Taylor formula

Let us now consider other methods of studying a function in a small neighborhood of some point.

These methods are based on the use of derivatives up to the n-th order.

Let the point x_0 for the function $y(x)$ be an interior point of the domain of definition.

That is, there is a neighborhood of x_0 entirely contained in the domain of definition of this function.

And let the function $y(x)$ at this point have *continuous derivatives up to and including the n-th order.*

Let us assume that an analytical description (for example, in the form of a formula) of the function $y(x)$ is too complex for study. Or it is unknown at all. And let us be interested in the properties of $y(x)$ only in a relatively *small* neighborhood of the point x_0.

Then it seems appropriate to use an approximate description of this function in the form of a linear combination of *power* functions of order no greater than n :

$$y(x) \approx \sum_{k=0}^{n} A_k (x - x_0)^k$$

is valid, where $A_k \; \forall k \in [0, n]$ are some constants.

Specifically, we will assume that in some neighborhood of the point x_0 the equality

$$y(x) = \sum_{k=0}^{n} A_k (x - x_0)^k + r(x, x_0) \tag{5.1}$$

is true. Here $r(x, x_0)$ is a function equal to the approximation error.

Formally, (5.1) can be written with any coefficients A_k. However, one can expect that the quality of the approximation (i.e. the magnitude of $|\, r(x, x_0) \,|$) will depend on the values of these coefficients.

Therefore, we will use (5.1) with values of A_k such that the value $|\, r(x, x_0) \,|$, is *minimal.*

To do this, we require that *the value* of $r(x, x_0)$ be equal to zero at $x = x_0$.

Relation (5.1) at $x = x_0$ turns into the equality

$$y(x_0) = A_0 + r(x_0, x_0)\ . \tag{5.2}$$

Whence it follows that, to satisfy the condition $r(x_0, x_0) = 0$, we must take $A_0 = y(x_0)\,$.

Next, we require that *the first derivative* of the remainder term also be equal to zero at $x = x_0$.

We know that if functions are equal, then their derivatives are also equal. Then the equality obtained from (5.1) by term-by-term differentiation will be

$$r'(x, x_0) = y'(x) - \sum_{k=1}^{n} kA_k(x - x_0)^{k-1}\ . \tag{5.3}$$

Hence we obtain that at $x = x_0$ the condition $r'(x_0, x_0) = 0$ will be true, subject to $A_1 = y'(x_0)\,$.

Reasoning similarly, we obtain that the derivative of the remainder term of order k will turn to 0 at $x = x_0$, subject to $f^{(k)}(x_0) = k!\,A_k$. From which we get

$$A_k = \frac{1}{k!} y^{(k)}(x_0)\,. \tag{5.4}$$

As a result, we come to the conclusion that the «best» approximation of the function $y = f(x)$ has the form

$$y(x) = \sum_{k=0}^{n} \frac{y^{(k)}(x_0)}{k!}\,(x - x_0)^k + r(x, x_0)\,.$$

Let $y(x)$ have derivatives up to order $n-1$ inclusive in a neighborhood of the point x_0 and let there also exist $y^{(n)}(x_0)$, then the following *theorem* is true:

> **If in the power approximation its coefficients are chosen according to formulas (5.4), then** $r(x,x_0)=o(x-x_0)^n$.

In this case, the equality

$$y(x)=\sum_{k=0}^{n}\frac{y^{(k)}(x_0)}{k!}(x-x_0)^k+o(x-x_0)^n \tag{5.5}$$

is called *the expansion of the function* $y(x)$ *in a neighborhood of the point* x_0 *according to the Taylor formula with the remainder term in Peano form.*

If $y(x)$ has derivatives up to order $n+1$ inclusive in a neighborhood of the point x_0, then for any x in this neighborhood there exists ξ such that

$$r(x,x_0)=\frac{y^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

In this case $r(x,x_0)$ is called as *the remainder term of the Taylor formula in Lagrange form.*

Equality (5.5) in the case when $x_0=0$ is usually called *the Maclaurin formula.*

Finally, we note that $x-x_0=dx$ and (5.5) can be written as

$$y(x)=y(x_0)+\sum_{k=1}^{n}\frac{1}{k!}d^k y+o(x-x_0)^n.$$

We present Maclaurin formulas for some basic elementary functions.

