Differentials

When solving applied problems, it is often possible to obtain the necessary information about the local properties of function $y = f(x)$, using only its linear approximation.

In other words, it is possible to represent the function under study in the form

$$y(x) = y(x_0) + A(x_0)(x - x_0) + o(x - x_0)\,.$$

Hence the magnitude of the change in value of $f(x)$ at a small deviation x from x_0 can be estimated as

$$y(x) - y(x_0) \approx A(x_0)(x - x_0)\,.$$

In this case, $A(x_0)(x - x_0)$ is usually called the *differential* of the function $y = f(x)$ and denoted as dy. It is also said that the function $y = f(x)$ is *differentiable* at the point x_0.

If we accept by definition that the differential of an independent variable x is equal to its increment, i.e. $dx = \Delta x = x - x_0$, then $dy = A(x_0)dx$.

In other words, the differential dy can be considered as a function of *two independent* variables: x_0 and dx, where this function depends on dx *directly proportionally.*

It can be shown (this is a theorem!) that $A(x_0) = f'(x_0)$ and hence $dy = f'(x_0)dx$. Moreover, for the function $y = f(x)$ to be differentiable at the point x_0 it is necessary and sufficient that there exists a finite $f'(x_0)$.

For two differentiable functions $f(x)$ and $g(x)$ and arbitrary constants λ and μ the following equalities hold

$$\begin{aligned} d(\lambda f + \mu g) &= \lambda df + \mu dg\,, \\ d(fg) &= g df + f dg\,, \\ d\left(\frac{f}{g}\right) &= \frac{g df - f dg}{g^2} \quad g \neq 0\,. \end{aligned}$$

Now let the derivative $f'(x)$ be differentiable. Then, considering $dy = f'(x)dx$ as a function of x for a fixed dx and using dx as the *increment* x again, we can get the differential of dy.

This new differential is called the *second differential* for function $f(x)$ and is denoted as d^2y.

According to this definition, the following equalities are true

$$d^2y = d(dy) = d(f'(x)dx) = d(f'(x))dx = f''(x)dxdx = f''(x)(dx)^2\,,$$

which is usually written as $d^2y = f''(x)dx^2$.

Arguing similarly, for functions with a higher-order derivative we can defined the differential of order n with the form $d^{(n)}y = f^{(n)}(x)dx^n$.

We must keep in mind an important circumstance (called *invariance of the form of the first differential*). There is $dy = f'(x)dx$ to be always valid. But the formula $d^{(n)}y = f^{(n)}(x)dx^n$ is true only if $n = 1$ or if x is an independent variable.

Indeed, the second differential for $y(t) = f(x(t))$ is given by another formula:

$$d^2y = f''_{xx}dx^2 + f'_x d^2x\,.$$

Mean Value Theorems for Differentiable Functions

We will now consider methods for studying a function in a small neighborhood of a certain point, based on the use of the values of its derivatives. The basis of these methods are the following theorems, called *mean value theorems*.

Rolle's Theorem

If a function $f(x)$

1) is continuous on $[a, b]$,

2) has at each point (a, b) **a finite or infinite derivative of a definite sign,**

3) and the equality $f(a) = f(b)$ **is true,**

then $\exists \xi \in (a, b)$ **such that** $f'(\xi) = 0$.

Note that under the conditions of Rolle's theorem, among the points for which $f'(\xi) = 0$, there is always an extremum point of the function $f(x)$.

`Problem 4.1` *Prove that between two real roots of an algebraic polynomial with real coefficients there is a root of its derivative function.*

`Solution.` Let $y(x) = P_n(x)$ be an algebraic polynomial with real coefficients, for which a and b are adjacent real roots.
The function $y(x) = P_n(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then, by Rolle's theorem, there exists a point $\xi \in (a, b)$ such that $P'_n(\xi) = 0$.

`Solution obtained.` That is, ξ is a root of the derivative of the algebraic polynomial.

