Calculating Limits
Using Taylor's Formula

A representation of functions using Taylor's formula can be an effective method for calculating limits. More precisely, this representation can be used to expand uncertainties of the form «$\frac{0}{0}$» and «1^{∞}» . The basis for this use is the following theorem (*theorem* 6.1).

Let functions $f(x)$ **and** $g(x)$ **be such that** $\lim\limits_{x\to 0} f(x) = 0$, $\lim\limits_{x\to 0} g(x) = 0$ **and** $\exists m \in \mathbb{N}$ **such that the equalities** $f(x) = ax^m + o(x^m)$ **and** $g(x) = bx^m + o(x^m)$ **c** $b \neq 0$. **Then**

$$\textbf{1)} \qquad \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{ax^m + o(x^m)}{bx^m + o(x^m)} = \frac{a}{b},$$

$$\textbf{2)} \qquad \lim_{x\to 0} \Big(1 + f(x)\Big)^{\frac{1}{g(x)}} =$$

$$= \lim_{x 0} \Big(1 + ax^m + o(x^m)\Big)^{\overline{\frac{1}{bx^m + o(x^m)}}} = e^{\frac{a}{b}} .$$

Recall that the equality

$$y(x) = \sum_{k=0}^{n} \frac{y^{(k)}(x_0)}{k!}(x - x_0)^k + o\big((x - x_0)^n\big) \tag{6.1}$$

is called *the expansion of the function* $y(x)$ *in the neighborhood of the point* x_0 *by the Taylor formula with the remainder term in Peano form.* Equality (6.1) in the case when $x_0 = 0$ is called *the Maclaurin formula.*

We present a table of Maclaurin formulas for some basic elementary functions.

$$1)\quad e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) = 1 + x + \frac{x^2}{2!} + o(x^2)\,,$$

$$2)\quad \operatorname{sh} x = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)\,,$$

$$3)\quad \operatorname{ch} x = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)\,,$$

$$4)\quad \sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)\,,$$

$$5)\quad \cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5),\,,$$

$$6)\quad (1+x)^a = 1 + \sum_{k=1}^{n} C_a^k x^k + o(x^n) = 1 + ax + \frac{a(a-1)}{2!}x^2 + o(x^2)\,,$$

$$7)\quad \frac{1}{1+x} = \sum_{k=0}^{n} (-1)^k x^k + o(x^n) = 1 - x + x^2 + o(x^2)\,,$$

$$8)\quad \frac{1}{1-x} = \sum_{k=0}^{n} x^k + o(x^n) = 1 + x + x^2 + o(x^2)\,,$$

$$9)\quad \frac{1}{\sqrt{1-x}} = 1+\sum_{k=1}^{n}\frac{(2k-1)!!}{2^k k!}x^k + o(x^n) = 1+\frac{1}{2}x+\frac{3}{8}x^2+o(x^2)\,,$$

$$10)\quad \ln(1+x) = k=1^n\frac{(-1)^{k-1}}{k}x^k + o(x^n) = x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\,,$$

$$11)\quad \operatorname{arctg} x = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{2k+1}+o(x^{2n+2}) = x-\frac{x^3}{3}+\frac{x^5}{5}+o(x^6)\,.$$

They may also be useful

$$12)\quad \operatorname{tg} x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+o(x^6)\,,$$

$$13)\quad \arcsin x = x+\frac{1}{6}x^3+\frac{3}{40}x^5+o(x^6)\,.$$

Let us explain the idea of this method with the following example.

`Example` 6.1. Find $A = \lim\limits_{x\to 0} \dfrac{\operatorname{sh} 2x - 2\sin x}{x^3}$.

`Solution.` According to formulas 2) and 4), the following equalities will be true

$$\operatorname{sh} 2x = 2x - \frac{8x^3}{3!} + o(x^4) \qquad \text{and} \qquad \sin x = x - \frac{x^3}{3!} + o(x^4)\,.$$

Therefore

$$A = \lim_{x\to 0} \frac{2x + \frac{4}{3}x^3 + o(x^4) - 2x + \frac{1}{3}x^3 + o(x^4)}{x^3} =$$

$$= \lim_{x\to 0} \frac{\frac{5}{3}x^3 + o(x^4)}{x^3} = \lim_{x\to 0} \left(\frac{5}{3} + \frac{o(x^4)}{x^3} \right) =$$

$$= \frac{5}{3} + \lim_{x\to 0} \frac{o(x^4)}{x^3} = \frac{5}{3}.$$

The above transformations do not require any special comments. Let us note only the equalities

$$\lim_{x\to 0}\frac{o(x^4)}{x^3}=\lim_{x\to 0} x\cdot\frac{o(x^4)}{x^4}=0 \qquad \text{and} \qquad o(x^4)+o(x^4)=o(x^4).$$

The first of which follows from the definition of the function $o(x)$. In the second (it looks strange) all three functions $o(x)$ *are different.* A more precise notation would be $o_{(1)}(x^4)+o_{(2)}(x^4)=o_{(3)}(x^4)$.

