Preliminary remarks to the introduction to calculus

Modern mathematical texts allow some standard notations not often found in manuals, used in the study of mathematics in secondary school. Let's look at some of them.

Symbols of community, existence and logical connection

Most Common Special Characters (these are sometimes called *quantifiers*) are given in the following table:

SYMBOL	VALUE
$\forall$	for everyone, for all
$\exists$	exists, be found
$\exists!$	is found and, moreover, in a unique way
$:$	such that
$\longrightarrow$	is true, is valid

For example, using them to define a limited numerical sequence:

> a sequence $\{x_n\}$ is called *bounded* if there is a non-negative number C such that for any number n the inequality $|x_n| \leq C$,

can be written like this:

> a sequence $\{x_n\}$ is called *bounded* if

$$\exists C : \quad C \geq 0 \longrightarrow \forall n \quad |x_n| \leq C \ .$$

Summation and multiplication symbols

Suppose we need to write down an expression for the sum in which the number of terms is arbitrary. It is known how the value of each term depends on its number. Then you can use the special summation symbol $\sum\limits_{k=1}^{n} a_k$, indicating the general form of the summand and the range of variation of the summation index.

This symbol can be considered to replace words «sum of terms of the form a_k with k in the range from 1 to n ».

For example, with this symbol the amount

$$\sin 1 + \sin 2 + \sin 3 + \cdots + \sin(n-1) + \sin m$$

is written as

$$\sum_{j=1}^{m} \sin j,$$

and the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)n}$$

can be represented as

$$\sum_{k=2}^{n} \frac{1}{(k-1)k} \, .$$

Similarly, Newton's binomial formula using the summation symbol can be written as

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k,$$

where $\quad C_n^k = \dfrac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-(k-1))}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}$.

For example, when $n = 4$ we have

$$(a+b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4 \ .$$

A similar type of notation exists for the *multiplication* operation. The symbol $n!$ — *factorial* n — denoting

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n! \ ,$$

can be written in the form

$$\prod_{k=1}^{n} k = n! \, .$$

For illustration, we also present the following formulas

$$1+2+3+\cdots+(n-1)+n=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}\ ;$$

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2=\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}\ ;$$

$$1^3+2^3+3^3+\cdots+(n-1)^3+n^3=\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}\ ;$$

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\cdots+\sin n\alpha=\sum_{k=1}^{n}\sin\alpha k=\frac{\sin\ \dfrac{(n+2)\alpha}{2}\cdot\sin\dfrac{n\alpha}{2}}{\sin\dfrac{\alpha}{2}}.$$

Check that these relations imply the equality

$$\sum_{k=1}^{n}k^3=\left(\sum_{k=1}^{n}k\right)^2\ .$$

Useful inequalities

For any two non-negative numbers a and b *Cauchy's inequality* is true

$$\frac{a+b}{2} \geq \sqrt{ab}\ .$$

The inequality is sometimes used in the following form: for any two real numbers x and y the following relation holds true:

$$x^2 + y^2 \geq 2|xy|\ .$$

Cauchy's inequality is also true for a larger number of non-negative numbers:

$$\frac{a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_{n-1} a_n}\ .$$

Note that using the summation and multiplication symbols the last relation can be written as

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \sqrt[n]{\prod_{k=1}^{n} a_k}\ .$$

In a number of applied problems we can use the *Bernoulli inequality*

$$(1+a)^x \geq 1 + xa\,,$$

which is true $\forall x$ and $\forall a > -1$,

Notes on the role of precision of definitions and formulations

In the process of studying mathematics, you should pay special attention to the completeness and accuracy of *definitions, statements of theorems* and *descriptions of properties.* Unacceptable as excessive verbosity in similar terms, as well as the loss of any of their parts.

Let us illustrate this with the following examples.

1°. *Arithmetic square root.*

As has already been noted, By definition of the arithmetic square root it is considered that that $\sqrt{a^2} = |a|$. The question may arise: «Isn't it simpler to assume that $\sqrt{a^2} = a$?»
To show the incorrectness of such a definition, Consider the following chain of transformations:

for *any pair* of numbers x and y the equalities will be true

$$\begin{gathered} x^2 - 2xy + y^2 = y^2 - 2yx + x^2, \\ \Downarrow \\ (x-y)^2 = (y-x)^2, \\ \Downarrow \\ \sqrt{(x-y)^2} = \sqrt{(y-x)^2}. \end{gathered}$$

If we now apply the definition of the form $\sqrt{a^2} = a$, then we get

$$x - y = y - x \qquad \Rightarrow \qquad x = y,$$

which is obviously false. While using the definition $\sqrt{a^2} = |a|$ gives

$$|x - y| = |y - x| \qquad \Rightarrow \qquad 0 = 0,$$

which is true for any pair of numbers x and y.