`Lagrange's theorem` **If a function $f(x)$**

1) is continuous on $[a, b]$,

2) has at each point (a, b) a finite or infinite derivative of a certain sign,

then $\exists \xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

The statement of Lagrange's theorem is often called the *finite increment formula.*

It follows from Lagrange's theorem that if $f(x)$ is continuous in some neighborhood of x_0, differentiable in a punctured neighborhood of this point and $\lim\limits_{x \to x_0} f'(x)$ exists, then $f'(x)$ is continuous at x_0.

Problem 4.2 *Prove that* $\forall x > 0 \;\; \exists \theta(x)$ *such that*

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}},$$

$$\textit{with} \lim_{x\to+0} \theta(x) = \frac{1}{4} \qquad \textit{and} \qquad \lim_{x \;\; to+\infty} \theta(x) = \frac{1}{2}.$$

Solution. By Lagrange's theorem, applied to the differentiable function $y = \sqrt{x} \;\; x \in [x_0, x_0+1]$, we have the equality

$$\sqrt{x_0+1} - \sqrt{x_0} = \frac{1}{2\sqrt{\xi}}.$$

Putting $\xi = x_0 + \theta(x)$, we obtain

$$\sqrt{x_0+1} - \sqrt{x_0} = \frac{1}{2\sqrt{x_0+\theta(x)}}$$

$$\sqrt{x_0+1} + \sqrt{x_0} = 2\sqrt{x_0+\theta(x)}\,.$$

Next, squaring both sides of the equality, we find that

$$\theta(x_0) = \frac{1}{4} + \frac{\sqrt{x_0^2+x_0} - x_0}{2} = \frac{1}{4} + \frac{1}{2}\frac{x_0}{\sqrt{x_0^2+x_0}+x_0}$$

Solution obtained. and, therefore, $\lim\limits_{x\to+0} \theta(x) = \dfrac{1}{4} \qquad \lim\limits_{x\to+\infty} \theta(x) = \dfrac{1}{2}.$

`Cauchy's theorem` **If functions** $\phi(t)$ **and** $\psi(t)$

1) are continuous on $[a,b]$,

2) have finite derivatives at each point (a,b)**, and** $\phi'(t) \neq 0 \ \ \forall t \in (a,b)$,

then $\exists \xi \in (a,b)$ **such that**

$$\frac{\psi(b)-\psi(a)}{\phi(b)-\phi(a)} = \frac{\psi'(\xi)}{\phi'(\xi)}.$$

Cauchy's theorem implies a useful rule for unraveling uncertainties of the form «$\frac{0}{0}$» and «$\frac{\infty}{\infty}$», called L'Hopital's rule.

Theorem L'Hopital's rule

If functions $f(x)$ and $g(x)$

1) are differentiable in a punctured neighborhood of point a, and $g'(x) \neq 0$ in this neighborhood,

2) functions $f(x)$ and $g(x)$ are simultaneously either infinitesimal or infinitely large as $x \to a$,

3) there exists a finite $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

Then the equality $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ is valid and for differentiable and infinitesimal at the point a functions $f(x)$ and $g(x)$ we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

We illustrate the application of L'Hopital's rule with the following examples.

`Problem 4.3.` Find $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}$.

`Solution.` According to L'Hopital's rule $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}}{1}=0$.

`Problem 4.4.` Find $\lim\limits_{x\to3}\dfrac{3^x-x^3}{x-3}$.

`Solution.` In this case, we have an uncertainty of the form «$\dfrac{0}{0}$», and since $(3^x-x^3)'=3^x\ln3-3x^2$ and $(x-3)'=1\neq0$, then according to L'Hopital's rule

$$\lim_{x\to3}\frac{3^x-x^3}{x-3}=\lim_{x\to3}\frac{3^x\ln3-3x^2}{1}=27\ln\frac{3}{e}.$$