$$
\begin{array}{lll}
1)\quad e^x & = & \sum\limits_{k=0}^{n} \dfrac{x^k}{k!} + o(x^n) = \\
 & = & 1 + x + \dfrac{x^2}{2!} + o(x^2)\,,
\end{array}
$$

$$
\begin{array}{lll}
2)\quad \operatorname{sh} x & = & \sum\limits_{k=0}^{n} \dfrac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = \\
 & = & x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + o(x^6)\,,
\end{array}
$$

$$
\begin{array}{lll}
3)\quad \operatorname{ch} x & = & \sum\limits_{k=0}^{n} \dfrac{x^{2k}}{(2k)!} + o(x^{2n+1}) = \\
 & = & 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + o(x^5)\,,
\end{array}
$$

$$
\begin{array}{lll}
4)\quad \sin x & = & \sum\limits_{k=0}^{n} (-1)^k \dfrac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = \\
 & = & x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + o(x^6)\,,
\end{array}
$$

$$
\begin{array}{lll}
5)\quad \cos x & = & \sum\limits_{k=0}^{n} (-1)^k \dfrac{x^{2k}}{(2k)!} + o(x^{2n+1}) = \\
 & = & 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + o(x^5)\,,
\end{array}
$$

$$
\begin{array}{lll}
6)\quad (1+x)^a & = & 1 + \sum\limits_{k=1}^{n} C_a^k x^k + o(x^n) = \\
 & = & 1 + ax + \dfrac{a(a-1)}{2!} x^2 + o(x^2)\,,
\end{array}
$$

$$
\begin{array}{lll}
7)\quad \dfrac{1}{1+x} & = & \sum\limits_{k=0}^{n} (-1)^k x^k + o(x^n) = \\
 & = & 1 - x + x^2 + o(x^2)\,,
\end{array}
$$

$$
\begin{array}{lll}
8)\quad \dfrac{1}{1-x} & = & \sum\limits_{k=0}^{n} x^k + o(x^n) = \\
 & = & 1 + x + x^2 + o(x^2)
\end{array}
$$

$$
\begin{aligned}
9)\quad \frac{1}{\sqrt{1-x}} &= 1+\sum_{k=1}^{n}\frac{(2k-1)!!}{2^k k!}x^k+o(x^n)= \\
&= 1+\frac{1}{2}x+\frac{3}{8}x^2+o(x^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
10)\quad \ln(1+x) &= \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}x^k+o(x^n)= \\
&= -\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\,,
\end{aligned}
$$

$$
\begin{aligned}
11)\quad \operatorname{arctg} x &= \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{2k+1}+o(x^{2n+2})= \\
&= x-\frac{x^3}{3}+\frac{x^5}{5}+o(x^6)\,.
\end{aligned}
$$

They may also be useful

$$
12)\qquad \operatorname{tg} x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+o(x^6)\,,
$$

$$
13)\qquad \arcsin x = x+\frac{1}{6}x^3+\frac{3}{40}x^5+o(x^6)\,.
$$

`Example` 5.01. Expand by Taylor formula $y(x) = \dfrac{x^2+5}{x^2+x-12}$ in the neighborhood of $x_0 = -1$ up to $o((x+1)^n)$.

`Solution.` 1) To use Maclaurin's table formulas, we make a change of variable $t = x+1$, then $x = t-1$ and

$$f(t) = \frac{(t-1)^2+5}{(t-1)^2+(t-1)-12} = \frac{t^2-2t+6}{t^2-t-12} = \frac{t^2-2t+6}{(t-4)(t+3)}.$$

2) Expand $f(t)$ into simple fractions

$$f(t) = A + \frac{B}{t-4} + \frac{C}{t+3}.$$

From the condition $A(t-4)(t+3)+B(t+3)+C(t-4) = t^2-2t+6$ we find that $A = 1,\ B = 2,\ C = -3$, that is

$$f(t) = 1 + \frac{2}{t-4} - \frac{3}{t+3}.$$

3) Transform the entry $f(t)$ to a form convenient for using tabular decompositions

$$f(t) = 1 - \frac{1}{2}\frac{1}{1-\frac{t}{4}} - \frac{1}{1+\frac{t}{3}}$$

and use formulas 7) and 8), we obtain

$$f(t) = 1 - \frac{1}{2}\sum_{k=0}^{n}\frac{t^k}{4^k} - \sum_{k=0}^{n}\frac{(-1)^k t^k}{3^k} + o(t^n) =$$

$$= \left(1 - \frac{1}{2} - 1\right) + \sum_{k=1}^{n}\left(-\frac{1}{2\cdot 4^k} + \frac{(-1)^{k+1}}{3^k}\right)t^k + o(t^n)\,.$$