2°. *How many roots can a quadratic equation have?*

Consider the following three statements A), B) and C):

A) The equation $ax^2 + bx + c = 0$ is quadratic.

B) A quadratic equation cannot have more than two roots.

C) For any pairwise unequal numbers $\alpha,\ \beta$ and γ equation

$$\frac{(x-\alpha)(x-\beta)}{(\gamma-\alpha)(\gamma-\beta)} + \frac{(x-\beta)(x-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(x-\gamma)(x-\alpha)}{(\beta-\gamma)(\beta-\alpha)} = 1$$

can be reduced to the form A) and at the same time it has three different roots $x_1 = \alpha,\ x_2 = \beta,\ x_3 = \gamma$.

It is obvious that statements A), B) and C) are contradictory in their entirety. In other words, one of them is erroneous and, at first glance, the greatest statement C) raises doubts. However, it is actually true and the error is contained in statement A). The fact is that it is called square equation $ax^2 + bx + c = 0$ with $a \neq 0$. And it is for him that statement B) is true.

In our case, if we bring equation C) to the form indicated in Statement A), the coefficient of x^2 will be equal to zero.

Moreover, this equation will take the form $1 = 1$, that is, it is an *identity* — true equality for any value of x (including for $x_1 = \alpha,\ x_2 = \beta,\ x_3 = \gamma$).

3°. *Is it possible to arbitrarily group terms in a sum?*

It would seem that the associativity of the addition operation for numbers allows us to give a positive answer to this question. However, this is true only for sums with *a finite* number of terms. If the number of terms in the sum is not limited, then a situation similar to the following may arise.

Let us agree «on faith» with the statement, that the sum of an unlimited number of zeros equals zero, and consider a sum of the form

$$A = 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + \dots$$

grouping the terms first as

$$A = (1+2-3)+(1+2-3)+(1+2-3)+(1+2-3)+(1+2-3)+\dots$$

we conclude that $A = 0$, since each sum in parentheses equals zero. However, with another grouping method

$$A = 1+(2-3+1)+(2-3+1)+(2-3+1)+(2-3+1)+(2-3+\dots$$

we get $A = 1$. This means that the associativity rule does not apply to sums with an *unlimited* number of terms.

The last example clearly demonstrates that infinity cannot be treated like an ordinary number.

It is also worth noting that methodologically similar problems may arise and in case of mutual replacement of concepts «lack of certainty» and «existence of probability», which is often allowed when reasoning on an intuitive level.

Functional dependencies

In practice, quite often we have to deal with the so-called *variable quantities*, that is, numerical characteristics that can take on different values. Such quantitative characteristics are usually called simply *variables*.

For example, to describe a specific person you can use variables: age, height, weight, IQ, etc. In this case, it often turns out that the values of one variable may be related to the values of another. Let's say a person's weight depends on his height, his height depends on his age, currency exchange rate — from time, etc.

In some cases dependence one variable from another turns out to be *unambiguous*. In these cases, for each valid value of one variable the value of other exists and is unique. For example, the area of a circle is unique depends on its radius, the age of a person has the only meaning at every moment of time, The mass of a homogeneous body is uniquely determined by its volume.

Function definition

Unambiguous dependencies between variables commonly called *functional dependencies* or, simply, *functions.* They are the subject of study in calculus. They also play an important role in a large number of theoretical and applied disciplines.

definition 1.1. We will say that *function* is given, if a **rule** is specified according to which **each** number x, belonging to the numerical set X, **unique** number y is assigned, belonging to the numerical set Y.

The set X is usually called *domain of definition* of a function, and the set Y is *domain of its values.* The function itself can be denoted, for example, like this:

$$y = f(x) \quad x \in X,\ y \in Y.$$

In this case the variable x is called *argument* of the function, and the variable y is *the value* of the function $f(x)$.

