Function of one variable

Let us recall the definition of a function as a special way of describing the relationship between variable quantities.

Definition 2.1. We will say that *functional dependence* or, simply *function*, is given if a **rule** is specified according to which **each** number x, belonging to the number set X, is assigned a **singular** number y, belonging to the numerical set Y.

The set X is usually called the *domain of definition* of a function, and the set Y is called *the domain of its values*. The function itself is usually denoted

$$y = f(x),\ x \in X,\ y \in Y.$$

Finally, x is the independent variable, called *the argument*, and y is the dependent variable, *the value of the function* or, simply, *function*.

Quite often, a function is specified only by a formula.

In this case *is assumed* that the area definition is a set of numbers for which *all* operations used in writing this formula are feasible.

For the range of values in this case a set of numbers is accepted that are the values of the function corresponding to *all possible* values of the argument.

Let us give examples of the use of this definition.

`Example` 2.1. 1) $y = x^2 + \dfrac{1}{x}$.

Domain of definition: obviously $X : \{\forall x \neq 0\}$.

Range of values: notice, that in a small neighborhood of zero the value of the function under study differ little from the function values $y = \dfrac{1}{x}$. Therefore, in the range of Y there are all real numbers, except maybe zero. Moreover, it is also obvious: $y = 0$ at $x = -1$. This means that Y is $\mathbb{R}$.

Sometimes it is useful to estimate a set of values function under study on some subset of the domain of definition. For example, for this function for $x > 0$ it is true Cauchy's inequality

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}, \qquad a, b, c > 0\,.$$

Hence

$$x^2 + \frac{1}{x} = x^2 + \frac{1}{2x} + \frac{1}{2x} \geq 3\sqrt[3]{x^2 \cdot \frac{1}{2x} \cdot \frac{1}{2x}} = \frac{3}{\sqrt[3]{4}}\,.$$

2) $y = \dfrac{x}{x^2 + x + 1}$

Domain of definition: found from the condition

$$x^2 + x + 1 \neq 0 \quad \Longleftrightarrow \quad X \equiv \mathbb{R}.$$

Range of values: the range of a given function can be found using the inequality

$$\left| x + \frac{1}{x} \right| \geq 2\,.$$

Indeed, we have $y(0) = 0$, and for $x \neq 0$

$$y = \frac{x}{x^2 + x + 1} = \frac{1}{x + \dfrac{1}{x} + 1}\,.$$

Where does $-1 \leq y \leq \dfrac{1}{3}$.

Let us recall that functions are usually classified according to the presence or absence of the property *periodicity* and the property *parity*.

`Definition` 2.2. Function $y = f(x)$ will be called *periodic* if there is a number $T > 0$ such that for any $x \in X$ done $x \pm T \in X$ and

$$f(x+T) = f(x).$$

The minimum of these numbers T is called the *period* of the function $y = f(x)$.

`Example` 2.2. Periodic functions include the following:

$$\begin{array}{lll} y = \cos x & \text{with period } T = & 2\pi, \\ y = e^{\sin 3\pi x} & \text{with period } T = & \frac{2}{3}, \\ y = \operatorname{ctg} x & \text{with period } T = & \pi, \\ y = \arcsin \sin \pi x & \text{with period } T = & 1. \end{array}$$

`Definition 2.3.` Let X be the domain of definition of the function $y = f(x)$, symmetric with respect to the point $x = 0$, then this function is called:

even, if $\forall x \in X : \quad f(-x) = f(x)$,
odd, if $\forall x \in X : \quad f(-x) = -f(x)$,

`Example` 2.3. Classification of functions by parity:

$y = |x|$ – even,
$y = \sqrt[3]{x}$ – odd,
$y = \operatorname{tg} x$ – odd,
$y = \cos x$ – even,
$y = 3^x$ – is neither even nor odd.

There are functions that are neither even nor odd. In a symmetric domain of definition each of them can be represented as the sum of some even function and some odd one. To do this you can use, for example, formula

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}\,.$$

Thus, for the function $y = 3^x$, the expansion into the sum of even and odd will have the form

$$y = \frac{3^x + 3^{-x}}{2} + \frac{3^x - 3^{-x}}{2}\,.$$

Limit of a function and its properties

Let us consider the values of some function $f(x)$ in a small neighborhood of the point $x = a$. In a large number of cases, you will notice that these values turn out to be closer to a certain number A, the less the value x differs from a. This number A may not be equaled to the value of $f(a)$. It can exist even in cases where the point a does not belong to the domain of definition of the function $f(x)$.

