Numerical sequence and its properties

Numerical sequence

We will say that *a numerical sequence* is given, if a rule is specified according to which *for each* natural number n *unique* value x_n is assigned. x_n is called the value of the n-th term of the sequence.

The numerical sequence is usually denoted as $\{x_n\}$.

According to this definition, the numerical sequence may be considered as *function of the natural number series.* In other words, as a function, whose domain of definition is the set of all natural numbers $\mathbb{N}$.

Note that the argument of the formula defining the sequence may not be the number of sequence members.

For example, for sequence with $x_k = \dfrac{1}{k-1}$ $\forall k \geq 2$ the number of members is not k, but $n = k - 1$.

Numerical sequences can be specified in various ways.

1) *By listing the values of its members.* For example, the sequence $\{x_n\}$, in which all terms with even numbers are equal to one, and all terms with odd numbers are -1, can be written as $\{-1, 1, -1, 1, -1, 1, \ldots\}$

2) *By functional rules,* which for each member of the sequence allows uniquely determine value by its number. For example, for the sequence considered in 1) such a rule could be formula

$$x_n = (-1)^n \qquad \text{or} \qquad x_n = \sin\left(\frac{\pi}{2} + \pi n\right) \ , \quad n = 1, 2, \ldots$$

3) *By recurrent rules,* according to which the value of each term sequences can be uniquely determined by the value of one or more previous members. For example, for the sequence considered in 1) such a rule can be the ratios

$$x_{n+1} = (-1) \cdot x_n \ , \qquad x_1 = -1 \ , \qquad n = 1, 2, \ldots$$

   Another example of recurrently defined sequences are arithmetic and geometric progressions.

As an exercise, suggest different ways to describe a number sequence with odd-numbered terms are equaled to one, and with even numbers they are equaled to zero.

Sometimes members of one numerical sequence are expressed through members of another. Examples here are

1) $\{S_n\}$ — sequence *partial sums* for sequence $\{a_k\}$, where $S_n = \sum\limits_{k=1}^{n} a_n$,
2) $\{b_n\}$ — *subsequence* for sequence $\{a_k\}$, where $b_n = \{a_{k_n}\}$. $\{k_n\}$ is a sequence of natural numbers, for which $k_n < k_{n+1} \ \ \forall n \in \mathbb{N}$.

Classification of numerical sequences

Numerical sequences can be distinguished according to the set of values of their members. For example,

— sequence, all members of which have the same sign are called *of constant signs.*

— sequence, all members of which have a value not exceeding in absolute value some fixed number is called *bounded.*

Note that it is convenient to give definition of a bounded sequence, using logical symbols. For example, a sequence $\{x_n\}$ is called *bounded* if

$$\exists C \geq 0 : \forall n : |x_n| \leq C\,.$$

In other words, there is a non-negative number C such that for any natural n inequality $|x_n| \leq C$ is valid.

If the last inequality has the form $x_n \leq C$ (or $x_n \geq C$), then they say about *bounded above* (or, correspondingly, *bounded below*) number sequence.

Let us now formulate the definition of *unbounded* numerical sequence. Recall that *negation* of some definition must be constructed in compliance with the rules of formal logic.

For example, the formulation «there is no number C such that ...» is not erroneous, but it is not suitable for definition, because it is impossible to be sure that this number C *does not exist* (to complete such search is physically impossible!).

A constructive version of the definition of an unbounded sequence could be, say, the following: a sequence $\{x_n\}$ is called *unbounded* if

$$\forall C \geq 0 : \exists N_C : |\, x_{N_C} | > C \,.$$

That is, for every non-negative number C there is a number N_C such that inequality $|\, x_{N_C} | > C$ is valid.

Numerical sequences can also be distinguished by their nature changes in the values of their members when the number of the members changes. For example,

— sequence, in which the change in its member number is 1 changes the sign of this term to the opposite one, called *alternating,*

— sequence, for which the inequality $x_{n+1} > x_n$ holds for any n is called *monotonically increasing.* If $\forall n$ inequality $x_{n+1} < x_n$ is valid, then the sequence is called *monotonically decreasing.*

Let us explain these definitions with the following examples.

`Example` 1.1. 1) Numerical sequence $x_n = 1 - \dfrac{1}{n}$ is bounded since $\forall n :\ 0 \leq x_n < 1$. In addition, it will be monotonically increasing due to the inequality

$$x_{n+1} - x_n = \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right) =$$

$$= -\frac{1}{n+1} + \frac{1}{n} = \frac{1}{n(n+1)} > 0 \qquad \forall n.$$

2) Numerical sequence $x_n = n^{(-1)^n}$, for which

$$x_1 = 1,\ x_2 = 2,\ x_3 = \frac{1}{3},\ x_4 = 4,\ x_5 = \frac{1}{5},\ x_6 = 6,\ \ldots,$$

is bounded from below (by zero), is not bounded from above and is not neither monotonically increasing nor monotonically decreasing.