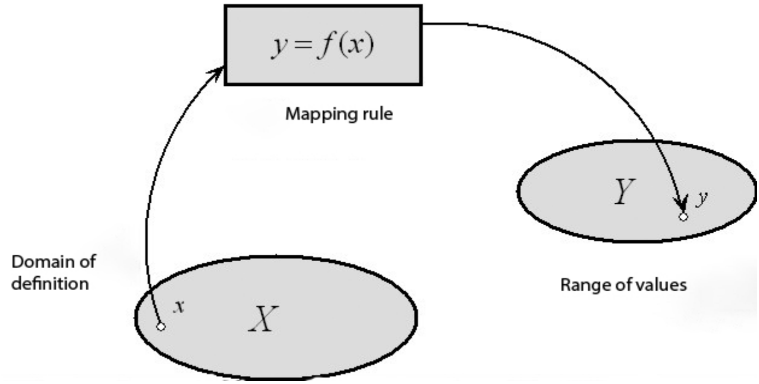

Fig.1. definition of functional dependency

Schematically, the functional dependence can be represented as an object consisting of three components (see Fig. 1):

domain of definition,
range of values and
a *mapping rule.*

According to the mapping rule each number from the domain of definitions is assigned a single number from the range of values.

Both the domain of definition and the range of values are numerical sets. The mapping rule may have various forms of presentation:

as a table,
as a mathematical formula,
as a plot
or be a solution to some problem (say, to some equation).

Finally, this rule can simply be described verbally.

Note that all of them must be supplemented by descriptions of sets X and Y.

However, quite often the function is specified only by a *formula.*

In this case *is assumed* that the domain of definition consists of the numbers for which all operations used in writing this formula are feasible. The range of values consists of numbers that are the value of the formula for all feasible arguments. It is also assumed that the mapping rule is described by this formula.

According to this convention, we can say that X, for example, for function $y = \sqrt{x-3}$ is the set of all real numbers no less than 3. Indeed the arithmetic square root is taken possible only from a non-negative number.

The value set Y contains all non-negative numbers.

Symbolically, all this can be written as:

$$X : \{\forall x \geq 3\},\ Y : \{\forall y \geq 0\} \qquad \text{or} \qquad X : \{[3, +\infty)\},\ Y : \{[0, +\infty)\}.$$

Note that the problem of finding a domain of definition and a range of values does not always turn out to be so trivial.

Let us illustrate this with the following examples.

Example 1.1. Find the domain of definition and range of values for functions: a) $y = \sqrt{\dfrac{2x+3}{x-2}}$.

1) *Let's find the domain of definition*:

solving inequality $\dfrac{2x+3}{x-2} \geq 0$, we get $\left[\begin{array}{ccc} x & \leq & -\dfrac{3}{2}, \\ x & > & 2, \end{array}\right.$ that is, finally,

$X : \left\{ \left(-\infty, -\dfrac{3}{2} \right] \cup \left(2, +\infty \right) \right\}$, since taking the square root is only possible from a non-negative number.

We also note that the number 2 does not belong to the domain of definition, since with such a value of x the denominator of the radical expression will turn into 0 .

Range of values:

to find the range of values, consider the formula $y = \sqrt{\dfrac{2x+3}{x-2}}$ as an *equation with unknown x and parameter y* . Let's find out for what values of y there exists x — a real root this equation.

Simple calculations lead to $x = \dfrac{2y^2+3}{y^2-2}$, what does existence mean real x for any $y \neq \pm\sqrt{2}$.

On the other hand, the value of the function in the example under consideration is arithmetic square root. Therefore, y is non-negative. Combining the found restrictions on the value y , we get that

$$Y : \left\{ [0, \sqrt{2}) \bigcup (\sqrt{2}, +\infty) \right\} .$$

b) $\quad y = x + \dfrac{1}{x} \quad .$

Domain of definition:

obviously $X : \{\forall x \neq 0\}$ or, which is the same $\{(-\infty, 0) \bigcup (0, +\infty)\}$.

Range of values:

We obtain a description of the range of values of this function in two steps. First, consider the case $x > 0$. For any positive x the following inequality is true:

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0\,, \text{ or } \left(x - 2\sqrt{x}\frac{1}{\sqrt{x}} + \frac{1}{x}\right) \geq 0 \quad \Longrightarrow \quad x + \frac{1}{x} \geq 2.$$

For the case $x < 0$, an estimate of the range of values can be obtained using the equality $(-x) + \dfrac{1}{(-x)} = -\left(x + \dfrac{1}{x}\right)$, from which, due to the inequality

$$x + \frac{1}{x} \geq 2 \quad \forall x > 0 \qquad \text{we have} \qquad x + \frac{1}{x} \leq -2 \quad \forall x < 0.$$