An example is the function $f(x) = \dfrac{\sin x}{x}$, which has no value at the point $x = 0$, but is defined in any punctured neighborhood of it. You may notice that value of $f(x) = \dfrac{\sin x}{x}$ differs less from 1 the smaller the absolute value of x.

If such a number A exists, it is called *the limit of the function* $f(x)$ *as* x *tends to* a. This fact is symbolically designated as

$$\lim_{x \to a} f(x) = A.$$

Let us now give strict definitions for the concept of the limit of a function.

`Definition` 2.4. The number A is called *the limit of the function* $f(x)$ *as* x *tends to* a, if *for any* numerical sequence $\{x_n\}$ is such that $\lim\limits_{n\to\infty} x_n = a$ and $x_n \neq a$, numerical sequence $\{f(x_n)\}$ *converges* to A, that is, the equality $\lim\limits_{n\to\infty} f(x_n) = A$ holds

In symbolic form, this definition can be written as follows: $\lim\limits_{x\to a} f(x) = A$, if $\forall\{x_n\} \underset{n\to\infty}{\longrightarrow} a \ : \ \{f(x_n)\} \underset{n\to\infty}{\longrightarrow} A.$

or

`Definition` 2.5. The number A is called *the limit of the function* $f(x)$ *as* x *tends to* a, if *for any* $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that for *all* x satisfying the inequality $0 < |x - a| < \delta_\varepsilon$ fair $|\,f(x) - A\,| < \varepsilon$,

which is symbolically written like this:

$$\forall\varepsilon > 0 \ \ \exists\delta_\varepsilon > 0 : \quad \forall x : \ 0 < |x - a| < \delta_\varepsilon \quad \longrightarrow \quad |\,f(x) - A\,| < \varepsilon.$$

Definition 2.4 is usually called *by determining the limit of a function according to Heine*, and Definition 2.5 — *Cauchy definition.* There is a theorem stating *the equivalence* of these definitions.

The *negations* of Definitions 2.4 and 2.5 also play an important role.

Denial 2.1. The number A is not *the limit of the function $f(x)$ as x tends to a*, if *there is* a numerical sequence $\{x_n^*\}$ such that $\lim\limits_{n\to\infty} x_n^* = a$ and $x_n^* \neq a$, and the numerical sequence $\{f(x_n^*)\}$ *does not converge* to A, that is, the condition $\lim\limits_{n\to\infty} f(x_n^*) \neq A$. is satisfied

In symbolic form, this definition can be written as follows: $\lim\limits_{x\to a} f(x) \neq A$, if $\exists\{x_n^*\} \underset{n\to\infty}{\longrightarrow} a \ : \ \lim\limits_{n\to\infty} f(x_n^*) \neq A.$

similarly constructed

Denial 2.2. The number A is not *the limit of the function $f(x)$ as x tends to a*, if *there is* $\varepsilon_0 > 0$ such that for any $\delta > 0$ *exists* x_0 satisfying the condition $0 < |x_0 - a| < \delta$, at which $|\,f(x_0) - A\,| \geq \varepsilon_0$,

which is symbolically written like this:

$$\exists\varepsilon_0 > 0 \ \forall\delta > 0 : \quad \exists x_0 : \ 0 < |x_0 - a| < \delta \quad \longrightarrow \quad |\,f(x_0) - A\,| \geq \varepsilon_0.$$

In order to make sure, that the limit of the function $f(x)$ as x tends to a does not exist, it is enough to find one sequence $\lim\limits_{n\to\infty} x_n^* = a$ for which $\lim\limits_{n\to\infty} f(x_n^*)$ does not exist.

Or construct two converging sequences $\lim\limits_{n\to\infty} x_n^* = a$ and $\lim\limits_{n\to\infty} y_n^* = a$ such that $\lim\limits_{n\to\infty} f(x_n^*) \neq \lim\limits_{n\to\infty} f(y_n^*)$.

In computing practice, there are often cases when the passage to the limit to a point a is performed only at its *left* or only at *right* semi-neighborhood. Such limits are called *one-sided*. The first case is usually denoted as $x \to a - 0$, and the second case is $x \to a + 0$.

