Derivative of a function at a point

The value of a function and its limit are local numerical characteristics that make it possible to quantitatively describe the function both at a certain point and in its small vicinity.

However, these characteristics are not enough when it is necessary to evaluate not only the function values themselves, but also the relative magnitude of their change. To do this, a special quantitative method is used. This characteristic of a function is called *derivative of a function at a point.* Let's give its definition.

`Definition 3.1.` *The derivative of a function at a point* is the limit of the ratio of the increment in the value of the function to the increment of its argument, when the last increment tends to zero.

In other words, for a function $y = f(x)$ its derivative at the point x_0, is denoted as $f'(x_0)$ or $y'_x(x_0)$, is equal to

$$f'(x_0) = \lim_{t\to 0} \frac{f(x_0 + t) - f(x_0)}{t}\,. \tag{3.1}$$

Indeed, if the value of the argument was x_0 , and became $x_0 + t$, then its increment is obviously equal to $\Delta x = (x_0 + t) - x_0 = t$. Similarly, if the function value was $f(x_0)$ and it became $f(x_0 + t)$, then its corresponding increment is $\Delta f = f(x_0 + t) - f(x_0)$.

From this definition it follows that the function $y = f(x)$ must have values in some neighborhood of the point x_0, and also be *continuous* at point x_0.

The last condition is necessary (but not sufficient!) for the existence of a derivative of a function at a point, since only for continuous functions the limit of the increment of the value of the function is equal to zero as the increment of the argument tends to zero.

However, even for a continuous function the limit (3.1) is an uncertainty of the form «$\dfrac{0}{0}$». That is, a conclusion about the existence (or non-existence) of a derivative at a point can be made only after the «disclosure» of this uncertainty.

Let us explain Definition 3.1 with the following examples.

`Example` 3.1. Find the derivative of the function $y = x^3$ at the point $x_0 = 2$.

First, let's solve this problem for an arbitrary fixed point x_0 . Let the increment of the argument at point x_0 be equal to t, Let's find the corresponding increment in the value of this function using the formula «cube of the sum of two numbers»,

$$\Delta y = f(x_0{+}t){-}f(x_0) =$$

$$= (x_0 + t)^3 - x_0^3 = \left(x_0^3 + 3x_0^2 t + 3x_0 t^2 + t^3\right) - x_0^3 = 3x_0^2 t + 3x_0 t^2 + t^3 \ .$$

Then

$$f'\left(x_0\right) = \lim_{t\to 0} \frac{3x_0^2 t + 3x_0 t^2 + t^3}{t} = \lim_{t\to 0} \left(3x_0^2 + 3x_0 t + t^2\right) = 3x_0^2 \ .$$

Substituting $x_0 = 2$ into the resulting expression, we find that the desired value of $y'_x(2)$ for the function $y = x^3$ at the point 2 is 12.

`Example` 3.2. Find the derivative of function $y = \sqrt{x+1}$ at the point $x_0 = 3$.

As in the previous example, we first solve this problem for an arbitrary fixed point x_0 . Let the increment of the argument at point x_0 be equal to t. Let's find the corresponding increment in the value of function

$$\Delta y = f(x_0 + t) - f(x_0) = \sqrt{x_0 + t + 1} - \sqrt{x_0 + 1}\ .$$

Then, after multiplying by the conjugate expression, we get

$$f'\,(x_0) = \lim_{t\to 0} \frac{\sqrt{x_0 + t + 1} - \sqrt{x_0 + 1}}{t} =$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{x_0 + t + 1} + \sqrt{x_0 + 1}} = \frac{1}{2\sqrt{x_0 + 1}}\ .$$

Substituting $x_0 = 3$, into the resulting expression we find, that the desired value is $y'_x(3)$ for the function $y = \sqrt{x+1}$ at point 3 will be equal to $\frac{1}{4}$.

`Example` 3.3. Find the derivative of function $y = |x|$ at the point $x_0 = 0$.