4) Finally, we return to the original independent variable x :

$$f(x) = -\frac{1}{2} + \sum_{k=1}^{n}\left(\frac{(-1)^{k+1}}{3^k} - \frac{1}{2\cdot 4^k}\right)(x+1)^k + o((x+1)^n)\,.$$

`Example` 5.02. Expand by Taylor's formula $y(x) = e^{2x^2-12x}$ in the neighborhood of $x_0 = 3$ up to $o((x-3)^{2n+1})$.

`Solution.` 1) To use expansions by Maclaurin's formula, introduce a new variable $t = x - 3$, and express $y(x)$ through t by substituting $x = t + 3$, we get

$$f(t) = e^{2(t+3)^2-12(t+3)} = e^{2t^2-18}.$$

Applying formula 1), we get

$$f(t) = e^{-18}\sum_{k=0}^{n}\frac{(2t^2)^k}{k!}+o((t^2)^n) = e^{-18}\sum_{k=0}^{n}\frac{2^k t^{2k}}{k!}+o(t^{2n+1})$$

2) Returning to the original variable x, we obtain the desired expansion according to the Taylor formula

$$y(x) = e^{-18}\sum_{k=0}^{n}\frac{2^k}{k!}(x-3)^{2k} + o((x-3)^{2n+1})\,.$$

`Example` 5.03. Obtain an expansion by Maclaurin's formula for the function $y(x)=\cos 2x\sin x$ up to $o(x^{2n+2})$.

`Solution.` 1) Applying the trigonometric formula

$$\cos\alpha\sin\beta=\frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2},$$

we obtain $y(x)=\frac{1}{2}\sin 3x-\frac{1}{2}\sin x$. Then according to formula 4)

$$y(x)=\frac{1}{2}\left(\sum_{k=0}^{n}(-1)^k\,\frac{(3x)^{2k+1}}{(2k+1)!}-\sum_{k=0}^{n}(-1)^k\,\frac{x^{2k+1}}{(2k+1)!}\right)+o(x^{2n+2})$$

And finally,

$$y(x)=\frac{1}{2}\sum_{k=0}^{n}\frac{(-1)^k\left(3^{2k+1}-1\right)}{(2k+1)!}x^{2k+1}+o(x^{2n+2})$$

`Example` 5.04. Represent the function

$$y(x) = \ln\sqrt[4]{\frac{x-2}{5-x}}$$

in the neighborhood of the point $x_0 = 3$ up to $o((x-3)^n)$ by the Taylor formula.

`Solution.` 1) To apply the expansions according to Maclaurin's formula, we introduce a new variable $t = x - 3$, and express $y(x)$ through t by substituting $x = t + 3$, we get

$$f(t) = \ln\sqrt[4]{\frac{1+t}{2-t}} = \frac{1}{4}(\ln(1+t) - \ln(2-t)) =$$

$$= \frac{1}{4}\ln(1+t) - \frac{1}{4}\ln\left(2\left(1-\frac{t}{2}\right)\right)\,.$$

Applying formula 10) twice, we find that

$$f(t) = \frac{1}{4}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}t^k - \frac{1}{4}\ln 2 + \frac{1}{4}\sum_{k=1}^{n}\frac{1}{k2^k}t^k + o((t)^n) =$$

$$= -\frac{1}{4}\ln 2 + \sum_{k=1}^{n}\left(\frac{(-1)^{k-1}}{4k} + \frac{1}{k2^{k+2}}\right)t^k + o((t)^n)$$

2) Returning to the original variable x, we obtain the desired expansion by Taylor's formula

$$y(x) = -\frac{1}{4}\ln 2 + \sum_{k=1}^{n}\frac{(-1)^{k-1}2^k + 1}{4k2^k}(x-3)^k + o((x-3)^n)\,.$$

`Example` 5.05. Expand by Maclaurin's formula $y(x) = (x-1)e^{\frac{x}{2}}$ up to $o(x^n)$.