In addition, if the value of the function when passing to the limit there remains more than A (*tendency from above*), then the limit value is denoted as $A + 0$, and, if less than A (*tendency from below*), then as $A - 0$.

Finally, we note that the definitions of limit apply not only when a denotes some finite number, but can also be modified for cases in which a is one of the following three characters ∞ , $+\infty$ or $-\infty$. It is also possible that A is not a finite number, and takes one of the symbolic meanings ∞ , $+\infty$ или $-\infty$.

The following table provides examples of notation various types of limit passages and their formulations in Cauchy quantifiers.

Designation	Formulation
$\lim\limits_{x\to 3+0} f(x) = 7$	$\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0 :\ \forall x :\ 0 < x-3 < \delta_\varepsilon \longrightarrow \lvert f(x) - 7 \rvert < \varepsilon$
$\lim\limits_{x\to +\infty} f(x) = 5-0$	$\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0 :\ \forall x :\ x > \delta_\varepsilon \longrightarrow -\varepsilon < f(x) - 5 < 0$
$\lim\limits_{x\to 4-0} f(x) = -\infty$	$\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0 :\ \forall x :\ -\delta_\varepsilon < x-4 < 0 \longrightarrow f(x) < -\varepsilon$
$\lim\limits_{x\to 2} f(x) \neq -1$	$\exists\varepsilon_0 > 0\ \forall\delta > 0 :\ \exists x_0 :\ 0 < \lvert x_0 - 2\rvert < \delta \longrightarrow \lvert f(x_0) + 1 \rvert \geq \varepsilon_0$
$\lim\limits_{x\to -\infty} f(x) \neq \infty$	$\exists\varepsilon_0 > 0\ \forall\delta > 0 :\ \exists x_0 :\ x_0 < -\delta \longrightarrow \lvert f(x_0) \rvert < \varepsilon_0$

We emphasize: the limit of a function as x tends to a is a *local* numerical characteristic of the function (same as $f(a)$ — its value), that is, related to the point a.

For the same point, the value of the function and the value of its limit, as well as their existence, independent of each other. They can exist simultaneously and be equal or unequal to each other. They may also not exist, both together and separately.

Let us explain this with the following examples.

`Example` 2.4. Find, using Definition 2.4, $\lim\limits_{x\to 3}\dfrac{2x}{x^2+1}$.

Let us take an numerical sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_n = 3$, then the n-th term of the sequence $\{f(x_n)\}$ will have the form $\dfrac{2x_n}{x_n^2+1}$.

Let's find its limit using the properties of limits of numerical sequences known to us.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\frac{2x_n}{x_n^2+1} = \frac{\lim\limits_{n\to\infty} 2x_n}{\lim\limits_{n\to\infty}(x_n)^2+1} =$$

$$= \frac{2\cdot\lim\limits_{n\to\infty} x_n}{(\lim\limits_{n\to\infty} x_n)\cdot(\lim\limits_{n\to\infty} x_n)+1} = \frac{2\cdot 3}{3\cdot 3+1} = \frac{3}{5}\,.$$

It is easy to see that the function $f(x) = \dfrac{2x}{x^2+1}$ is defined (has a value) for any finite x, including for $x = 3$.

In this case, due to $f(3) = \dfrac{2\cdot 3}{3^2+1} = \dfrac{3}{5}$, the value of the function and its limit at the point $x = 3$ coincide.

However, if we consider the behavior of the same function as x tends to ∞, then we get a different case.

On the one hand, the function $f(x) = \dfrac{2x}{x^2+1}$ is undefined, that is, it has no value at $x = \infty$. But, on the other hand, for $\lim\limits_{n\to\infty} x_n = \infty$,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{2x_n}{x_n^2+1} = \lim_{n\to\infty} \frac{1}{x_n} \cdot \frac{2}{1+\dfrac{1}{x_n^2}} =$$

$$= \lim_{n\to\infty} \frac{1}{x_n} \cdot \lim_{n\to\infty} \frac{2}{1+\dfrac{1}{x_n^2}} = 0 \cdot 2 = 0\ .$$

So, this function at $x \longrightarrow \infty$ has a limit equal to 0, that is, $f(x) \longrightarrow 0$, but $f(x)$ doesn't have a value.