This function is continuous at the zero point. Moreover, due to $x_0 = 0$

$$f'(x_0) = \lim_{t \to 0} \frac{|x_0 + t| - |x_0|}{t} = \lim_{t \to 0} \frac{|t|}{t} = \lim_{t \to 0} \begin{cases} 1 \ , & \text{if } t > 0, \\ -1 \ , & \text{if } t < 0. \end{cases} \qquad (3.2)$$

How can we conclude that the function in question does not have a derivative at $x_0 = 0$.

Indeed, it is possible to specify two different numerical sequences, for example,

$$\left\{ t_n = \frac{1}{n} \right\} \underset{n \to \infty}{\longrightarrow} 0 \qquad \text{and} \qquad \left\{ \tau_n = -\frac{1}{n} \right\} \underset{n \to \infty}{\longrightarrow} 0 \ ,$$

such that

$$f'(t_n) = f'\left(\frac{1}{n}\right) \underset{n \to \infty}{\longrightarrow} 1 \qquad \text{and} \qquad f'(\tau_n) = f'\left(-\frac{1}{n}\right) \underset{n \to \infty}{\longrightarrow} -1 \ .$$

This means that limit (3.1) does not exist due to the denial of the Heine definition of the limit.

Note that from the existence of the derivative of function $y = f(x)$ at the point x_0 the continuity of this function follows at x_0. Indeed, let there be a finite limit

$$A = \lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t} \, .$$

Then

$$f(x_0 + t) - f(x_0) = At + o(t) \, , \tag{3.3}$$

since the value $\dfrac{f(x_0 + t) - f(x_0)}{t} - A$ infinitely small. And passing in equality (3.3) to the limit at $t \to 0$, we obtain the fulfilled continuity condition $\lim\limits_{t \to 0} f(x_0 + t) = f(x_0)$.

The converse is not true. From the continuity of function the existence of derivative at a point may not follow (see Example 3.3).

Concluding the discussion of Definition 3.1, we note that mathematical texts use various methods of notation derivative of a function at a point. In addition to those used above, the most commonly used designations are

$$y'_x(x)\big|_{x=x_0} \; , \qquad \left.\frac{dy}{dx}\right|_{x=x_0} \; , \quad f'(x)\big|_{x=x_0} \; .$$

The concept of derivative of a function at a point allows its *geometric interpretation*, the meaning of which is illustrated by the task constructing a tangent to the graph of a function at some point.

Let's say we need to draw a tangent to the graph of function $y = f(x)$ at point A with coordinates $\left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\|$. Let's select another point B , having a coordinate representation $\left\| \begin{matrix} x_1 \\ y_1 \end{matrix} \right\|$. Since both of these points lie on the graph, then the equalities $y_0 = f(x_0)$ and $y_1 = f(x_1)$ are valid.

Let us draw the secant AB through the selected points (see Fig. 1.) It is easy to see that the equation of line passing through these points is $y = k(x - x_0) + y_0,$ where the value of *slope* $k = \dfrac{y_1 - y_0}{x_1 - x_0} = \operatorname{tg} \alpha$.

Let's start now, «sliding» according to the plot, bring point B closer to point A .

Then $x_1 \longrightarrow x_0$ and, therefore, $t = x_1 - x_0 \longrightarrow 0$. At the end this secant will become the desired tangent, the value of slope for which is equal to

$$k = \lim_{x \to x_0} \frac{y(x) - y(x_0)}{x - x_0} = \lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t} = f'(x_0) \ .$$

Thus, we come to the conclusion that *the value of the derivative of the function at a point is equal to slope of the tangent at this point.*