Note that Theorem 6.1 does not provide a way to find the value of m. However, it is clear that this value is determined by the first nonzero terms in the expansions for the functions $f(x)$ and $g(x)$.

When solving problems, one can only recommend starting the construction of the expansion with the simplest of these two. The following examples illustrate the expediency of this approach.

`Example` 6.2. Find $A = \lim\limits_{x\to 0} \dfrac{x\sqrt{1+\sin x} + \ln(1-x)}{\operatorname{tg} x - \sin x}$.

`Solution.` 1) The denominator here is clearly simpler than the numerator. Therefore, we will start with the first. From table formulas 4) and 12) we have

$$\operatorname{tg} x = x + \frac{x^3}{3} + o(x^4) \qquad \text{and} \qquad \sin x = x - \frac{x^3}{6} + o(x^4)\,.$$

Therefore, the denominator is $\dfrac{1}{2}x^3 + o(x^4) \ \to \ m = 3\,.$

2) In the numerator, the first term is equal to the product of x and a complex (i.e. superposition) function $\sqrt{1+\sin x}$. Therefore, it is sufficient to expand the latter to $o(x^2)$, since $x\,o(x^2) = o(x^3)$.

By tabular formula 4) , where $a = \dfrac{1}{2}$

$$\sqrt{1+t} = 1 + \frac{1}{2}t + \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2!}t^2 + o(t^2) =$$

$$= 1 + \frac{1}{2}t - \frac{1}{8}t^2 + o(t^2).$$

Therefore

$$\sqrt{1+\sin x} = 1 + \frac{1}{2}\left(x - \frac{1}{6}x^3 + o(x^4)\right) -$$

$$\frac{1}{8}\left(x - \frac{1}{6}x^3 + o(x^4)\right)^2 + o(x^2) =$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)\,.$$

Note that here we used the well-known formula

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc. \qquad (6.2)$$

Then we collected into $o(x^2)$ *all* terms of the second or higher order of smallness.

3) The table formula 10) for the logarithmic function yields

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + o(x^3)\,,$$

which, together with the expansion obtained in 2), allows to write the Maclaurin expansion for the entire numerator

$$x\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)\right) - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + o(x^3) =$$

$$= -\frac{11}{24}x^3 + o(x^3)\,.$$

And finally, $\quad A = \lim\limits_{x\to 0} \dfrac{-\dfrac{11}{24}x^3 + o(x^3)}{\dfrac{1}{2}x^3 + o(x^3)} = -\dfrac{11}{12}\,.$

`Example 6.3.` Find $A = \lim\limits_{x\to 0} \dfrac{e^x + \ln(1-\sin x) - 1}{\sqrt[3]{8-x^4} - 2}$.

`Solution.` 1) The denominator in this problem is also simpler than the numerator. For the denominator, using table formula 6) with $a = \dfrac{1}{3}$, we have

$$\sqrt[3]{8-x^4} - 2 \quad = \quad 2\left(\sqrt[3]{1-\frac{1}{8}x^4} - 1\right) =$$

$$= 2\left(1 - \frac{1}{3}\cdot\frac{x^4}{8} + o(x^4) - 1\right) \quad = \quad -\frac{1}{12}x^4 + o(x^4)\,.$$

So, $m = 4$.

2) To construct the expansion of the numerator, we need the following table formulas

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+o(x^4)\,, \\
\ln(1-x) &= -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+o(x^4)\,, \\
\sin x &= x-\frac{x^3}{6}+o(x^4)\,.
\end{aligned}
$$

Specifically, for the second term in the numerator, again using formula (6.2), we find

$$\ln(1-\sin x) =$$

$$= -\left(x-\frac{x^3}{6}+o(x^4)\right) \; -\frac{1}{2}\left(x-\frac{x^3}{6}+o(x^4)\right)^2 -$$

$$-\frac{1}{3}\left(x-\frac{x^3}{6}+o(x^4)\right)^3 -\frac{1}{4}\left(x-\frac{x^3}{6}+o(x^4)\right)^4 +o(x^4) =$$

$$= -x+\frac{x^3}{6}-\frac{x^2}{2}+\frac{x^4}{6}-\frac{x^3}{3}-\mathfrak{f}racx^4 4+o(x^4) =$$

$$= -x-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{12}+o(x^4)\,.$$

As a result, the formula for the numerator takes the form

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}-x-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{12}-1+o(x^4)=$$

$$=-\frac{x^4}{24}+o(x^4)\,.$$

3) Thus, finally, we get

$$A=\lim_{x\to 0}\frac{-\dfrac{1}{24}x^4+o(x^4)}{-\dfrac{1}{12}x^4+o(x^4)}=\frac{1}{2}\,.$$

Sometimes, to construct the Maclaurin approximation, it is expedient to combine the use of tabular expansions and its definition, i.e. formula (6.1). Let us give an example of such a situation.