The limit of a numerical sequence and its properties

To describe a numerical sequence it is necessary to specify a rule that allows to find the values of sequence members by their numbers.

In addition, the numerical sequence may have a quantitative characteristic not associated with specific members, but with the entire sequence as a whole.

This characteristic is called *the limit of a numerical sequence* and defined as follows.

`Definition` 1.1. The number A is called *the limit of a numerical sequence* $\{x_n\}$, if *for any* positive number ε *exists* a number N_ε such that *for all* members of the sequence with numbers $n \geq N_\varepsilon$ the inequality $|\, x_n - A\,| < \varepsilon$ is valid.

The fact that the number A is the limit of the number sequence $\{x_n\}$, is symbolically written as $\lim\limits_{n\to\infty} x_n = A$. Sometimes the definition is also written as $\{x_n\} \underset{n\to\infty}{\longrightarrow} A$.

The number A (if it exists!) can be either contained or not to be contained in the set of sequence members values.

Using quantifiers, the condition $\lim\limits_{n\to\infty} x_n = A$ can be written like this:

$$\forall\varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \quad \longrightarrow \quad |x_n - A| < \varepsilon\,.$$

Obviously, not every numerical sequence has a limit, which is equaled to the number A. Definition of a statement $\lim\limits_{n\to\infty} x_n \neq A$ should be formulated in the form of a testable condition, for example as

`Definition` 1.2. Number A *is not a limit numerical sequence* $\{x_n\}$, if *exists* a positive number ε_0 such that *for any* number $N \in \mathbb{N}$ *there is* a member of this sequence with number $n_0 \geq N$, for which the inequality $|x_{n_0} - A| \geq \varepsilon_0$ is valid.

Or in quantifiers:

$$\exists \varepsilon_0 > 0 \quad \forall N \in \mathbb{N} : \exists n_0 \geq N \quad \longrightarrow \quad |x_0 - A| \geq \varepsilon_0 \, .$$

Sequences whose limit is 0 (that is, for which $\lim\limits_{n\to\infty} x_n = 0$) are called *infinitesimal.*

Sequences whose members (starting from a certain number) take values modulo values greater than any predetermined number are called *infinitely large* .

`Definition` 1.3. We will say, that the sequence $\{x_n\}$ has as its limit ∞, if

$$\forall\varepsilon > 0 \;\; \exists N_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geq N_\varepsilon \longrightarrow \Big|x_n\Big| > \varepsilon \, .$$

This fact is denoted as $\lim\limits_{n\to\infty} x_n = \infty$.

Similarly, we can define the limits of the form $\lim\limits_{n\to\infty} x_n = \pm\infty$. For example, equality $\lim\limits_{n\to\infty} x_n = -\infty$ means that

$$\forall\varepsilon > 0 \;\; \exists N_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geq N_\varepsilon \longrightarrow x_n < -\varepsilon \, .$$

In other words, sequences having as their limit ∞, infinitely large. In this case, one should distinguish between infinitely large sequences from unbounded sequences, having no limit.

If a numerical sequence has a finite limit, then it is called *convergent,* otherwise — *divergent.*

An example of a divergent sequence would be $x_n = (-1)^n$, that is, $\{-1,\ 1,\ -1,\ 1, \ldots\}$. An example of an infinitely large sequence is $x_n = n$.

`Example` 1.2. Numerical sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$, for which $x_n = \frac{1}{n}$, has a limit equal to zero. That is, $A = 0$ or $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Let us prove this using Definition 1.1. Let us note firstly, that this numerical sequence is monotonically decreasing, since $\forall n$ the following inequality is true: $\frac{1}{n} > \frac{1}{n+1}$, that is, $x_n > x_{n+1}$.

We have

$$| x_n - A | = \left| \frac{1}{n} - 0 \right| = \frac{1}{n}.$$

Therefore, for any given positive ε you can choose a number [1]

$$N_\varepsilon = \left[\frac{1}{\varepsilon} \right] + 1, \tag{1.1}$$

for which $\dfrac{1}{N_\varepsilon} < \varepsilon$ and $| x_{N_\varepsilon} - 0 | = \left| \dfrac{1}{N_\varepsilon} - 0 \right| = \dfrac{1}{N_\varepsilon} < \varepsilon$. Then, due to the monotonic decrease of the sequence under consideration, *for all* numbers $n \geq N$ the following inequality will also be true

$$| x_n - 0 | = \frac{1}{n} < \varepsilon \qquad \forall n \geq N_\varepsilon .$$

So $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. The number 0 is the limit of the number sequence $x_n = \dfrac{1}{n}$.

See Fig.1 as an illustration.

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[1] Here $\left[\dfrac{1}{\varepsilon} \right]$ denotes the *integer part* of the fraction $\dfrac{1}{\varepsilon}$. For the integer part of a number x the notation is also used $[x] = \text{floor}(x)$.