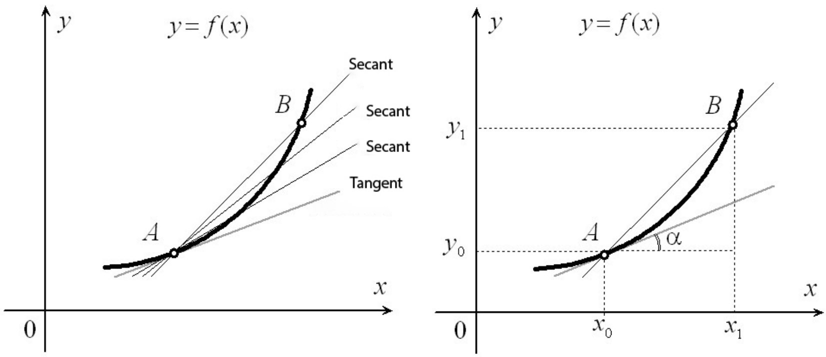

Fig.1. Geometric meaning of the derivative of a function at a point

In conclusion, we note that the horizontality of tangent to the plot of function at its extreme points is geometrically obvious.

From the above considerations it follows that *the value of the derivative of the function equals zero at these points* (if, of course, this derivative exists).

Derived function

Finding the values of derivative functions at points using definition 3.1 usually is quite a difficult task.

In practice it is more convenient to use a different approach based on the following considerations.

In definition 3.1, the passage to the limit is performed with respect to the auxiliary variable t, while the value x_0 is a fixed numeric parameter.

If we change x_0, then the value of the limit (3.1), generally speaking, it will change. However, for each specific x_0 it is *one,* because if the limit exists, then it is unique.

Therefore, definition 3.1 can be considered as a rule according to which *each* value $x \in X$ is assigned *unique* number $f'(x)$. In other words, here some new *function* is specified whose value at points x is equal to $f'(x)$.

This function is usually called *derivative function* of $y = f(x)$ and denoted as $f'(x)$.

Search of $f'(x)$ for known $f(x)$ is called *differentiation.* In the case when for $f(x)$ there exists $f'(x)$, they also say, that the function $f(x)$ *is differentiable.*

On the other hand, a function $y = F(x)$ such that $F'(x) = f(x)$, is called *antiderivative* for $y = f(x)$. The operation of finding it for known $f(x)$ is called *integration.*

For example, using the solution to Example 3.1, we can state that the derivative function of $y = x^3$ is $y = 3x^2$.

On the other hand, if we know the derivative function for $y = f(x)$, then the values of its derivative at each point x_0 are also known.

Derivative functions of $y = f(x)$ are also usually denoted as y'_x or $\dfrac{dy}{dx}$.

There are cases where a function depends on more than one variable. The identifier of the variable with respect to which the derivative is taken can be indicated explicitly as a subscript. For example, for a function $f(x,p)$ depending on x and p, the notation $f'_x(x,p)$ means the derivative with respect to the variable x. Variable p is considered as a fixed parameter here.

This derivative is called *partial derivative* and denoted as $\dfrac{\partial f}{\partial x}$.

It is also worth noting that in Russian mathematical texts the concepts «derivative of a function at a point» and «derivative function» are often denoted by the same word «derivative», assuming that it is clear from the context what is being said.

Obviously, it is much more convenient to use derivatives of functions to find the values of the derivative of a function at a point, rather than to use definition 3.1 for this purpose. But then the question arises: how to find derivatives of functions?

The answer is: you need to use

1) *table of derivatives* for some small set of elementary functions obtained directly using definition 3.1,
   and

2) *differentiation rules* expressing derivatives of some functions through derivatives of others.

Table 3.1

$f(x)$	$f'(x)$
x^a	ax^{a-1}
e^x	e^x
$a^x, \quad a > 0, a \neq 1$	$a^x \ln a$
$\ln \lvert x \rvert$	$\dfrac{1}{x}$
$\log_a \lvert x \rvert, \quad a > 0, a \neq 1$	$\dfrac{1}{x \ln a}$
$\sin x$	$\cos x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\operatorname{arctg} x$	$\dfrac{1}{1+x^2}$

An example of the first table is numbered 3.1.

