Cramer's rule

We will consider a system of linear equations n with n unknowns:

$$\begin{cases} \alpha_{11}\xi_1 + \alpha_{12}\xi_2 + \ldots + \alpha_{1n}\xi_n = \beta_1, \\ \alpha_{21}\xi_1 + \alpha_{22}\xi_2 + \ldots + \alpha_{2n}\xi_n = \beta_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{n1}\xi_1 + \alpha_{n2}\xi_2 + \ldots + \alpha_{nn}\xi_n = \beta_n \end{cases} \qquad (*)$$

in an unexpanded form $\sum_{i=1}^{n} \alpha_{ji}\xi_i = \beta_j$; $j = [1, n]$ or in matrix form $\| A \| \| x \| = \| b \|$, where the square matrix $\|A\|$ has components α_{ji}, and the columns $\|x\|$ and $\|b\|$ are respectively ξ_i and β_j.

Definition	We will call an ordered set of numbers $\{\xi_1, \xi_2, \ldots, \xi_n\}$ a *particular solution* (or, simply, a *solution*) of a system of linear equations if, when substituting these numbers into each of the equations of the system, we obtain an identity.

There is a

Theorem (Cramer's rule). **In order for the system of linear equations (*) to have a unique solution, it is necessary and sufficient that $\Delta = \det\| A \| \neq 0$, and in this case the solution of this system will have the form**

$$\xi_i = \frac{\Delta_i}{\Delta} \qquad i = 1,2,\dots,n\,,$$

where Δ_i is the determinant of the matrix obtained from the matrix $\|A\|$ by replacing its i-th column with a column of free terms $\| b \|$:

$$\Delta_i = \det\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \dots & \beta_1 & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \beta_2 & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \beta_n & \dots & \alpha_{nn} \end{Vmatrix}.$$

$\uparrow$

i- th column

Task 12.01. *Find all solutions of a system of linear equations* $\begin{cases} \mu x_1 + 8x_2 = \mu + 4, \\ 2x_1 + \mu x_2 = \mu \end{cases}$ *for any values of the parameter* $\mu \in \mathbf{R}$.

Solution: 1) Cramer's theorem states: in order for a system of linear equations $\begin{cases} \alpha_{11}x_1 + \alpha_{12}x_2 = \beta_1, \\ \alpha_{21}x_1 + \alpha_{22}x_2 = \beta_2 \end{cases}$ to have a unique solution $\{x_1^*; x_2^*\}$, it is necessary and sufficient that $\Delta \neq 0$, while $x_1^* = \dfrac{\Delta_1}{\Delta}$ and $x_2^* = \dfrac{\Delta_2}{\Delta}$, where

$$\Delta = \det\begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix}; \ \Delta_1 = \det\begin{Vmatrix} \beta_1 & \alpha_{12} \\ \beta_2 & \alpha_{22} \end{Vmatrix}; \ \ \Delta_2 = \det\begin{Vmatrix} \alpha_{11} & \beta_1 \\ \alpha_{21} & \beta_2 \end{Vmatrix}.$$

In the case when $\Delta = 0$, a special study is required.

2). In our case

$$\Delta = \det\begin{Vmatrix} \mu & 8 \\ 2 & \mu \end{Vmatrix} = \mu^2 - 16\,,$$

$$\Delta_1 = \det\begin{Vmatrix} \mu+4 & 8 \\ \mu & \mu \end{Vmatrix} = \mu^2 - 4\lambda \quad \text{and}$$

$$\Delta_2 = \det\begin{Vmatrix} \mu & \mu+4 \\ 2 & \mu \end{Vmatrix} = \mu^2 - 2\mu - 8\,.$$

Therefore, when $\mu \in (-\infty,-4) \cup (-4,4) \cup (4,+\infty)$ by Cramer's theorem the system has a unique solution

$$x_1^* = \frac{\mu}{\mu+4}\,; \quad x_2^* = \frac{\mu+2}{\mu+4}\,.$$

3). When $\mu = -4$ the given system has the form $\begin{cases} -4x_1 + 8x_2 = 0, \\ \quad x_1 - 2x_2 = -4. \end{cases}$. There are no solutions here.

4). Finally, when $\mu = 4$ the given system will be $\begin{cases} 4x_1 + 8x_2 = 8, \\ 2x_1 + 4x_2 = 4. \end{cases}$ It has an infinite set of solutions described by the formula

$$\begin{cases} x_1^* = 2 - 2\tau, \\ x_2^* = \tau \end{cases} \quad \forall \tau \in (-\infty,+\infty)\,.$$

Solution is found

Rank of a matrix

Consider a matrix $\|A\|$ of size $m \times n$. Let the number k be such that $1 \le k \le \min\{m,n\}$. We choose in $\|A\|$ in some way k columns and k rows whose intersections contain elements that form a square matrix of a *minor* of order k.

Let all minors of order higher than k be zero, then all minors of order higher than k will be zero, since each minor of order $k+1$ can be represented as a linear combination of minors of order k.

Definition	The maximum of the orders of minors of matrix $\|A\|$, *different from zero*, is called the *rank* of the matrix and is denoted by $\text{rg}\|A\|$.
	Any nonzero minor of the matrix whose order is equal to its rank is called a *basic minor*.
	The columns (rows) of the matrix that are part of the matrix of a basic minor are called *basic*.