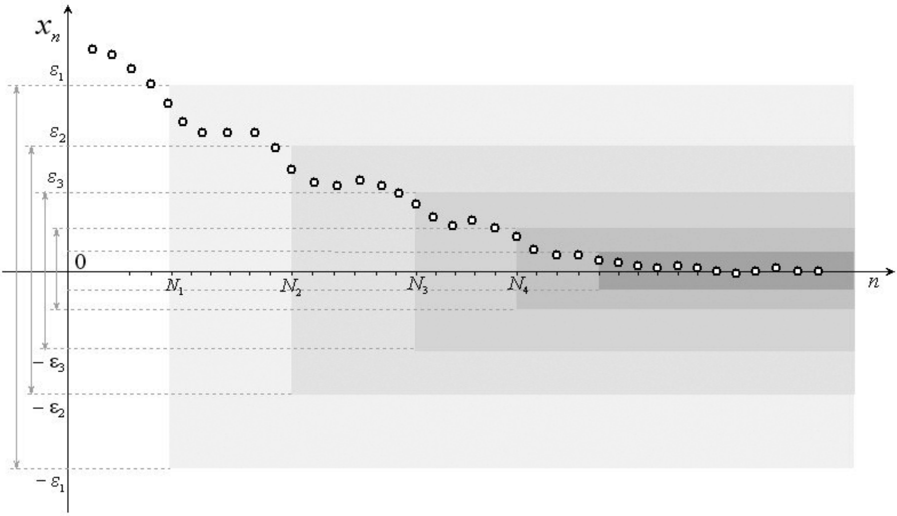

Fig.1. Example of a numerical sequence graph with a zero limit.

`Example` 1.3. Numerical sequence $x_n = (-1)^{n+1}$ has no limit.

Let us prove this using definition 1.1. Let's pretend that A is the limit of this sequence.

For convergence it is necessary so that $\forall \varepsilon > 0$ and $\forall n \in \mathbb{N}$

$$\begin{cases} |\, x_n - A \,| & < \varepsilon, \\ |\, x_{n+1} - A \,| & < \varepsilon. \end{cases}$$

Let us take some even natural number as n. Let $\varepsilon = \dfrac{1}{2}$. Then the necessary condition for convergence takes the form

$$\begin{cases} |\, 1 - A \,| & < \dfrac{1}{2}, \\ |\, -1 - A \,| & < \dfrac{1}{2}. \end{cases}$$

or

$$\begin{cases} -\dfrac{1}{2} < & 1 - A & < \dfrac{1}{2}, \\ -\dfrac{1}{2} < & 1 + A & < \dfrac{1}{2}. \end{cases}$$

If we add these inequalities term by term, we get an incorrect consequence like $-1 < 2 < 1$. This means that such a number A does not exist and the sequence has no limit.

Definition 1.1 can be interpreted as some kind of «game», in which one «player» sets *arbitrary* (as small as desired) positive number ε.

His «enemy» based on the value of this number, must select (or just guess) the number N_ε , so that $|x_n - A| < \varepsilon$. (see Fig. 1).

If the «winner» in this game is the second player $\forall\varepsilon > 0$, then the number A is the limit of the number sequence $\{x_n\}$.

Note that the number selection rule of N_ε may be *different for different* ε. This ambiguity is emphasized by the subscript of the searched N_ε. It is clear that this fact makes it easier for the second player to play for victory.

On the other hand, from the point of view of formal logic, the fact that «the second player» fails at some $\varepsilon > 0$ to find a value N_ε does not mean that A is no limit to this sequence.

Finally, as an exercise, answer the following question. Is the dependence of variable N_ε on the variable ε functional or not?

Let us note the main ones that are useful for solving practical problems: properties of limits of number sequences.

Let everything used in the entries 1°–6°, sequences are convergent and C is some constant, then

1°. $\lim\limits_{n\to\infty}(x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$.

2°. $\lim\limits_{n\to\infty}(C \cdot x_n) = C \cdot \lim\limits_{n\to\infty} x_n$.

3°. $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

4°. If, in addition,

$$y_n \neq 0\ \forall n \ \text{ and } \ \lim_{n\to\infty} y_n \neq 0, \text{ then } \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n} .$$

5°. If the sequence increases (decreases) monotonically and is limited from above (from below), then it has a limit.

6°. If $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = A$ and $\forall n \quad x_n \geq z_n \geq y_n$, then we have $\lim\limits_{n\to\infty} z_n = A$.

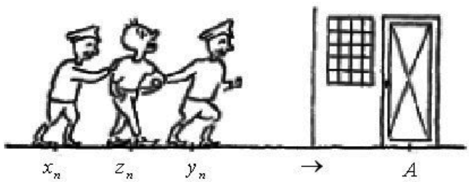

Fig.2. Theorem 6° «about two policemen».

Note that the 5° property, which is a *sufficient* condition the existence of a limit, allows us to draw a conclusion about the convergence of a sequence, *without finding the value of its limit.*

An important example. Consider the numerical sequence, whose values are equal to the perimeter of a regular polygon inscribed in a circle of radius R, with an unlimited doubling of the number of its sides.