We finally obtain that the range of values of this function is the set $Y : \{(-\infty, -2] \bigcup [2, +\infty)\}$.

c) $\quad y = \dfrac{2x^2+2x+1}{x^2+3x+2}$

Domain of definition:

is found from

$$x^2+3x+2 \neq 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} x \neq -2 \\ x \neq -1, \end{array} \right.$$

since the denominator of a fraction cannot have zero values. There are no other restrictions on the calculation of function values, so the domain of definition will be

$$X : \left\{ (-\infty, -2) \bigcup (-2, -1) \bigcup (-1, +\infty) \right\}.$$

Range of values:

It is convenient to find the range of values of a given function using the fact that that its domain of definition is set of arbitrary real numbers (with the exception of -2 and -1.)

Let us consider the equality $y = \dfrac{2x^2+2x+1}{x^2+3x+2}$ as an equation with unknown x and solve it, considering y to be some fixed parameter. To do this, we transform this equality to the form, standard for quadratic equations

$$(y-2)x^2 + (3y-2)x + (2y-1) = 0\,, \qquad (1.1)$$

whose roots are determined by the well-known formula

$$x_{1,2} = \frac{-\,(3y-2) \pm \sqrt{D}}{2(y-2)} \qquad y \neq 2\,,$$

where the discriminant of the quadratic equation (1.1) is

$$D = (3y-2)^2 - 4(y-2)(2y-1)\,.$$

Value of x will be real when $D \geq 0$, or, in our case,

$$(3y-2)^2 - 4(y-2)(2y-1) = y^2 + 8y - 4 \geq 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad y \in \Big\{\,(-\infty, -2\sqrt{5}-4] \bigcup [2\sqrt{5}-4, +\infty)\Big\}\ .$$

In other words, x can take real values only or at $y \leq -2\sqrt{5}-4 \approx -8.4$, or at $y \geq 2\sqrt{5}-4 \approx 0.4$. Consequently, the range of values of this function is formed by the numbers y, satisfying either the first or second of the resulting inequalities and not equal to 2.

Finally, note that the above reasoning does not apply for $y = 2$, because in this case equation (1.1) is not quadratic.

Nevertheless, the number 2 belongs to the range of values, since for this case equation (1.1) has a real solution $x = -\frac{3}{4}$. It means that value of y is equal to 2 , when $x = -\frac{3}{4}$. Hence,

$$Y : \left\{ (-\infty, -2\sqrt{5} - 4] \bigcup [2\sqrt{5} - 4, +\infty) \right\} .$$

Classification of functions

Functions are usually classified according to the presence or it lacks the property *periodicity* and the property *parity*.

definition 1.2. The function $y = f(x)$ is called *periodic*, if there is a number $\tau > 0$ such that $\forall x \in X \Longrightarrow x \pm \tau \in X$ and $f(x \pm \tau) = f(x)$. The smallest of such numbers, i.e. $T = \min \tau$, is called the *period* of the function $y = f(x)$.

Example 1.2. Examples of periodic functions include:

$$
\begin{array}{ll}
y = \sin x & \text{with period } T = 2\pi, \\
y = \cos 3\pi x & \text{with period } T = \frac{2}{3}, \\
y = \operatorname{tg} x & \text{with period } T = \pi, \\
y = x - \operatorname{floor}(x) & \text{with period } T = 1.
\end{array}
$$

`definition` 1.3. Let X be the domain of definition for the function $y = f(x)$, symmetrical about the point $x = 0$, then this function is called:

even, if $\forall x \in X$ Done $f(-x) = f(x)$,
odd, if $\forall x \in X$ $f(-x) = -f(x)$.

`Example` 1.3. Classifying functions by parity gives:

$y = x^2$ — even,
$y = x^3$ — odd,
$y = \sin x$ — odd,
$y = |x|$ — even,
$y = 3^x$ — is neither even nor odd.

Note that in the symmetric domain of definition each function can be represented as the sum of some even function and some odd. For this you can use, for example, formula

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Thus, for the function $y = 3^x$, the expansion into the sum of even and odd will have the form

$$y = \frac{3^x + 3^{-x}}{2} + \frac{3^x - 3^{-x}}{2}.$$

Check for yourself what this function can be written in the form

$$y = \operatorname{ch}(x \ln 3) + \operatorname{sh}(x \ln 3).$$

Summary of the basic concepts of calculus

In the final part of the topic, we will briefly describe some important concepts of calculus. In the future, we will consider all of them more strictly and in detail. Their discussion here is for informational purposes only. This material is intended for optional information. It may be useful in studying physics and other applied disciplines.