Next, we consider n pieces of m-component columns of the form

$$\| a_1 \| = \left\| \begin{matrix} \alpha_{11} \\ \alpha_{21} \\ \dots \\ \alpha_{m1} \end{matrix} \right\| ; \ \| a_2 \| = \left\| \begin{matrix} \alpha_{12} \\ \alpha_{22} \\ \dots \\ \alpha_{m2} \end{matrix} \right\| ; \dots ; \ \| a_n \| = \left\| \begin{matrix} \alpha_{1n} \\ \alpha_{2n} \\ \dots \\ \alpha_{mn} \end{matrix} \right\|$$

and columns $\| b \| = \left\| \begin{matrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_m \end{matrix} \right\| ; \ \| o \| = \left\| \begin{matrix} 0 \\ 0 \\ \dots \\ 0 \end{matrix} \right\|$.

Since for columns (as a special case of matrices) the operations of comparison, addition and multiplication by a number are defined, we will say that a column $\|b\|$ is a linear combination of columns

$$\| a_1 \|, \| a_2 \|, \dots, \| a_n \| ,$$

if there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\| b \| = \sum_{j=1}^{n} \lambda_j \| a_j \|$.

Theorem (On the basic minor). **Every column (row) of a matrix is a linear combination of the basic columns (rows) of this matrix.**

Definition

We will call columns $\|a_1\|,\|a_2\|,\dots,\|a_n\|$ *linearly dependent* if there exist numbers $\lambda_1,\lambda_2,\dots,\lambda_n$ that are not equal to zero simultaneously, such that

$$\sum_{j=1}^{n}\lambda_j\|a_j\|=\|o\|,\quad (\sum_{j=1}^{n}|\lambda_j|>0).$$

Lemmas

1) For the columns (rows) of a matrix to be linearly dependent, it is necessary and sufficient that one of them be a linear combination of the others.

2) If among the columns of a matrix there is a linearly dependent subset, then the set of all columns of this matrix is also linearly dependent.

Theorem

For the determinant to be equal to zero, it is necessary and sufficient that the columns (rows) of its matrix be linearly dependent.

Theorem
(On the rank of a matrix)

The maximum number of linearly independent columns of a matrix is equal to the maximum number of linearly independent rows and is equal to the rank of this matrix.

Task 12.02. *Find the rank of a* 100x200 *matrix, all elements of which are equal to* 5.

Solution: 1). On the one hand, the sought rank is not less than one, since there is a non-zero minor of the first order different from zero. For example, this is the determinant of a square submatrix of size 1x1, which is the first element in the first row and first column.

2). On the other hand, any minor of the second order in this matrix has the form $\det\begin{Vmatrix} 5 & 5 \\ 5 & 5 \end{Vmatrix} = 0$. Therefore, the rank of this matrix is strictly less than 2.

3). Comparison of points 1 and 2 leads to the conclusion that the sought rank is equal to one.

Solution is found

For practical calculation of rank, the *Gauss method* is used, which consists of successive changes in the matrix, in which the value of the determinants of square submatrices (and, therefore, the value of the rank) does not change, and the calculation of the rank of the final matrix is easy to perform according to its definition.

Task 12.03. *Using the Gauss method, find the rank of the matrix*

$$\|A\| = \begin{Vmatrix} 2 & -6 & 10 & -10 \\ 3 & -6 & 12 & -12 \\ -7 & 3 & -17 & 17 \\ 2 & -6 & 10 & -10 \\ 8 & 2 & 14 & -14 \end{Vmatrix}.$$

Solutioon: 1). Поскольку исходная матрица имеет две одинаковые строки, то, заменив четвертую строку разностью первой и четвертой, получим

$$\text{rg}\|A\| = \text{rg}\begin{Vmatrix} 2 & -6 & 10 & -10 \\ 3 & -6 & 12 & -12 \\ -7 & 3 & -17 & 17 \\ 2 & -6 & 10 & -10 \\ 8 & 2 & 14 & -14 \end{Vmatrix} = \text{rg}\begin{Vmatrix} 2 & -6 & 10 & -10 \\ 3 & -6 & 12 & -12 \\ -7 & 3 & -17 & 17 \\ 0 & 0 & 0 & 0 \\ 8 & 2 & 14 & -14 \end{Vmatrix}.$$

The zero row can be discarded, since it does not affect the value of the rank of the matrix.

$$\text{rg}\|A\| = \text{rg}\begin{Vmatrix} 1 & -3 & 5 & -5 \\ 1 & -2 & 4 & -4 \\ -7 & 3 & -17 & 17 \\ 4 & 1 & 7 & -7 \end{Vmatrix}.$$

2). Next, we zero out all the elements of the first column, except for the one in the first row. To do this, we replace the second row with the difference of the first and second. We replace the third row with the sum of the third and second, multiplied by 7. We replace the fourth row with the difference of the fourth and second, multiplied by 4.

$$\operatorname{rg}\|A\| = \operatorname{rg}\begin{Vmatrix} 1 & -3 & 5 & -5 \\ 0 & -1 & 1 & -1 \\ 0 & -18 & 18 & -18 \\ 0 & 13 & -13 & 13 \end{Vmatrix}.$$

Then, taking 18 out of the third row, and 13 out of the fourth, we get

$$\operatorname{rg}\|A\| = \operatorname{rg}\begin{Vmatrix} 1 & -3 & 5 & -5 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{Vmatrix}$$

At the last step, we replace the third row with the difference of the third and second. Finally, we replace the fourth row with the sum of the fourth and second. As a result, we get a matrix with an obvious rank value

$$\operatorname{rg}\|A\| = \operatorname{rg}\begin{Vmatrix} 1 & -3 & 5 & -5 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} = 2.$$

Solution is found