This sequence has a limit, since it is geometrically obvious that

— it is monotonically increasing (due to the «triangle» rule)

— and bounded above by the perimeter of a square, described around the same circle.

The limit value of this sequence is taken (by definition!) to be *the circumference,* which is denoted as $2\pi R$.

In conclusion, it is also worth noting that student folklore calls property 6° by the “two policemen” theorem (see Fig. 2).

Definition 1.1. is inconvenient for practical use because of you need to know A.

This difficulty may be overcome by the concept of fundamental sequence.

`Definition` 1.4. We will say that the sequence $\{x_n\}$ *is fundamental* if

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} \quad \text{such that}$$

$$\forall n \geq N_\varepsilon \text{ and } p \in \mathbb{N} \longrightarrow \left| x_{n+p} - x_n \right| < \varepsilon \,.$$

An important statement turns out to be true (*Cauchy Criterion*):

A sequence converges if and only if it is fundamental.

The Cauchy criterion is both a necessary and sufficient condition for convergence. Therefore, his denial also turns out to be useful:

`Definition` 1.5. The sequence $\{x_n\}$ is not *fundamental* if

$$\exists \varepsilon_0 > 0 \ : \ \forall N \in \mathbb{N} \quad \text{such that}$$

$$\forall n_0 \geq N_\varepsilon \text{ and } p_0 \in \mathbb{N} \longrightarrow \left| x_{n_0+p_0} - x_{n_0} \right| < \varepsilon_0 \,.$$

`Example` 1.4 Using the Cauchy criterion, prove the convergence of the sequence $x_n = \sum\limits_{k=1}^{n} \frac{1}{k^2}$.

`Solution:` Let us use the estimate that is correct for $m > 1$

$$\frac{1}{m^2} < \frac{1}{(m-1)m} = \frac{1}{m-1} - \frac{1}{m}.$$

We get

$$\left| x_{n+p} - x_n \right| = \sum_{k=n+1}^{n+p} \frac{1}{k^2} < \sum_{k=n+1}^{n+p} \frac{1}{k(k+1)} =$$

$$= \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \ldots + \frac{1}{n+p} - \frac{1}{n+p+1} =$$

$$= \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1} < \frac{1}{n}.$$

Repeating the reasoning of Example 1.2, we obtain that to satisfy the inequality $\frac{1}{n} < \varepsilon \;\; \forall n > N_\varepsilon$, enough to take

$$N_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1\,.$$

This proves that the sequance is fundamental, and therefore the convergence of sequence $\{x_n\}$.

`Note`: It will be shown later in the harmonic analysis course that what's in this problem $\lim\limits_{n\to+\infty} x_n = \frac{\pi^2}{6}$.

`Example` 1.5 Prove using the Cauchy criterion that the sequence called in mathematics *harmonic series*, of the form $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ diverges.

`Solution:` Let us use the negation of the Cauchy criterion. We have an assessment

$$\Big| x_{n+p} - x_n \Big| = \sum_{k=n+1}^{n+p} \frac{1}{k} =$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+p-1} + \frac{1}{n+p} >$$

$$> \frac{1}{n+p} + \frac{1}{n+p} + \ldots + \frac{1}{n+p} + \frac{1}{n+p} = \frac{p}{n+p} .$$

$\forall N \in \mathbb{N}$ we can take $n_0 = N \geq N$ and $p_0 = N$, for which $\dfrac{p_0}{n_0 + p_0} = \dfrac{1}{2} = \varepsilon_0$.

This proves that $\{x_n\}$ is non-fundamental. Hence, the sequence $\{x_n\}$ diverges.

We will say that the sequence $\{y_k\}$ is a *subsequence* for $\{x_n\}$, if $y_k = x_{n_k} \quad n_k \in \mathbb{N}$, where $\{n_k\}$ is *strictly mototonic* increasing sequence, consisting of natural numbers.

Let $\{y_k\}$ is some convergent subsequence of the numerical sequence $\{x_n\}$. Then $A = \lim\limits_{n\to\infty} y_n$ is called the *partial limit* of the sequence $\{x_n\}$. In quantifiers this definition can be written as follows:

$$\forall \varepsilon > 0 \;\; \exists N_\varepsilon \in \mathbb{N} \quad \text{such that} \quad \forall k \geq N_\varepsilon \;\longrightarrow\; \Big| x_{n_k} - A \Big| < \varepsilon\,.$$

The following statements are true for subsequences.

1) If for the sequence $\{x_n\}$ $\lim\limits_{n\to\infty} x_n = A$, then any subsequence of $\{x_n\}$ has A as its limit.

2) (*Bolzano-Weierstrass Theorem.*) If a sequence $\{x_n\}$ is bounded, then it always has a convergent subsequence.

From statement 1) it follows that if the sequence has at least two *different* partial limits, then it diverges.

Let us illustrate this fact by examining the convergence sequences with $x_n = \sin n$.

Consider a sequence of segments on the real axis of the form $\left\{ \left[\frac{\pi}{4} + 2\pi k, \ \frac{3\pi}{4} + 2\pi k \right] \right\} \quad k \in \mathbb{N}$. The length of each of them is greater than 1, and therefore this segment contains at least one natural number. In addition, these segments do not have common points.