Limit of function

In addition to its *value*, a function may have other *local* numeric characteristics. One of which is a number called *the limit of a function at a point.*

Let us consider values of a function $f(x)$ in a small neighborhood of the point x_0, and let x tends to x_0 in some way. In a large number of cases it can be seen that these values are close to some number A, *for any* method of approaching x to x_0. In other words, the value $f(x)$ in this neighborhood differs as little as possible from A.

An example is the function $f(x) = \dfrac{\sin x}{x}$, which has no value at the point $x_0 = 0$, but is defined in any of its neighborhoods. In each neighborhood of $x_0 = 0$, the function $f(x)$ has values that differ as little as desired from $A = 1$.

Let such a number A exists *for any* way x tends to x_0, then A is called *the limit of the function $f(x)$ with x tending to x_0*. This fact is symbolically designated as $\lim\limits_{x\to x_0} f(x) = A\,.$

In the example considered this will be $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1\,.$

Here we do not give a strict definition of the limit of a function. We will only notice that this concept makes sense not only when x_0 is some finite number, but also is symbolically represented as ∞ , $+\infty$ or $-\infty$.

Let us emphasize once again: the limit of a function as x tends to x_0 is a *local* numerical characteristic of the function, that is, related to the point x_0.

Moreover, for the same point, the value of the function and its limit independent of each other: they can exist simultaneously and be equal or unequal to each other, and may also not exist, both together and separately.

Let us explain this with the following example.

Example 1.4. Consider a function called *number signature,* which is denoted as $y = \operatorname{sgn} x$ and defined by the formula

$$\operatorname{sgn} x = \left[\begin{array}{rl} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{array} \right.$$

The graph of this function is shown in Fig. 2.

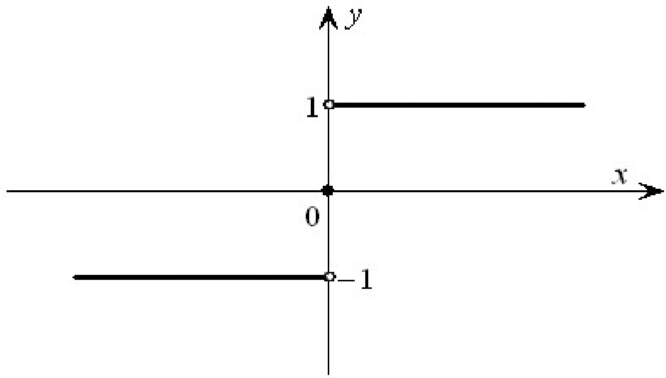

Fig. 2. Graph of the function $y = \operatorname{sgn} x$

This function has a zero value at $x = 0$, but the limit $\lim\limits_{x\to 0} \operatorname{sgn} x$ does not exist. Indeed, if you are approaching 0 from the right, then the value of this function will remain equal to $+1$. While when approaching the zero point from the left, this value will be -1.

Check for yourself that the function $y = |\operatorname{sgn} x|$ at point 0 has both limit equal to 1 and value equal to zero.

Finally, we point out that in the case where at the point x_0 as a limit, so the function values exist and *are equal to each other*, then the function is said to be *continuous* at this point.

Derivative of a function at a point

The value of a function and its limit are local numerical characteristics, allowing to quantitatively describe the function in a small neighborhood of a certain point. However, these characteristics are not enough, when you need to evaluate relative change in the value of a function when the argument changes.

For example, the functions $y(x) = 2x$ and $z(x) = 5x$ have at the point $x_0 = 0$ zero value, but relative changes in their values in a small neighborhood of zero are significantly different.

Really,

$$\frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0} = \frac{2x}{x} = 2\,,$$

$$\frac{\Delta z}{\Delta x} = \frac{z - z_0}{x - x_0} = \frac{5x}{x} = 5\,.$$

To estimate the magnitude of the relative change in the value of a function, it is used special numerical characteristic: *derivative of a function at a point.*

`definition` 1.4. *The derivative of a function at a point* is the number equal to the limit of the ratio of the value increment of the function to the increment of its argument when the increment of the argument tends to zero.