Obviously, this table is not enough, to obtain derivatives of *any* functions given by formulas. That's why Table 3.2 should also be used, expressing *derivatives from some functions, through derivatives from others.*

Table 3.2

1°	$(\ f(x)+g(x)\ )' = f'(x)+g'(x)$
2°	$(\ C\cdot f(x)\ )' = C\cdot f'(x)$, where C − const
3°	$(\ f(x)\cdot g(x)\ )' = f'(x)\cdot g(x)+f(x)\cdot g'(x)$
4°	$(\ f(g(x))\ )'_x = f'_g(g)\cdot g'_x(x)$

Let us illustrate the use of tables 3.1 and 3.2 with the following examples.

`Example` 3.4. Suppose we need to find the derivatives of functions

$$y = \frac{\sqrt[3]{x^2} + \sqrt{x} + 1}{x}, \quad y = \frac{\sin x}{x} \quad \text{and} \quad y = e^{\arccos x}.$$

`Solution.`

1) A comparison of rules 1° and 3° of Table 3.2 convinces that it is easier to differentiate the sum of functions than their product. Therefore, first we will perform term-by-term division, that is, we will look for the derivative of the function

$$\left(\frac{\sqrt[3]{x^2} + \sqrt{x} + 1}{x} \right)' = \left(x^{-\frac{1}{3}} + x^{-\frac{1}{2}} + x^{-1} \right)' =$$

and, using the first formulas from Tables 3.1 and 3.2, we get

$$= -\frac{1}{3} \cdot x^{-\frac{4}{3}} - \frac{1}{2} \cdot x^{-\frac{3}{2}} - x^{-2} = -\frac{1}{3x\sqrt[3]{x}} - \frac{1}{2x\sqrt{x}} - \frac{1}{x^2}.$$

2) Table 3.2 does not contain a formula for differentiating a fraction, although you can find it in many textbooks. Therefore, we first transform this function into a product, and only then apply rule 3° of Table 3.2

$$\left(\frac{\sin x}{x}\right)' = \big((\sin x)\cdot(x^{-1})\big)' \; = \; (\sin x)'\cdot(x^{-1}) + (\sin x)\cdot(x^{-1})' \; =$$

$$= \; (\cos x)\cdot x^{-1} + (\sin x)\cdot(-x^{-2}) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x\cos x - \sin x}{x^2}\,.$$

3) First, recall that

$$g(x) = \arccos x = \frac{\pi}{2} - \arcsin x \ ,$$

and therefore, according to the first rule of Table 3.2 and the last line of Table 3.1, we have

$$g'_x(x) = -\frac{1}{\sqrt{1-x^2}} \ .$$

Then,

$$y'_x = \left(e^{\arccos x}\right)'_x = \left(e^{g(x)}\right)'_x = \left(e^{g}\right)'_g \cdot g'_x(x) =$$

$$= e^{g} \cdot g'_x(x) = e^{g} \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{e^{\arccos x}}{\sqrt{1-x^2}} \ .$$

Derivatives of functions defined in a special way

Derivative of the inverse function

If the function $y = f(x)$ is continuous and strictly monotone in some neighborhood of the point x_0 and if this function has a *non-zero* derivative at the point x_0, then *function* $x = f^{-1}(y)$ (inverse to $y = f(x)$) has a derivative at the point $y_0 = f(x_0)$. The value of which

$$\left.\frac{df^{-1}}{dy}\right|_{y=y_0} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=x_0}}$$

`Example.` Consider two inverse functions $y = \sin x$ and $x = \arcsin y$ in a small (guaranteeing strict monotonicity) neighborhood of the point $\left\{ x_0 = \dfrac{\pi}{4},\ y_0 = \dfrac{1}{\sqrt{2}} \right\}$.