Let us take one natural number n_k from each such segment and use them to construct the subsequence x_{n_k}. At the same time, it is obvious that $\frac{1}{\sqrt{2}} \leq x_{n_k} \leq 1$. Since this sequence is bounded, it has a partial limit A, for which $\frac{1}{\sqrt{2}} \leq A \leq 1$.

Let us now consider another sequence of segments of the form $\left\{ \left[\frac{5\pi}{4} + 2\pi m, \ \frac{7\pi}{4} + 2\pi m \right] \right\} \quad m \in \mathbb{N}$. Similarly above, we obtain the subsequence x_{n_m}, which has a partial limit B and at the same time $-1 \leq B \leq -\frac{1}{\sqrt{2}}$.

So the sequence x_n has at least two different partial limits, and therefore she disperses.

Using the properties of sequence limits, you can get some useful for solving problems, ratios.

`Example` 1.6 Prove that for $a > 1$

$$1)\ \lim_{n\to\infty} \sqrt[n]{a} = 1, \qquad 2)\ \lim_{n\to\infty} \sqrt[n]{n} = 1,$$

$$3)\ \lim_{n\to\infty} \frac{n}{a^n} = 0, \qquad 4)\ \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

`Solution.` 1) Consider the sequence $x_n = \sqrt[n]{a} - 1$. All its terms are positive and for them $a = (1 + x_n)^n \geq n x_n$ is true. Last the inequality is *Bernoulli's inequality*, which follows from Newton's binomial formula.
That's why

$$0 < x_n \leq \frac{a}{n} \longrightarrow \lim_{n\to\infty} x_n = 0 \longrightarrow \lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} (1+x_n) = 1.$$

2) All members of the sequence $x_n = \sqrt[n]{n} - 1$ are positive and for them $n = (1 + x_n)^n \geq \dfrac{n(n-1)}{2} x_n^2 istrue$. The last inequality again follows from Newton's binomial formula. Therefore, for $n \geq 2$ we have $4n \geq n^2 x_n^2$. From

$$0 \leq x_n \leq \frac{2}{\sqrt{n}} \longrightarrow \lim_{n\to\infty} x_n = 0 \longrightarrow \lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (1+x_n) = 1.$$

3) Due to $a > 1$ all terms of the sequence

$$x_n = a^n = (1 + a - 1)^n$$

are positive and for them, as in 2) is true for $n \geq 2$

$$x_n \geq \frac{n(n-1)}{2}(a-1)^2 \geq \frac{n^2}{4}(a-1)^2.$$

Therefore we have $4n \geq n^2(a-1)^2$ and then

$$0 \leq \frac{n}{a^n} \leq \frac{4}{\sqrt{n}}(a-1)^2 \qquad \longrightarrow \qquad \lim_{n\to\infty} x_n = 0\,.$$

4) Prove this equality yourself, using limit 3) and the statement that $\forall \varepsilon > 0$ and $\forall n \in \mathbb{N}$ inequality $\dfrac{\ln n}{n} < \varepsilon$ is equivalent inequality $n < (e^{\varepsilon})^n$.

In conclusion, we note that a bounded sequence can have infinitely many partial limits.

For example, it is known that the set of rational numbers on the segment $[0, 1]$ is countable, and therefore from them we can form numerical sequences. On the other hand, on this segment there are an uncountable set real numbers, in any neighborhood of each of which there are infinitely many rational numbers.

That is, every real number on a given interval is a partial limit of some sequences of rational numbers.

Calculating the limits of numerical sequences

Search for the value of the limit of numerical sequences based only on on its definition, it can turn out to be a rather complex computational procedure.

In practice it is more convenient to use *properties of limits* of sequences in combination with some small set of limits found in advance.

Within the framework of this course, combination of a set of properties 1°—6° and the following three limits

$$\lim_{n\to\infty}\frac{1}{n}=0\ ;\qquad \lim_{n\to\infty}\left(n\cdot\sin\frac{1}{n}\right)=1\qquad\text{and}\qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e\ .$$

will be quite sufficient.

The validity of the first equality was shown in example 1.1.

Consider the second equality, often called *is the first remarkable limit.*

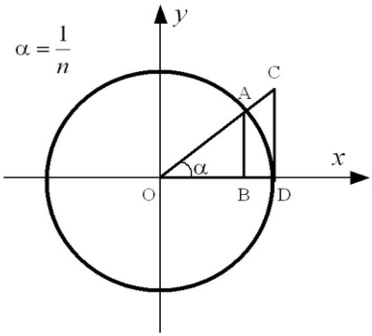

Fig.3. Towards the proof of equality $\lim\limits_{n\to\infty}\left(n\cdot\sin\dfrac{1}{n}\right)=1.$

On a trigonometric circle of unit radius we plot the angle the value of which (in radian measure) is equal to $\alpha=\dfrac{1}{n}$ (Fig. 3.3), and construct right triangles OAB and OCD.

