# Differentials

When solving applied problems, it is often possible to obtain the necessary information about the local properties of function $y = f(x)$, using only its linear approximation.

In other words, it is possible to represent the function under study in the form

$$y(x) = y(x_0) + A(x_0)(x - x_0) + o(x - x_0)\,.$$

Hence the magnitude of the change in value of $f(x)$ at a small deviation $x$ from $x_0$ can be estimated as

$$y(x) - y(x_0) \approx A(x_0)(x - x_0)\,.$$

In this case, $A(x_0)(x - x_0)$ is usually called the *differential* of the function $y = f(x)$ and denoted as $dy$. It is also said that the function $y = f(x)$ is *differentiable* at the point $x_0$.

If we accept by definition that the differential of an independent variable $x$ is equal to its increment, i.e. $dx = \Delta x = x - x_0$, then $dy = A(x_0)dx$.

In other words, the differential $dy$ can be considered as a function of *two independent* variables: $x_0$ and $dx$, where this function depends on $dx$ *directly proportionally.*

It can be shown (this is a theorem!) that $A(x_0) = f'(x_0)$ and hence $dy = f'(x_0)dx$. Moreover, for the function $y = f(x)$ to be differentiable at the point $x_0$ it is necessary and sufficient that there exists a finite $f'(x_0)$.

For two differentiable functions $f(x)$ and $g(x)$ and arbitrary constants $\lambda$ and $\mu$ the following equalities hold

$$\begin{aligned} d(\lambda f + \mu g) &= \lambda df + \mu dg\,, \\ d(fg) &= gdf + fdg\,, \\ d\left(\frac{f}{g}\right) &= \frac{gdf - fdg}{g^2} \quad g \neq 0\,. \end{aligned}$$

Now let the derivative $f'(x)$ be differentiable. Then, considering $dy = f'(x)dx$ as a function of $x$ for a fixed $dx$ and using $dx$ as the *increment* $x$ again, we can get the differential of $dy$.

This new differential is called the *second differential* for function $f(x)$ and is denoted as $d^2y$.

According to this definition, the following equalities are true

$$d^2y = d(dy) = d(f'(x)dx) = d(f'(x))dx = f''(x)dxdx = f''(x)(dx)^2\,,$$

which is usually written as $d^2y = f''(x)dx^2$.

Arguing similarly, for functions with a higher-order derivative we can defined the differential of order $n$ with the form $d^{(n)}y = f^{(n)}(x)dx^n$.

We must keep in mind an important circumstance (called *invariance of the form of the first differential*). There is $dy = f'(x)dx$ to be always valid. But the formula $d^{(n)}y = f^{(n)}(x)dx^n$ is true only if $n = 1$ or if $x$ is an independent variable.

Indeed, the second differential for $y(t) = f(x(t))$ is given by another formula:

$$d^2y = f''_{xx}dx^2 + f'_x d^2x\,.$$

# Mean Value Theorems for Differentiable Functions

We will now consider methods for studying a function in a small neighborhood of a certain point, based on the use of the values of its derivatives. The basis of these methods are the following theorems, called *mean value theorems.*

`Rolle's Theorem` **If a function $f(x)$**

1) **is continuous on $[a, b]$,**
2) **has at each point $(a, b)$ a finite or infinite derivative of a definite sign,**
3) **and the equality $f(a) = f(b)$ is true,**

**then $\exists \xi \in (a, b)$ such that $f'(\xi) = 0$.**

Note that under the conditions of Rolle's theorem, among the points for which $f'(\xi) = 0$, there is always an extremum point of the function $f(x)$.

`Problem 4.1` *Prove that between two real roots of an algebraic polynomial with real coefficients there is a root of its derivative function.*

`Solution.` Let $y(x) = P_n(x)$ be an algebraic polynomial with real coefficients, for which $a$ and $b$ are adjacent real roots.

The function $y(x) = P_n(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then, by Rolle's theorem, there exists a point $\xi \in (a, b)$ such that $P'_n(\xi) = 0$.

`Solution obtained.` That is, $\xi$ is a root of the derivative of the algebraic polynomial.