Example 2.5. Consider a function called *number signature,* denoted as $y = \operatorname{sgn} x$ and defined by the formula (see Fig. 1)

$$\operatorname{sgn} x = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

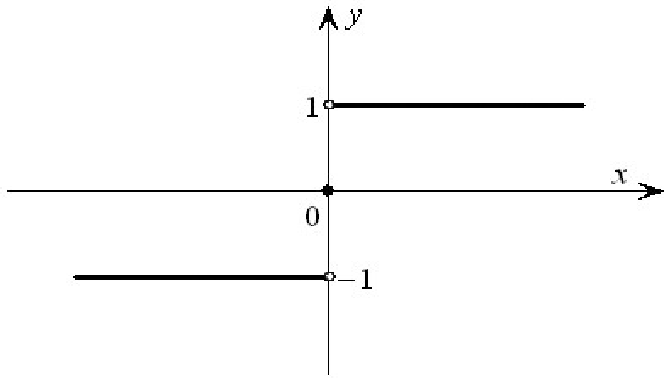

Fig.1. Graph of the function $y = \operatorname{sgn} x$

This function at $x = 0$ has a zero value, that is, $\operatorname{sgn} 0 = 0$, but the limit $\lim\limits_{x\to 0} \operatorname{sgn} x$ does not exist.

Indeed, let's take two number sequences with $x_n = \dfrac{1}{n}$ and $y_n = -\dfrac{1}{n}$. They both have the number 0 as their limit. But, by definition of the signature,

$$\lim_{n\to\infty} \operatorname{sgn} x_n = \lim_{n\to\infty} \operatorname{sgn}\left(\frac{1}{n}\right) = 1 \text{ and } \lim_{n\to\infty} \operatorname{sgn} y_n = \lim_{n\to\infty} \operatorname{sgn}\left(-\frac{1}{n}\right) = -1.$$

This contradicts Definition 2.4, since the value of the limit $\operatorname{sgn} x_n$ must be *the same for everyone* sequences $\{x_n\} \underset{n\to\infty}{\longrightarrow} 0$.

Finally, it is possible that the function at some point has unequal to each other, both meaning and limit. An example would be function $y = \operatorname{sgn}^2 x$, whose value at zero is 0 and the limit is 1.

It is also possible that the function at some point there is no value and there is no limit. Using negation of the definition according to Heine, we will make sure that such a situation occurs for $f(x) = \sin\dfrac{1}{x}$ at $x = 0$.

Indeed, let us consider two sequences $\{x_n\}$ and $\{y_n\}$, whose

$$x_n = \frac{1}{2\pi n} \qquad \text{and} \qquad y_n = \frac{1}{\dfrac{\pi}{2} + 2\pi n} \qquad n \in \mathbb{N}.$$

Both of these sequences are infinitesimal. For them the limits are $\lim\limits_{n\to\infty} f(x_n) = 0$ and $\lim\limits_{n\to\infty} f(y_n) = 1$. This means that $\lim\limits_{x\to 0} f(x)$ does not exist due to negating definition of the limit according to Heine.

The properties of function limit are similar to the properties of sequence limit.

Let us present the main ones, assuming that used in the formulations limits exist.

1°. $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$.

2°. $\lim\limits_{x\to a}(C \cdot f(x)) = C \cdot \lim\limits_{x\to a} f(x)$.

3°. $\lim\limits_{x\to a}(f(x) \cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$.

4°. If, in addition, $\lim\limits_{x\to a} g(x) \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$.

5°. An analogue of the theorem «about two policemen» if for all x, belonging to some neighborhood of the point a, $f(x) \geq h(x) \geq g(x)$ is true and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = A$, then $\lim\limits_{x\to a} h(x) = A$.

Application of function limit definitions 2.4 and 2.5 (same as in the case of numerical sequences) complicated by the fact that they use the value of the limit A.

There is necessary and at the same time sufficient condition of existence of a finite function limit, which does not have this drawback. It is called *Cauchy criterion* and formulated in the form of the following theorem.

Cauchy criterion. In order for the function $f(x)$ to have at the point a finite limit, it is necessary and sufficient that for any positive ε there existed a positive number δ_ε is such that for any $x_1 \in (a-\delta_\varepsilon, a+\delta_\varepsilon)$ and any $x_2 \in (a-\delta_\varepsilon, a+\delta_\varepsilon)$, the inequality $|\, f(x_1) - f(x_2)\,| < \varepsilon$ to be valid.