Note that the sector OAD, on the one hand, contains the triangle OAB, and on the other hand, it itself is contained in the triangle OCD. This means that for the *areas* of these three figures the following inequalities are valid:

$$S_{\triangle \mathrm{OAB}} \le S_{\cup \mathrm{OAD}} \le S_{\triangle \mathrm{OCD}} .$$

Since $S_{\triangle \mathrm{OAB}} = \frac{1}{2} \cdot |\mathrm{OB}| \cdot |\mathrm{AB}|$, $S_{\triangle \mathrm{OCD}} = \frac{1}{2} \cdot |\mathrm{OD}| \cdot |\mathrm{CD}|$, and the area of the circular sector $S_{\cup \mathrm{OAD}} = \frac{1}{2} \cdot |\mathrm{OD}| \cdot \alpha$, then taking into account $|\mathrm{OD}| = 1$ we arrive at the inequalities

$$\frac{1}{2} \cdot \sin\alpha \cdot \cos\alpha \quad \le \quad \frac{1}{2} \cdot 1 \cdot \alpha \quad \le \quad \frac{1}{2} \cdot 1 \cdot \mathrm{tg}\,\alpha .$$

or

$$\frac{1}{2} \cdot \sin\frac{1}{n} \cdot \cos\frac{1}{n} \quad \le \quad \frac{1}{2} \cdot 1 \cdot \frac{1}{n} \quad \le \quad \frac{1}{2} \cdot 1 \cdot \mathrm{tg}\,\frac{1}{n} .$$

Transforming, we get

$$\frac{1}{\sin\frac{1}{n}\cdot\cos\frac{1}{n}} \quad\geq\quad n \quad\geq\quad \frac{\cos\frac{1}{n}}{\sin\frac{1}{n}}.$$

Whence it finally follows that

$$\frac{1}{\cos\frac{1}{n}} \quad\geq\quad n\cdot\sin\frac{1}{n} \quad\geq\quad \cos\frac{1}{n}.$$

Now you can use the properties of limits of number sequences. We will assume that

$$x_n = \frac{1}{\cos\frac{1}{n}}\,; \quad z_n = n\cdot\sin\frac{1}{n}\,; \quad y_n = \cos\frac{1}{n}\,.$$

Then, by virtue of the obvious equality $\lim\limits_{n\to\infty}\cos\frac{1}{n} = 1$ and the theorems "about two policemen— properties 6°, we get that from $x_n \geq z_n \geq y_n$ should $\lim\limits_{n\to\infty} n\cdot\sin\frac{1}{n} = 1$.

Importance of calculating the third of the above limits $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ illustrates the problem «about a good bank and a greedy depositor», having the following formulation.

Let a certain «good» bank offers its depositors 100% per annum on time deposits, with uniform accrual of interest on the deposit over time. [2]

One of his clients has a sum of money at the beginning of the year the size of one ruble, which he wants to deposit in the bank before the beginning of next year.

An obvious calculation shows that the investor will receive an amount in rubles at the end of the year: your contribution is 1 ruble plus 100%, that is, another 1 ruble. So,

$$S_1 = 1 + 1 = 2\ .$$

However, this result seems insufficient to the investor and he reasons as follows: «if I put my ruble in the first half of the year, then on June 30 I will have $\left(1+\dfrac{1}{2}\right)$ ruble, which I will put aside for the remaining six months.» Then in a year the investor will have

$$S_2 = \left(1+\frac{1}{2}\right) + \frac{1}{2}\cdot\left(1+\frac{1}{2}\right) = \left(1+\frac{1}{2}\right)^2 = 2\frac{1}{4}.$$

[2] In reality, of course, no bank has ever done or does this.

Although the effect of this operation is obvious, this is not enough for the «greedy» investor. His next reasoning is as follows: « if I put my ruble for the first four months, then by May 1 I will have in my hands $\left(1+\dfrac{1}{3}\right)$ rub, which I will deposit for the next four months and receive on September 1st

$$\left(1+\frac{1}{3}\right)+\frac{1}{3}\cdot\left(1+\frac{1}{3}\right)=\left(1+\frac{1}{3}\right)^2 .$$

Then I invest this amount for the remaining four months and receive in the end

$$S_3=\left(1+\frac{1}{3}\right)^2+\frac{1}{3}\cdot\left(1+\frac{1}{3}\right)^2=\left(1+\frac{1}{3}\right)^3=2\frac{10}{27},$$

which is greater than S_2. »

It is easy to see that if the whole year is divided into n equal periods, then the amount received will be

$$S_n=\left(1+\frac{1}{n}\right)^{n-1}+\frac{1}{n}\cdot\left(1+\frac{1}{n}\right)^{n-1}=\left(1+\frac{1}{n}\right)^{n} .$$

Let us now examine the properties of the resulting numerical sequence $S_n = \left(1 + \dfrac{1}{n}\right)^n$.