This characteristic is denoted as $f'(x_0)$ and is equaled to

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\Delta f}{\Delta x}\,.$$

In other words, this definition means: for a function $y = f(x)$ its derivative at the point x_0, denoted as $f'(x_0)$ or $y'_x(x_0)$, is equal to

$$f'(x_0) = \lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t}. \tag{1.2}$$

Indeed, if the value of the argument was x_0, and became $x_0 + t$, then its increment is obviously equal to $\Delta x = (x_0 + t) - x_0 = t$. Similarly, if the value of the function was $f(x_0)$, and it became $f(x_0 + t)$, then its corresponding increment is $\Delta f = f(x_0 + t) - f(x_0)$.

From definition 1.4 it follows that the function $y = f(x)$ must have values in some neighborhood of the point x_0, and also be *continuous* at point x_0. The last condition is necessary (but not sufficient!) for the existence of a derivative of a function at a point, since only for a continuous function the limit of increment of the value of the function is equal to zero as the increment of the argument tends to zero.

However, even for a continuous function, limit (1.2) is an uncertainty of the form «$\frac{0}{0}$», that is, a conclusion about the existence (or non-existence) derivative of a function at a point can be done only after "disclosure" of this uncertainty.

Let us explain definition 1.4 and the term “uncertainty disclosure” with the following examples.

Example 1.5. Find the derivative of the function $y = x^3$ at the point $x_0 = 2$.

First, let’s solve this problem for an arbitrary fixed point x_0. Let the increment of the argument at point x_0 be equal to t. Then we find the corresponding increment in the value for this function

$$f(x_0+t) - f(x_0) =$$

$$= (x_0+t)^3 - x_0^3 = \left(x_0^3 + 3x_0^2 t + 3x_0 t^2 + t^3\right) - x_0^3 = 3x_0^2 t + 3x_0 t^2 + t^3 \ .$$

We have

$$f'(x_0) = \lim_{t\to 0} \frac{3x_0^2 t + 3x_0 t^2 + t^3}{t} = \lim_{t\to 0} \left(3x_0^2 + 3x_0 t + t^2\right) = 3x_0^2 \ .$$

Substituting $x_0 = 2$ into the resulting expression, we find, that the desired value is $y'_x(2)$ is equal to 12.

Example 1.6. Find the derivative of the function $y = |x|$ at point $x_0 = 0$.

For this function, due to $x_0 = 0$

$$f'(x_0) = \lim_{t\to 0} \frac{|x_0 + t| - |x_0|}{t} = \lim_{t\to 0} \frac{|t|}{t} = \lim_{t\to 0} \left[\begin{array}{rl} 1\ , & \text{if } t > 0, \\ -1\ , & \text{if } t < 0. \end{array} \right. \qquad (1.3)$$

How can we conclude that the function in question has no derivative at zero, since limit (1.3) does not exist.

Concluding our discussion of definition 1.4, we note that mathematical texts use different ways of notation derivative of a function at a point. In addition to those used above, The most frequently encountered designations include

$$y'_x(x)\Big|_{x=x_0} ; \qquad \frac{dy}{dx}\Big|_{x=x_0} ; \qquad f'(x)\Big|_{x=x_0} .$$

Also the concept of the derivative of a function at a point allows *a geometric interpretation*, the meaning of which is best illustrated by the problem constructing a tangent to the plot of a function at some point.

Solving this problem, we come to the conclusion (see Fig. 3) that *the value of the derivative of the function at a point equals tangent of the angle formed by the tangent at this point to the plot of the function with x-axis* (Ox) .

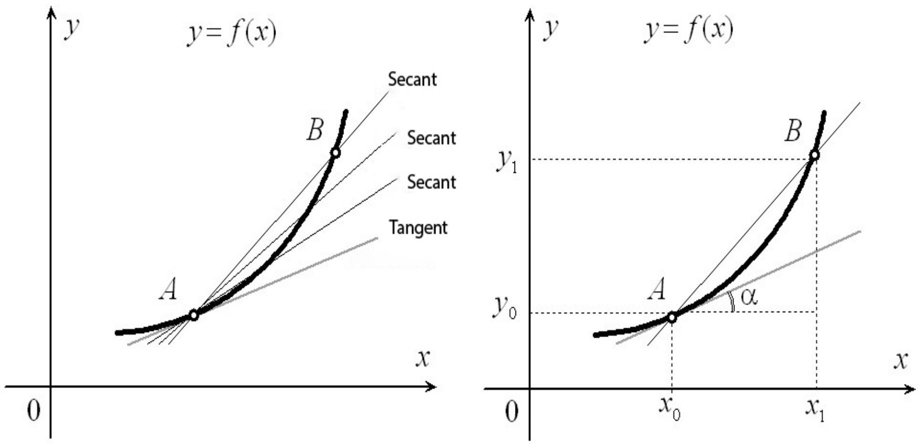

Fig. 3. Geometric meaning of the derivative of a function at a point.