At this point we have

$$\begin{array}{llll} y(x) = \sin x & y'_x = \cos x & x_0 = \dfrac{\pi}{4} & y'_x(x_0) = \dfrac{1}{\sqrt{2}} \\ x(y) = \arcsin y & x'_y = \dfrac{1}{\sqrt{1-y^2}} & y_0 = \dfrac{1}{\sqrt{2}} & x'_y(y_0) = \sqrt{2}\,. \end{array}$$

Derivative of a function specified parametrically

Let the functions $x = x(t)$ and $y = y(t)$ are defined in some neighborhood of the point $x_0 = x(t_0)$ and parametrically define the function $y = f(x)$.

If, in addition, $x = x(t)$ and $y = y(t)$ have derivatives at the point t_0 and $x'_t(t_0) \neq 0$, so then $y'_x(x_0) = \dfrac{y'_t(t_0)}{x'_t(t_0)}$.

`Example.` Let $\begin{cases} x = R\cos t\,, \\ y = R\sin t\,. \end{cases}$

In this case $x_0 = R\cos t_0 \quad \forall t_0 \neq k\pi,$ where $k \in (Z)$
And

$$y'_x(x_0) = -\frac{R\cos t_0}{R\sin t_0} = -\operatorname{ctg} t_0\,.$$

Derivative of a function specified implicitly

Let a function $y = f(x)$ be differentiable on the interval (a, b) is given implicitly by the condition $F(x, y(x)) = 0$. Then $y'_x(x)$ is found from the equation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y'_x(x) = 0 \, .$$

`Example.` Let $y = f(x)$ be defined implicitly by the condition

$$\sqrt{x} + \sqrt{y} = 3 \qquad x \in (0, 9) \, .$$

Find $y'_x(4)$, if $y(4) = 1$.

Because the

$$\frac{\partial F}{\partial x} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{\partial F}{\partial y} = \frac{1}{2\sqrt{y}} \, ,$$

That

$$y'_x(4) = -\sqrt{\frac{y}{x}} \, \Bigg|_{\substack{x\,=\,4 \\ y\,=\,1}} = -\frac{1}{2} \, .$$

Higher order derivatives

Suppose that the function $y = f(x)$ has a derivative function, which is also differentiable. Then the derivative function of the derivative is called *second-order derivative* for $y = f(x)$ and is denoted as

$$y''_{x=x_0} \ ; \quad \left.\frac{d^2y}{dx^2}\right|_{x=x_0} \ ; \quad y''(x)|_{x=x_0} \ .$$

Searching for the value of this numerical characteristic comes down to calculating a limit of the form

$$f''(x_0) = \lim_{t\to 0} \frac{f'(x_0+t) - f'(x_0)}{t} \ . \tag{3.4}$$

Since the second derivatives of a function at a point (as the limits of the function) are uniquely defined, then you can define a new function whose values are the numbers obtained from formula (3.4).

This function (the derivative of the derivative) is called *the second derivative function of the function* $y = f(x)$. To find it, you should use the same rules as for the first derivative of a function.

For example, if $y = \ln|x|$, then according to Table 3.1 $y' = \dfrac{1}{x}$. In turn, the derivative function of $\dfrac{1}{x} = x^{-1}$ according to the same Table 3.1 is equaled to $(-1)x^{-2} = -\dfrac{1}{x^2}$. That is, $(\ln|x|)'' = -\dfrac{1}{x^2}$.

Using similar reasoning, we can give a definition of a derivative of order n. We will denote this derivative as

$$y^{(n)}_{x=x_0}\ ; \qquad \left.\frac{d^n y}{dx^n}\right|_{x=x_0}\ ; \quad \left. y^{(n)}(x)\right|_{x=x_0}\ .$$