First, let us show that $\forall n : S_{n+1} > S_n$. Indeed,

$$\frac{S_{n+1}}{S_n} = \frac{\left(1 + \dfrac{1}{n+1}\right)^{n+1}}{\left(1 + \dfrac{1}{n}\right)^{n}} = \frac{\left(1 + \dfrac{1}{n+1}\right)^{n+1}}{\left(1 + \dfrac{1}{n}\right)^{n+1}} \cdot \left(1 + \frac{1}{n}\right) =$$

$$= \left(\frac{\dfrac{n+2}{n+1}}{\dfrac{n+1}{n}}\right)^{n+1} \cdot \left(1 + \frac{1}{n}\right) = \left[\frac{n(n+2)}{(n+1)^2}\right]^{n+1} \cdot \left(1 + \frac{1}{n}\right) =$$

$$= \left[\frac{n^2 + 2n}{(n+1)^2}\right]^{n+1} \cdot \left(1 + \frac{1}{n}\right) = \left[\frac{n^2 + 2n + 1 - 1}{(n+1)^2}\right]^{n+1} \cdot \left(1 + \frac{1}{n}\right) =$$

and, according to Bernoulli's inequality: $(1+x)^a > 1 + ax,\ x > -1$,

$$= \left[1 - \frac{1}{(n+1)^2}\right]^{n+1} \cdot \left(1 + \frac{1}{n}\right) > \left(1 - \frac{1}{n+1}\right) \cdot \left(1 + \frac{1}{n}\right) =$$

$$= \left(\frac{n}{n+1}\right) \cdot \left(1 + \frac{1}{n}\right) = 1.$$

Thus, $\dfrac{S_{n+1}}{S_n} > 1$ and the sequence S_n – *monotonically increasing*, that is, the investor's «nimbleness» is justified.

Nevertheless, now let's make sure that the investor will not be able to get an arbitrarily large profit. Let us perform the following estimate using Newton's binomial formula.

$$S_n = \left(1 + \frac{1}{n}\right)^n =$$

$$= 1^n + n \cdot 1^{n-1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot 1^{n-2} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot 1^{n-3} \cdot \frac{1}{n^3} + \ldots \leq$$

$$\leq 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + + \frac{1}{2^3} + \ldots \leq$$

and, according to the formula for the sum of all terms of an infinitely decreasing geometric progression, we get

$$\leq 1 + \frac{1}{1 - \dfrac{1}{2}} = 3.$$

This means that the sequence $\{S_n\}$ *is upper bounded* and no matter how much the investor fusses, he will not be able to get even three rubles.

According to the 5° property, the numerical sequence is monotonically increasing and bounded above has a limit. This means $\{S_n\}$ converges.

The limit of the numerical sequence $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ is an irrational (like π or $\sqrt{2}$) number, approximately equaled to $e \approx 2.718281828459045\ldots$ and denoted as e. In other words,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

This equality is usually called *the second remarkable limit.*

Note that the direct application of the 3° property for calculating the first remarkable limit is impossible, since of the two sequences $x_n = n$ and $y_n = \sin\dfrac{1}{n}$ only the second one converges. Its limit is 0, while the first one increases indefinitely.

Such a case is usually called *uncertainty* of the form «$0\cdot\infty$». Calculating the limit requires special research here.

Similar problems arise for uncertainties of the type «$\dfrac{0}{0}$» , «$\dfrac{\infty}{\infty}$» , «$\infty - \infty$» , «1^{∞}» .

The second remarkable limit is an example of the latter type of uncertainty.

Transformations of the formula notation of a general member of a numerical sequence in those cases when direct use properties of number sequences 1°—6° is impossible, is usually called the method of «uncertainty disclosure».

Let's consider the following problems.

Example 1.6. Find $\lim\limits_{n\to\infty}\dfrac{(3n+2)^2}{5n^2+3}$.

The formula for the value of a sequence term is a fraction, however, using the 4° property is not possible because

$$\lim_{n\to\infty}(3n+2)^2=+\infty \quad\text{and}\quad \lim_{n\to\infty}\left(5n^2+3\right)=+\infty\ .$$

That is, we have a case of uncertainty of the form «$\dfrac{\infty}{\infty}$».

For its «opening» (before moving to the limit!) transform the numerator using the formula «square of the sum of two numbers», and then divide as the numerator, so the denominator by n^2 and as a result we get

$$\lim_{n\to\infty}\frac{(3n+2)^2}{5n^2+3}=\lim_{n\to\infty}\frac{9n^2+12n+4}{5n^2+3}=\lim_{n\to\infty}\frac{9+\dfrac{12}{n}+\dfrac{4}{n^2}}{5+\dfrac{3}{n^2}}\ .$$

Now, due to $\lim\limits_{n\to\infty}\dfrac{1}{n}=0$ and $\lim\limits_{n\to\infty}\dfrac{1}{n^2}=\lim\limits_{n\to\infty}\dfrac{1}{n}\cdot\lim\limits_{n\to\infty}\dfrac{1}{n}=0$, it is obvious that the limits of the numerator and denominator exist, and we can use the properties 4°, 2° и 1° .