In conclusion, we note that from the above reasoning it follows that *the value of the derivative of the function equals zero at these points or its non-existence.* An example would be the functions $y = x^2$ and $y = |x|$, having a minimum at the point $x_0 = 0$.

On the other hand, this condition is not sufficient. Example: for function $y = x^3$, at point $x_0 = 0$ the derivative is equaled to zero, but it does not have an extremum here.

Derivative Function

Finding the values of derivative of a function at a point by definition 1.4 in most cases is a rather difficult task.

In practice it turns out to be much more convenient to use a different approach based on the following considerations.

The passage to the limit used in definition 1.4 is carried out by auxiliary variable t, while the value of x_0 is a fixed numeric parameter.

If you change x_0, then the value of the limit (1.2), generally speaking, it will change as well. However, for each x_0 it is *unique,* because if the limit exists, then he is the only one.

Therefore, definition 1.4 can be considered as a rule by which each value of x_0 the single value $f'(x_0)$ is assigned. That is, we can say that in this casey some *new function* is given. And the value of the function at point x_0 is equaled to $f'(x_0)$.

This new function is called *derivative function* for the function $y = f(x)$ and denoted as $f'(x)$. In this case, the search operation $f'(x)$ for a given $f(x)$ is called *differentiation.* In the case when for $f(x)$ there exists $f'(x)$, they also say that the function $f(x)$ *is differentiable.*

On the other hand, a function $y = F(x)$ such that $F'(x) = f(x)$, is called *antiderivative for* $y = f(x)$. The operation of finding $F(x)$ for a given $f(x)$ is called *integration.*

For example, using the solution to Problem 1.5, we can say that the derivative function for $y = x^3$ is $y = 3x^2$. Indeed, the method used for solving this problem is suitable for *any* point x_0.

Derivative functions are also usually denoted as

$$y'_x \; ; \qquad \frac{dy}{dx} \; ; \qquad y'(x) \; .$$

In cases where a function depends on more than one variable, identifier of the variable with respect to which the derivative is taken, is indicated explicitly as a subscript.

For example, for a function $f(x, p)$ depending on x and p, the notation $f'_x(x, p)$ means the derivative with respect to the variable x, under the assumption that p is a fixed parameter.

It is also worth noting that in mathematical texts the concepts «derivative of a function at a point» and «derivative function» often denoted by the same word «derivative», believing that it is clear from the context what is being said.

The search of derivative at a point by means of derivatives of functions is obviously much more convenient, than by using definition 1.4. But then the question arises: how to find derivatives of functions?

The answer to this question is:

it ought to use the information contained in the following two tables:

1) the first table contains derivative formulas for some small set of elementary functions, obtained directly by definition 1.4 and
2) the second table allows you to express the derivative of one function in terms of the derivative of another.

The first of these tables, usually called «Table derivatives» could look like this, for example

$f(x)$	$f'(x)$
x^a	ax^{a-1}
e^x	e^x
$a^x, \quad a > 0, a \neq 1$	$a^x \ln a$
$\ln \lvert x \rvert$	$\frac{1}{x}$
$\log_a \lvert x \rvert, \quad a > 0, a \neq 1$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\operatorname{arctg} x$	$\frac{1}{1+x^2}$

Table 1.1

The second table, called «Differentiation rules», can be written in the following form

1°	$(f(x) + g(x))' = f'(x) + g'(x)$
2°	$(C \cdot f(x))' = C \cdot f'(x)$, where C – const
3°	$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4°	$(f(g(x)))'_x = f'_g(g(x)) \cdot g'_x(x)$

Table 1.2

Tables 1.1 and 1.2 assume by default that all the derivatives used to write these formulas exist.

The following examples show how tables 1.1 and 1.2 can be used.