In quantifiers, the Cauchy criterion has the formulation

$$\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 : \forall\, x_1 \in (a-\delta_\varepsilon, a+\delta_\varepsilon) \text{ and } x_2 \in (a-\delta_\varepsilon, a+\delta_\varepsilon) \quad \longrightarrow$$

$$\longrightarrow \quad |\, f(x_1) - f(x_2)\,| < \varepsilon\,.$$

Accordingly, the negation of the Cauchy criterion can be formulated like this:

$$\exists \varepsilon_0 > 0 \; \forall \delta > 0 : \exists\, x_{10} \in (a-\delta, a+\delta) \text{ and } x_{20} \in (a-\delta, a+\delta) \quad \longrightarrow$$

$$\longrightarrow \quad |\, f(x_{10}) - f(x_{20})\,| \geq \varepsilon_0\,.$$

`Example` 2.6. Using the negation of the Cauchy criterion, prove that that $y = \sin\dfrac{1}{x}$ has no finite limit at point $x = 0$.

Really,

$$\exists\, \varepsilon_0 = 1\ \forall \delta > 0 : \text{at} \quad k \geq \left[\frac{1}{2\pi\delta}\right] + 1$$

$$\exists \quad x_{10} = \frac{1}{2\pi k} \in (-\delta, \delta) \quad \text{и} \quad x_{20} = \frac{1}{\dfrac{\pi}{2} + 2\pi k} \in (-\delta, \delta) \quad \longrightarrow$$

$$\longrightarrow \quad |\, f(x_{20}) - f(x_{10})\, | = |\, 1 - 0\, | = 1 = \varepsilon_0\, .$$

The concept of the limit of a function can be used for constructions approximating of other functions.

For example, if we are interested in the properties of a function only in a small neighborhood of a certain point, then a convenient approximation tool is special function $o(x)$, called «o-small».

`Definition` 2.6. The function $o(x)$ is defined in a certain neighborhood of a point $a = 0$, such as the equality $\lim\limits_{x\to 0} \frac{o(x)}{x} = 0$ is valid.

Note that the form of the function $o(x)$ does not play any role here.

Below are examples of constructions using $o(x)$ for approximation of some elementary functions.

$$\begin{array}{rcl} (1+x)^r & = & 1 + rx + o(x), \\ e^x & = & 1 + x + o(x), \\ \sin x & = & 1 + x + o(x^2), \\ \cos x & = & 1 - \frac{1}{2}\,x^2 + o(x^3), \\ \operatorname{tg} x & = & x + o(x^2), \\ \ln(1+x) & = & x + o(x), \\ \arcsin x & = & x + o(x^2). \end{array}$$

`Definition 2.7.` The function $f(x)$ is called *equivalent* at point x_0 to function $g(x)$, If $\lim\limits_{x\to x_0}\left|\dfrac{f(x)}{g(x)}\right| = 1$. Equivalence is usually denoted as follows: $f(x) \sim g(x)$.

A function $f(x)$ is said to be $O(g(x))$ at point x_0, if there is a number $C > 0$ such that $\lim\limits_{x\to x_0}\left|\dfrac{f(x)}{g(x)}\right| \leq C$.

For the equivalence of the functions $f(x)$ and $g(x)$, for example, at point x_0 it is necessary and sufficient, so that $f(x) = g(x) + o(x)$ for $x \to x_0$.

When calculating limits, equivalent functions you can replace one with the other.

Finally, the so-called «remarkable limits» of functions turn out to be useful for solving a lot of problems. These limits are:

1) *First* remarkable limit: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

2) *Second* remarkable limit:

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \text{or} \qquad \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \; .$$

Finding the limits of functions.
Disclosure of Uncertainties

To search for limits of functions, we can use the combination of the properties of limits and a set of «remarkable limits». In some cases it allows to perform «uncertainty disclosure»:

$$«0\cdot\infty» \ , \quad «\frac{0}{0}» \ , \quad «\frac{\infty}{\infty}» \ , \quad «\infty-\infty» \ , \quad «1^{\infty}» \ .$$

We will demonstrate the corresponding techniques using the following examples.

`Example` 2.7. Find $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^3-8}$.