$$\lim_{n\to\infty}\frac{9+\dfrac{12}{n}+\dfrac{4}{n^2}}{5+\dfrac{3}{n^2}}=\frac{\lim\limits_{n\to\infty}\left(9+\dfrac{12}{n}+\dfrac{4}{n^2}\right)}{\lim\limits_{n\to\infty}\left(5+\dfrac{3}{n^2}\right)}=\frac{9+12\lim\limits_{n\to\infty}\dfrac{1}{n}+4\lim\limits_{n\to\infty}\dfrac{1}{n^2}}{5+3\lim\limits_{n\to\infty}\dfrac{1}{n^2}}=\frac{9}{5}\ .$$

`Example` 1.7. Find $\lim\limits_{n\to\infty}\left(\sqrt{4n^2+3n}-2n\right).$

Formula for the general term of the sequence $a_n = \sqrt{4n^2+3n} - 2n$ represents the difference of two expressions. However we cannot use the 1° property, because

$$\lim_{n\to\infty}\sqrt{4n^2+3n} = +\infty \quad \text{and} \quad \lim_{n\to\infty} 2n = +\infty$$

and we are dealing with uncertainty of the form «$\infty - \infty$». To «reveal» it let's multiply and at the same time divide this difference for the amount of $\sqrt{4n^2+3}+2n$, followed by using the formula $(a-b)(a+b) = a^2 - b^2$.

$$\lim_{n\to\infty}\left(\sqrt{4n^2+3n}-2n\right) = \lim_{n\to\infty}\frac{(\sqrt{4n^2+3n}-2n)(\sqrt{4n^2+3n}+2n)}{\sqrt{4n^2+3n}+2n} =$$

$$= \lim_{n\to\infty}\frac{(4n^2+3n)-4n^2}{\sqrt{4n^2+3n}+2n} = \lim_{n\to\infty}\frac{3n}{\sqrt{4n^2+3n}+2n}.$$

The result is an uncertainty of the form «$\dfrac{\infty}{\infty}$», which we «reveal» by dividing the numerator and denominator by n.

$$\lim_{n\to\infty}\frac{3n}{\sqrt{4n^2+3n}+2n} = 3\cdot\lim_{n\to\infty}\frac{1}{\sqrt{4+\dfrac{3}{n}}+2} = \frac{3}{4}\,,$$

since $\displaystyle\lim_{n\to\infty}\sqrt{4+\frac{3}{n}} = 2.$

Indeed, on the one hand, $\sqrt{4+\dfrac{3}{n}} \geq 2$, but, on the other hand

$$\sqrt{4+\frac{3}{n}} = \sqrt{4+2\cdot 2\cdot\frac{3}{4n}+\frac{9}{16n^2}-\frac{9}{16n^2}} =$$

$$= \sqrt{\left(2+\frac{3}{4n}\right)^2-\frac{9}{16n^2}} \leq \sqrt{\left(2+\frac{3}{4n}\right)^2} = 2+\frac{3}{4n}.$$

That is,

$$2 \leq \sqrt{4+\frac{3}{n}} \leq 2+\frac{3}{4n}\,,$$

and based on the “two policemen” theorem, we come to the conclusion that $\displaystyle\lim_{n\to\infty}\sqrt{4+\frac{3}{n}} = 2$.

`Example` 1.8. Find $\lim\limits_{n\to\infty}\left(\dfrac{n+2}{n+1}\right)^{3n}$.

Here we have $\lim\limits_{n\to\infty}\dfrac{n+2}{n+1} = 1$ and $\lim\limits_{n\to\infty} 3n = \infty$, that is, an uncertainty of the form «1^∞».

To «expand» it, we transform the expression under the limit sign as follows.

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{3n} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{3n} = \lim_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^{k-1}\right]^{3} =$$

where $k = n+1$ and $n = k-1 \implies \lim\limits_{n\to\infty} k = \infty$

$$= \left[\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k}\right]^{3}\cdot\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{-3} = e^3\cdot 1 = e^3.$$

`Example` 1.9. Find the limit of the sequence given in the following recurrence form

$$x_1 = 1, \quad x_{n+1} = \sqrt{x_n + 6}\,.$$

`Solution.` All terms of this sequence are obviously non-negative.

Note also that if $x_n < 3$, then $x_{n+1} < 3$. But, since $x_1 = 1 < 3$, then all members of the sequence do not exceed 3. That is, the sequence is bounded above.

On the other hand, we have the estimate at $x_n < 3$

$$x_{n+1} - x_n = \sqrt{x_n + 6} - x_n = \frac{x_n + 6 - x_n^2}{\sqrt{x_n + 6} + x_n} > 0\,.$$

This means that this sequence increases monotonically. Then it converges according to the 5° property.

Let $\lim\limits_{n\to\infty} x_n = A > 0$. Then from

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = A$$

and the conditions of the problem we have $A^2 = A + 6$. Whence it follows that $A = 3$.