`Solution.` 1) According to formula 1), we have

$$y(x) = (x-1)\left(\sum_{k=0}^{n} \frac{1}{2^k k!}x^k + o(x^n)\right) .$$

This is correct, but it is not an answer to the problem.

2) Expanding the outer brackets, we get

$$y(x) = \sum_{k=0}^{n-1} \frac{1}{2^k k!}x^{k+1} - \sum_{k=0}^{n} \frac{1}{2^k k!}x^k + o(x^n)\,.$$

Introduce the summation index $m = k+1$. Then we have $k = m-1$ and

$$y(x) = \sum_{m=1}^{n} \frac{1}{2^{m-1}(m-1)!}x^m - \sum_{k=0}^{n} \frac{1}{2^k k!}x^k + o(x^n)\,.$$

3) The value of the sum does not depend on which letter denotes the summation index. In the first sum, replace m with k and write $y(x)$ in the form

$$y(x) = -1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}(k-1)!} x^k - \sum_{k=1}^{n} \frac{1}{2^k k!} + o(x^n) =$$

$$= -1 + \sum_{k=1}^{n} \left(\frac{1}{2^{k-1}(k-1)!} - \frac{1}{2^k k!} \right) x^k + o(x^n)\,.$$

The last expression can be simplified a bit by writing

$$y(x) = -1 + \sum_{k=1}^{n} \frac{1}{2^k (k-1)!} \left(2 - \frac{1}{k} \right) x^k + o(x^n)\,.$$

Finally we get

$$y(x) = -1 + \sum_{k=1}^{n} \frac{2k-1}{2^k k!} x^k + o(x^n)\,.$$

`Example` 5.06. Expand by Maclaurin's formula $y(x) = \dfrac{1}{1+x+x^2}$ up to $o(x^n)$.

`Solution.` 1) Using formula 7) we can write

$$\frac{1}{1+x+x^2} = \sum_{k=0}^{n} \left(x+x^2\right)^k + o(x^{2n})\,.$$

This is true, but it is not the answer, since it requires a formula of the view

$$\frac{1}{1+x+x^2} = \sum_{k=0}^{n} A_k x^k + o(x^n)\,.$$

2) First, we transform $y(x)$, multiplying the numerator and denominator by $1-x$.

$$y(x) = \frac{1}{1+x+x^2} = \frac{1-x}{(1-x)(1+x+x^2)} = \frac{1-x}{1-x^3}\,.$$

Then, according to formula 8), the equality will be true

$$y(x) = (1-x)\left(\sum_{k=0}^{n} x^{3k} + o(x^{3n})\right)\,.$$

3) Let's write the last formula without the summation symbol

$$y(x) = (1-x)\Big(1 + x^3 + x^6 + x^9 + \ldots + x^{3n} + o(x^{3n})\Big)$$

or, in ascending order of powers of x

$$y(x) = 1 - x + 0x^2 + x^3 - x^4 + 0x^5 + x^6 - x^7 + \ldots + x^{3n} + o(x^{3n})\,.$$

That is, the numerical sequence $\{A_k\}$ has the form

$$\{\,1,\ -1,\ 0,\ 1,\ -1,\ 0,\ 1,\ -1,\ 0,\ \ldots\}$$

with a periodically repeating triad of members $1, -1, 0$. This sequence can be defined functionally, for example, by the formula

$$A_k = \frac{2}{\sqrt{3}} \sin \frac{2\pi(k+1)}{3}\,,$$

which allows us to write the answer to the problem as

$$y(x) = \frac{2}{\sqrt{3}} \sum_{k=0}^{n} \sin \frac{2\pi(k+1)}{3} \cdot x^k + o(x^n)\,.$$

`Example` 5.07. Obtain the Maclaurin formula for

$$y(x) = \operatorname{arctg} x\,.$$

`Solution.` 1) Note that in this case the function $y'(x)$ can easily be written as a Maclaurin expansion using formula 7) . Indeed

$$y'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + o(x^{2n+1})\,.$$

Integrating this equality over x we get

$$\operatorname{arctg} x = C + \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})\,,$$

where C is a constant.

Taking into account that $\operatorname{arctg} 0 = 0$, we obtain

$$\operatorname{arctg} x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})\,.$$