In this problem it is necessary to reveal uncertainty of the form $«\frac{0}{0}»$.

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^3-8}=\lim_{x\to 2}\frac{(x-1)(x-2)}{(x-2)(x^2+2x+4)}=\lim_{x\to 2}\frac{x-1}{x^2+2x+4}=\frac{1}{12}\,.$$

`Example` 2.8 Find $\lim\limits_{x\to\infty} \dfrac{3x^2+2}{(2x+1)^2}$.

Here there is uncertainty of the form «$\dfrac{\infty}{\infty}$».

$$\lim_{x\to\infty} \frac{3x^2+2}{(2x+1)^2} = \lim_{x\to\infty} \frac{3x^2+2}{4x^2+4x+1} = \lim_{x\to\infty} \frac{3+\dfrac{2}{x^2}}{4+\dfrac{4}{x}+\dfrac{1}{x^2}} = \frac{3}{4}\,.$$

`Example` 2.9. Find $\lim\limits_{x\to+\infty} \left(\sqrt{x^2+3x}-x\right)$.

The type of uncertainty in this example is «$\infty-\infty$».

$$\lim_{x\to+\infty} \left(\sqrt{x^2+3x}-x\right) = \lim_{x\to+\infty} \frac{\sqrt{x^2+3x}-x}{1} =$$

$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{(\ sqrt x^2+3x+x)} =$$

$$= \lim_{x\to+\infty} \frac{x^2+3x-x^2}{(\sqrt{x^2+3x}+x)} = \lim_{x\to+\infty} \frac{3x}{(\sqrt{x^2+3x}+x)} =$$

now we are dealing with uncertainty of the form «$\dfrac{\infty}{\infty}$» — divide the numerator and denominator by x :

$$= \lim_{x\to+\infty} \frac{3}{\sqrt{1+\dfrac{3}{x}}+1} = \frac{3}{2}\,.$$

In some cases, in order to use the values of «remarkable» limits, it turns out to be advisable to perform a variable substitution.

Example 2.10. Find $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$.

To reveal uncertainty of the form «$\dfrac{0}{0}$» in this problem it is convenient to introduce a new variable $t = 5x$, which will obviously tend to zero as x tends to zero. Therefore, by virtue of the first “remarkable” limit

$$\lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{t\to 0}\frac{\sin t}{\left(\dfrac{t}{5}\right)} = \lim_{t\to 0} 5\cdot\frac{\sin t}{t} = 5\cdot\lim_{t\to 0}\frac{\sin t}{t} = 5\cdot 1 = 5\ .$$

`Example` 2.11. Find $\lim\limits_{x\to\infty}\left(1-\dfrac{3}{x}\right)^{x}$.

This problem leads to the need to disclose uncertainty of type «1^{∞}». Let's perform a variable change by putting $t = -\dfrac{x}{3} \Rightarrow x = -3t$. Substituting, we get

`Example` 2.11. Find $\lim\limits_{x\to\infty}\left(1-\dfrac{3}{x}\right)^{x}$.

This problem leads to the need to disclose uncertainty of type «1^{∞}». Let's perform a variable change by putting $t = -\dfrac{x}{3} \Rightarrow x = -3t$. Substituting, we get

$$\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{x} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{-3t} = \lim_{t\to\infty}\left(\left(1+\frac{1}{t}\right)^{t}\right)^{-3} =$$

$$= \left(\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\right)^{-3} = e^{-3} = \frac{1}{e^{3}}.$$

Continuity of a function at a point and on a set. Classification of break points

`Definition` 2.8. The function $f(x)$ is called *continuous at a point in domain of definition* $x = a$, if

$$\lim_{x\to a} f(x) = f(a).$$

If this condition is not satisfied, then the function $f(x)$ is said to have *discontinuity at the point* $x = a$.

If the function $f(x)$ is continuous at a point $x = a$, then for any number sequence $x_n \longrightarrow a$ equality is true

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(a).$$

Therefore, if $f(x)$ is continuous at the point $x = g(a)$, where $g(x)$ is another function continuous at the point $x = a$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(g(a)), \tag{2.1}$$

`Definition` 2.9. A function $f(x)$ is called *continuous on some numerical set* if it is continuous at *every* point of this set. A function $f(x)$ is called *discontinuous on some number set* if it is not continuous *at least one* of the points of this set.