When calculating higher order derivatives, the following formulas are often useful:

$$1)\quad (a^x)^{(n)} = a^x \ln^n a \qquad \text{in particular } (e^x)^{(n)} = e^x\,,$$

$$2)\quad (\sin\alpha x)^{(n)} = \alpha^n \sin\left(\alpha x + \frac{\pi n}{2}\right),$$

$$3)\quad (\cos\alpha x)^{(n)} = \alpha^n \cos\left(\alpha x + \frac{\pi n}{2}\right),$$

$$4)\quad \Big((ax+b)^p\Big)^{(n)} = a^n p(p-1)\ldots(p-n+1)(ax+b)^{p-n}$$

$$\text{in particular } \left(\frac{1}{x-a}\right)^{(n)} = \frac{(-1)^n n!}{(x-a)^{n+1}}\,,$$

$$5)\quad (\log_a|x|)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n \ln a}$$

$$\text{in particular } (\ln|x|)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n},$$

Finally, we present a formula called «*Leibniz formula*». This is the formula for the n-th derivative of the product of two functions $u(x)$ and $v(x)$, each of which has derivatives up to order n inclusive:

$$6)\quad \Big(u(x)\cdot v(x)\Big)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} v^{(k)}\,.$$

Let's look at examples of using these formulas

`Problem 3.1.` Find $f^{(n)}(x)$ for $f(x) = \dfrac{x+5}{x^2-2x-3}$.

`Solution.` We have $f(x) = \dfrac{x+5}{(x+1)(x-3)}$. Let's decompose this function into simple fractions $f(x) = \dfrac{A}{(x+1)} + \dfrac{B}{(x-3)}$.
Then

$$A(x-3) + B(x+1) = x+5 \quad \Longrightarrow \quad \begin{cases} A + B = 1, \\ -3A + B = 5. \end{cases}$$

Where $A = -1$ and $B = 2$.
For function

$$f(x) = -\frac{1}{(x+1)} + \frac{2}{(x-3)},$$

using the linearity of the differentiation operation and a special case of formula 4), we obtain

$$f^{(n)}(x) = (-1)^n n! \left(\frac{2}{(x-3)^{(n+1)}} - \frac{1}{(x+1)^{(n+1)}} \right).$$

`Problem 3.2.` Find $f^{(n)}(x)$ for $f(x) = (x+1)^2 \sin 3x$.

`Solution.` Let's apply formula 6) (i.e. Leibniz's formula). In this problem we put $u(x) = \sin 3x$ and $v(x) = (x+1)^2$.
Then

$$f^{(n)}(x) = C_n^{(0)} u^{(n)}(x) v^{(0)}(x) + \\ + C_n^{(1)} u^{(n-1)}(x) v^{(1)}(x) + C_n^{(2)} u^{(n-2)}(x) v^{(2)}(x) ,$$

since $v^{(k)}(x) = 0$ for $k \geq 3$.
According to formula 2) we have

$$\begin{aligned} (\sin 3x)^{(n)} &= 3^n \sin\left(3x + \frac{\pi n}{2}\right) , \\ (\sin 3x)^{(n-1)} &= 3^{n-1} \sin\left(3x + \frac{\pi n}{2} - \frac{\pi}{2}\right) , \\ (\sin 3x)^{(n-2)} &= 3^{n-2} \sin\left(3x + \frac{\pi n}{2} - \pi\right) , \end{aligned}$$

In addition, we have

$$\begin{array}{lll} v^{(0)}(x) = (x+1)^2 , & v^{(1)}(x) = 2(x+1) , & v^{(2)}(x) = 2 , \\ C_n^0 = 1 , & C_n^1 = n , & C_n^0 = \dfrac{n(n-1)}{2} . \end{array}$$

This (when using trigonometric reduction formulas) finally gives

$$\begin{aligned} f^{(n)}(x) &= 3^n (x+1)^2 \sin\left(3x + \frac{\pi n}{2}\right) - \\ &- 3^{n-1} 2n(x+1) \cos\left(3x + \frac{\pi n}{2}\right) - \\ &- 3^{n-2} n(n-1) \sin\left(3x + \frac{\pi n}{2}\right) . \end{aligned}$$