`Example` 1.1. Let it be necessary to find derivative functions for

$$1)\quad y=\frac{\sqrt[3]{x^2}+\sqrt{x}+1}{x}\,,\qquad 2)\quad y=\frac{\sin x}{x}\qquad \text{and}\qquad 3)\quad y=e^{\arccos x}.$$

`Solution.`

1) Comparison of rules 1° and 3° of table 1.2 shows that the sum of functions is easier to differentiate than the product, Therefore, first we perform term-by-term division, that is, we will look for the derivative of the function

$$\left(\frac{\sqrt[3]{x^2}+\sqrt{x}+1}{x}\right)'=\left(x^{-\frac{1}{3}}+x^{-\frac{1}{2}}+x^{-1}\right)'\;=$$

and, using the first formulas of tables 1.1 and 1.2, we obtain

$$=-\frac{1}{3}\cdot x^{-\frac{4}{3}}-\frac{1}{2}\cdot x^{-\frac{3}{2}}-x^{-2}=-\frac{1}{3x\sqrt[3]{x}}-\frac{1}{2x\sqrt{x}}-\frac{1}{x^2}\,.$$

2) Table 1.2 does not contain a formula for differentiating a fraction (although it can be found in many textbooks). Therefore, we first transform this function into a product, and only then apply rule 3° of table 1.2

$$\left(\frac{\sin x}{x}\right)' = \left((\sin x)\cdot(x^{-1})\right)' = (\sin x)'\cdot(x^{-1}) + (\sin x)\cdot(x^{-1})' =$$

$$= (\cos x)\cdot x^{-1} + (\sin x)\cdot(-x^{-2}) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x\cos x - \sin x}{x^2}\,.$$

3) First, remember that $g(x) = \arccos x = \dfrac{\pi}{2} - \arcsin x\,,$ whence, according to the first rule of table 1.2 and the penultimate line of table 1.1,

$$g'_x(x) = -\frac{1}{\sqrt{1-x^2}}\,.$$

Then,

$$y'_x = \left(e^{\arccos x}\right)'_x = \left(e^{g(x)}\right)'_x = \left(e^g\right)'_g \cdot g'_x(x) =$$

$$= e^g \cdot g'_x(x) = e^g \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{e^{\arccos x}}{\sqrt{1-x^2}}\,.$$

To conclude this topic, we will briefly consider two more mathematical objects, related to the concept of a derivative function.

Sometimes it turns out necessary to consider the rate of change the value *of the derivative function in the vicinity of a point* x_0. In other words, we need to find the derivative of a derivative function.

This characteristic is called *second-order derivative* or, simply, *second derivative.* It is usually denoted as

$$y''_{x=x_0} \qquad \left.\frac{d^2y}{dx^2}\right|_{x=x_0} ; \qquad y''(x)|_{x=x_0} ,$$

and the search for value of this numerical characteristic comes down to calculating limit of the form

$$f''(x_0) = \lim_{t\to 0} \frac{f'(x_0+t) - f'(x_0)}{t} . \tag{1.4}$$

Since the second derivatives of a function at a point (as the limits of a function) is uniquely defined then you can consider another function whose values are numbers, given by formula (1.4).

To find the second derivative you should use the same rules as when calculating the first one.

For example, function $y = \ln|x|$, has $y' = \dfrac{1}{x}$. Hence the second derivative of the function $y = \ln|x|$ will be (according to table 1.1) equaled to $y'' = (-1)x^{-2} = -\dfrac{1}{x^2}$.

That is, $(\ln|x|)'' = -\dfrac{1}{x^2}$.

Another mathematical object related to the derivative function is called *first differential* or, simply, *differential.*

The differential of a function $y = f(x)$ having a derivative at the point x, called the function df, depending on two variables x and dx, of the form

$$df(x, dx) = f'(x) \cdot dx.$$

Note that there are both dx and df are not products, but designations (symbols) of variable values.

By definition of a differential, its value depends directly proportionally on dx, while its dependence on x may be nonlinear.

The main property of the differential is described by the formula

$$f(x + dx) - f(x) = df(x, dx) + r(x, dx).$$

Here the remainder term $r(x, dx)$ satisfies equality (which is a statement of a theorem!)

$$\lim_{dx \to 0} \frac{r(x, dx)}{df(x, dx)} = 0\,.$$

This means that you can use the value $f'(x) \cdot dx$ as an *estimate* of the difference $f(x + dx) - f(x)$ for small enough $|dx|$ in small vicinity of x.

Indefinite and definite integrals

If the course program also requires preliminary acquaintance with these themes, then the relevant materials can be found in topics on the subject «Multidimensional analysis, integrals and series».