`Lagrange's theorem` **If a function $f(x)$**

**1) is continuous on $[a, b]$,**

**2) has at each point $(a, b)$ a finite or infinite derivative of a certain sign,**

**then $\exists \xi \in (a, b)$ such that**

$$f(b) - f(a) = f'(\xi)(b - a).$$

The statement of Lagrange's theorem is often called the *finite increment formula.*

It follows from Lagrange's theorem that if $f(x)$ is continuous in some neighborhood of $x_0$, differentiable in a punctured neighborhood of this point and $\lim\limits_{x \to x_0} f'(x)$ exists, then $f'(x)$ is continuous at $x_0$.

`Problem 4.2` *Prove that* $\forall x > 0 \;\; \exists \theta(x)$ *such that*

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x + \theta(x)}}\,,$$

$$\textit{with} \lim_{x\to +0} \theta(x) = \frac{1}{4} \qquad \textit{and} \qquad \lim_{x \;\; to +\infty} \theta(x) = \frac{1}{2}.$$

`Solution.` By Lagrange's theorem, applied to the differentiable function $y = \sqrt{x} \;\; x \in [x_0, x_0+1]$, we have the equality

$$\sqrt{x_0+1} - \sqrt{x_0} = \frac{1}{2\sqrt{\xi}}\,.$$

Putting $\xi = x_0 + \theta(x)$, we obtain

$$\sqrt{x_0+1} - \sqrt{x_0} = \frac{1}{2\sqrt{x_0 + \theta(x)}}$$

$$\sqrt{x_0+1} + \sqrt{x_0} = 2\sqrt{x_0 + \theta(x)}\,.$$

Next, squaring both sides of the equality, we find that

$$\theta(x_0) = \frac{1}{4} + \frac{\sqrt{x_0^2 + x_0} - x_0}{2} = \frac{1}{4} + \frac{1}{2}\frac{x_0}{\sqrt{x_0^2 + x_0} + x_0}$$

`Solution obtained.` and, therefore, $\displaystyle \lim_{x\to +0} \theta(x) = \frac{1}{4} \qquad \lim_{x\to +\infty} \theta(x) = \frac{1}{2}.$

`Cauchy's theorem` **If functions** $\phi(t)$ **and** $\psi(t)$

1. **are continuous on** $[a, b]$,
2. **have finite derivatives at each point** $(a, b)$**, and** $\phi'(t) \neq 0 \;\; \forall t \in (a, b)$,

**then** $\exists \xi \in (a, b)$ **such that**

$$\frac{\psi(b) - \psi(a)}{\phi(b) - \phi(a)} = \frac{\psi'(\xi)}{\phi'(\xi)}.$$

Cauchy's theorem implies a useful rule for unraveling uncertainties of the form «$\frac{0}{0}$» and «$\frac{\infty}{\infty}$», called L'Hopital's rule.

`Theorem L'Hopital's rule`

**If functions $f(x)$ and $g(x)$**

1. **are differentiable in a punctured neighborhood of point $a$, and $g'(x) \neq 0$ in this neighborhood,**
2. **functions $f(x)$ and $g(x)$ are simultaneously either infinitesimal or infinitely large as $x \to a$,**
3. **there exists a finite $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.**

**Then the equality $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ is valid and for differentiable and infinitesimal at the point $a$ functions $f(x)$ and $g(x)$ we have**

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

We illustrate the application of L'Hopital's rule with the following examples.

`Problem 4.3.` Find $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}$.

`Solution.` According to L'Hopital's rule $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}}{1}=0$.

`Problem 4.4.` Find $\lim\limits_{x\to 3}\dfrac{3^x-x^3}{x-3}$.

`Solution.` In this case, we have an uncertainty of the form «$\dfrac{0}{0}$», and since $(3^x-x^3)'=3^x\ln 3-3x^2$ and $(x-3)'=1\neq 0$, then according to L'Hopital's rule

$$\lim_{x\to 3}\frac{3^x-x^3}{x-3}=\lim_{x\to 3}\frac{3^x\ln 3-3x^2}{1}=27\ln\frac{3}{e}\,.$$