 0, \\ (\alpha_{21} - \alpha_{22} + 1)x_0 + \alpha_{22} = 0. \end{cases}$$

From where, taking into account $\forall x_0 \in \mathbf{R}$, from

$$\begin{cases} \alpha_{11} - \alpha_{12} - 1 = 0, \\ \alpha_{12} + 1 = 0, \\ \alpha_{21} - \alpha_{22} + 1 = 0, \\ \alpha_{22} = 0, \end{cases} \text{ we obtain the answer to the task: } \begin{cases} x^* = -y + 1, \\ y^* = -x + 1. \end{cases}$$

Solution is found

Task 10-2.05. Find all invariant lines of an affine transformation of the plane

$$\begin{cases} x^* = -28x + 18y + 65, \\ y^* = -45x + 29y + 107. \end{cases}$$

Solution: 1) Let the preimage line have an equation $Ax + By + C = 0$, $A^2 + B^2 > 0$, its image has an equation $Ax^* + By^* + C = 0$ or

$$A(-28x + 18y + 65) + B(-45x + 29y + 107) + C = 0 .$$

And this means that $(-28A - 45B)x + (18A + 29B)y + (65A + 107B + C) = 0$.

2) Since the equation of the line is determined up to a non-zero factor, the *invariance condition* will have the following form:

$$\begin{cases} A^* = kA = -28A - 45B, \\ B^* = kB = 18A + 29B, \\ C^* = kC = 65A + 107B + C. \end{cases}$$

3) Note that the first two equations form a linear homogeneous system:

$$\begin{cases} (-28-k)A - 45B = 0, \\ 18A + (29-k)B = 0. \end{cases}$$

Since A and B cannot be equal to 0 simultaneously, it is necessary to satisfy the condition (following from the theory of systems of linear equations):

$$\det \begin{vmatrix} -28-k & -45 \\ 18 & 29-k \end{vmatrix} = 0,$$

which gives

$$(k+28)(k-29) + 18 \cdot 45 = 0 \quad \Rightarrow \quad k^2 - k - 2 = 0.$$

From where either $k = 2$, or $k = -1$ and can be taken

for $k = 2$ $A = 3, B = -2, C = -7$, and for $k = -1$ $A = 5, B = -3, C = 24$.

Consequently, the desired invariant lines will be:

$$3x - 2y - 7 = 0 \quad \text{and} \quad 5x - 3y + 24 = 0.$$

Solution is found

Task 10-2.06. Find an affine transformation of the plane such that:

- 1) each of the lines $x - 2y - 3 = 0$ and $-x + y + 1 = 0$ is invariant,
- 2) the point $M = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ has as its image the point $M^* = \begin{pmatrix} -39 \\ -32 \end{pmatrix}$.

Solution:

1). Let us move to a *new* coordinate system in which the *origin* O' is the point of intersection of the given lines, and the *new basis* vectors are the direction vectors of these same lines.

The coordinates of O' are coordinates of the intersection point of the lines, determined from the system of equations:

$$\begin{cases} x_0 - 2y_0 = 3, \\ -x_0 + y_0 = -1. \end{cases}$$

Whence $x_0 = -1$ and $y_0 = -2$. This means that $\|\vec{OO'}\| = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

As the *direction vectors* of the new axes, we can obviously take $\|\vec{g}'_1\| = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and

$\|\vec{g}'_2\| = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, according to the theorem on *transition formulas*, we obtain that the ex-

pression of the old coordinates through the new ones has the form: $\begin{cases} x = 2x' + y' - 1, \\ y = x' + y' - 2. \end{cases}$

Whence we find the formulas for the *inverse* transition: $\begin{cases} x' = x - y - 1, \\ y' = -x + 2y + 3. \end{cases}$

2). Based on the conditions of the problem, we can assert that under the sought-for affine transformation:

- 1) , The point O' will be fixed,
- 2) and the coordinate axes of the new coordinate system will be invariant lines.

Then in the new coordinate system the formulas of the sought-for affine transformation

will have the following, very simple form: $\begin{cases} x^* = \lambda x', \\ y^* = \mu y', \end{cases}$ where λ and μ are some constants

stants

3). We will find the values of λ and μ from the condition $M \rightarrow M^*$ specified in the condition of the problem. Indeed, using the formulas of the inverse transition, we obtain that in the new coordinate system $M(8; -11)$, and $M^*(-8; -22)$, Whence it is obvious that $\lambda = -1$ and $\mu = 2$.

4). Finally, substituting the inverse transition formulas into the equalities $\begin{cases} x^* = -x', \\ y^* = 2y', \end{cases}$, we obtain the equations of the relationship between the coordinates of the image and the pre-image in the *original* coordinate system (that is, the formulas defining the desired affine transformation):

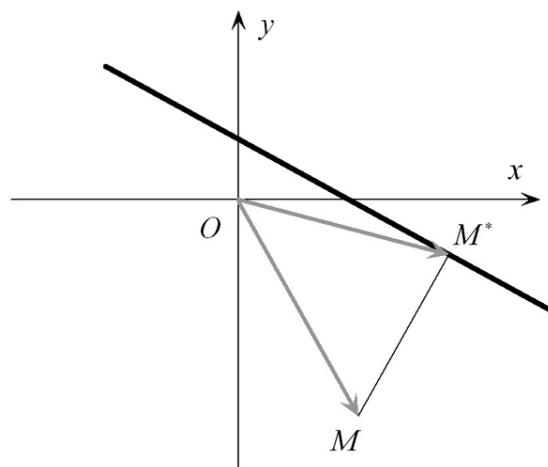
$$\begin{cases} x^* = -4x + 6y + 7, \\ y^* = -3x + 5y + 5. \end{cases}$$

Solution is found

Task 10-2.07. In an orthonormal coordinate system, find the matrix of the operator that orthogonally projects the position vectors of the points of the coordinate plane onto the line $x + 3y - 2 = 0$.

Solution:

1). Let the point-preimage M have the position vector $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, and the point-image M^* of the point M have its position vector $\vec{r}_0^* = \begin{pmatrix} x_0^* \\ y_0^* \end{pmatrix}$.



It follows from the figure that there is a point M^* of intersection of the line $x + 3y - 2 = 0$ and the perpendicular to it passing through M .

2) Since the normal vector of a straight line $x + 3y - 2 = 0$ is the direction vector of this perpendicular, the equation of the latter will have the form

$$\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x_0 \\ y_0 \end{vmatrix} + \tau \begin{vmatrix} 1 \\ 3 \end{vmatrix}.$$

From which it follows that the coordinates of the position vector of a point M^* will satisfy the system of equations

$$\begin{cases} x_0^* = x_0 + \tau, \\ y_0^* = y_0 + 3\tau, \\ x_0^* + 3y_0^* - 2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} x_0^* = \frac{9}{10}x_0 - \frac{3}{10}y_0 + \frac{1}{5}, \\ y_0^* = -\frac{3}{10}x_0 + \frac{1}{10}y_0 + \frac{3}{5}. \end{cases}$$

3) Using the rules of operations with matrices, we finally obtain that

$$\begin{pmatrix} x_0^* \\ y_0^* \end{pmatrix} = \begin{pmatrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \frac{1}{5} \\ \frac{3}{5} \end{pmatrix},$$

that is

$$\|\hat{A}\|_e = \begin{pmatrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{pmatrix}.$$

Solution is found

Note that the operator of orthogonal projection of points of a plane onto a fixed line is linear, but not affine, since there is no one-to-one relationship between the images and preimages. Therefore, $\det \|\hat{A}\|_e = 0$.