`Example` 2.12. Function $f(x) = \sin x$ is continuous on $X\ :\ (-\infty, +\infty)$, and function $f(x) = \dfrac{1}{x}$ is continuous on $X\ :\ (-\infty, 0)$ or on $X\ :\ (0, +\infty)$. But $f(x) = \dfrac{1}{x}$ is discontinuous on $X\ :\ ((-\infty, +\infty))$.

`Definition` 2.10. Function $f(x)$ is said to have a *removable discontinuity* at a point $x = a$, if $\lim\limits_{x\to a} f(x)$ exists, but $\lim\limits_{x\to a} f(x) \neq f(a)$. If $\lim\limits_{x\to a} f(x)$ does not exist when the point $x = a$ belongs to the area definition of the function $f(x)$, then the point $x = a$ is called the *irremovable discontinuity point* of the function $f(x)$.

`Example` 2.11. Examine for continuity and classify its function break points:

1) $y = \operatorname{sgn} x$.

For this function, the discontinuity point $x = 0$ is irremovable, since the limit does not exist at this point (see Fig. 1).

2) $y = |\operatorname{sgn} x|$.

For this function, the discontinuity point $x = 0$ is removable, since the limit at this point exists, $\lim\limits_{x\to 0} |\operatorname{sgn} x| = 1$, but not equal to the function value: $|\operatorname{sgn} 0| = 0 \neq 1$.

3)

$$y = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ a \quad , & \text{if } x = 0. \end{cases}$$

This function is continuous $\forall x$, if $a = 1$, since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and has a removable discontinuity point at $x = 0 \ \forall a \neq 1$.

4)

$$y = \frac{\sin(x-1)}{x^2-3x+2}$$

This function has in $x = 1$ removable break point, and in $x = 2$ — irremovable.

Indeed, if we transform the notation of this function to the form $y = \dfrac{\sin(x-1)}{x-1} \cdot \dfrac{1}{x-2}$, then at $x = 1$ we have

$$\lim_{x\to 1} \frac{\sin(x-1)}{x-1} \cdot \frac{1}{x-2} = \lim_{x\to 1} \frac{\sin(x-1)}{x-1} \cdot \lim_{x\to 1} \frac{1}{x-2} = 1 \cdot \frac{1}{-1} = -1.$$

While at the point $x = 2$, although $\lim\limits_{x\to 2} \dfrac{\sin(x-1)}{x-1} = \sin 1$, but $\lim\limits_{x\to 2} \dfrac{1}{x-2}$ does not exist, and we have a gap of an irremovable type.

Using the property of continuity in many cases simplifies the procedure for disclosing uncertainties when finding limits of functions.

`Example` 2.13. Find $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$.

This task leads to uncertainty of the type «$\dfrac{0}{0}$». To expand it, we perform the following transformations

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\left(\ln(1+x)^{\dfrac{1}{x}}\right)=\ln\left(\lim_{x\to 0}(1+x)^{\dfrac{1}{x}}\right)=\ln e=1\ .$$

Here we have used the continuity of the logarithmic function and the «second remarkable limit».

Properties of functions continuous on an interval

If the function $f(x)$ is continuous at every point (a, b) and has values at the ends of the segment $[a, b]$, coinciding with one-sided limits, then

1) $f(x)$ is bounded on $[a, b]$.

2) There exist $x_1 \in [a, b]$ and $x_2 \in [a, b]$ such that

$$f(x_1) = \sup_{x\in[a,b]} f(x) \qquad f(x_2) = \inf_{x\in[a,b]} f(x)\,.$$

3) Let $f(a) = A$ and $f(b) = B$. Then $\forall C \in [A, B]\ \exists c \in [a, b]$ such, that $f(c) = C$.

4) If a function $f(x)$ is strictly monotone on $[a, b]$, then it has the inverse function, which is also a strictly monotonic one.

Note that these properties may not hold for functions continuous on an interval. For example, $f(x) = \dfrac{1}{x}$ is continuous on the interval $(0, 1)$, but it is unbounded on this interval.

Function defined on a segment and having a corresponding set of values in the form of a segment, may not be continuous. This is, for example,

$$f(x) = \begin{cases} x^2 & \text{at} \quad x \in [-1, 0], \\ x + 1 & \text{at} \quad x \in \quad (0, 1]. \end{cases}$$