`Example` 6.4. Find $A = \lim\limits_{x\to 0} \dfrac{\operatorname{arctg}(3 + x^2) - \operatorname{arctg}(2 + \cos x)}{\ln(1 + x) - e^x + 1}$.

`Solution.` 1) Since the expansions are valid

$$e^x = 1 + x + \frac{x^2}{2!} + o(x^2) \qquad \ln(1 + x) = x - \frac{x^2}{2} + o(x^2)\,,$$

then for the denominator we easily obtain

$$x - \frac{x^2}{2} + o(x^2) - 1 - x - \frac{x^2}{2!} + o(x^2) \quad = \quad -x^2 + o(x^2)\,.$$

2) In this example $m = 2$ and the expansions for the numerator are easy to calculate using formula (6.1). To do this, at the point $x = 0$ we need to find the values of both the numerator itself and its derivatives up to and including the second order.

The value of the numerator $\Phi(x)$ at zero is obviously zero. Let us write out the formula for its first derivative, without making any simplifications. We have

$$\Phi'(x) = \frac{2x}{1+(3+x^2)^2} - \frac{-\sin x}{1+(2+\cos x)^2}$$

It is also obvious that $\Phi'(0) = 0$.

3) $\Phi''(x)$ is sought by the differentiation rule. We have for the first term in $\Phi'(x)$

$$\left(\frac{2x}{1+(3+x^2)^2}\right)'_x = \frac{2(1+(3+x^2)^2) - 2x(1+(3+x^2)^2)'_x}{(1+(3+x^2)^2)^2} =$$

$$= \frac{2(1+3^2)-0}{(1+3^2)^2} = \frac{1}{5}.$$

Similarly, we find for the second term in $\Phi'(x)$

$$\left(\frac{-\sin x}{1+(2+\cos x)^2}\right)'_x = -\frac{1}{10}.$$

Check it yourself.

4) Now, using (6.1), we write the expansion for the entire numerator

$$\Phi(x) = \Phi(0) + \Phi'(0)x + \frac{1}{2}\Phi''(0)x^2 + o(x^2)\,.$$

Substituting, we get

$$\Phi(x) = 0+0\,x+\frac{1}{2}\left(\frac{1}{5}-\left(-\frac{1}{10}\right)\right)x^2+o(x^2) = \frac{3}{20}x^2+o(x^2)\,.$$

Finally $$A = \lim_{x\to 0}\frac{\frac{3}{20}x^2 + o(x^2)}{-x^2 + o(x^2)} = -\frac{3}{20}\,.$$

`Example` 6.5 Find the first three terms of the expansion in the Maclaurin formula for the function $\operatorname{tg} x$.

`Solution.` 1) We use tabular expansions for the functions $\sin x$ and $\cos x$, as well as the expansion with undetermined coefficients for the *odd* function $\operatorname{tg} x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)\,,$$

$$\operatorname{tg} x = ax + bx^3 + cx^5 + o(x^5)\,.$$

2) We find the values of the undetermined coefficients using the equality
$$\cos x \cdot \operatorname{tg} x = \sin x\,,$$
which in the introduced notations has the form
$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)\right)\left(ax + bx^3 + cx^5 + o(x^5)\right) =$$
$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right)\,.$$

3) Multiplying the polynomials on the left side and equating the coefficients at equal powers of x on both sides, we obtain the values of the desired coefficients
$$a = 1\,, \qquad b = \frac{1}{3}, \qquad c = \frac{2}{15}\,.$$

Let us now consider examples of calculating a limit of the type specified in item 2) of Theorem 6.1.

`Example 6.6` Find $A = \lim\limits_{x\to 0}\left(\sqrt{1+2x}\cdot\cos x - \dfrac{\arcsin x}{1+x}\right)^{\dfrac{x}{(1-\operatorname{ch} x)^2}}$.

`Solution.` 1) Since the exponent is simpler than the base, we begin to find an expansion for the exponent. We have

$$1-\operatorname{ch} x = -\frac{1}{2}x^2+o(x^2) \quad\Longrightarrow\quad (1-\operatorname{ch} x)^2 = \frac{1}{4}x^4+o(x^4)\,.$$

Then

$$\frac{x}{(1-\operatorname{ch} x)^2} = \frac{x}{\dfrac{1}{4}x^4+o(x^4)} = \frac{1}{\dfrac{1}{4}x^3+o(x^3)}\,.$$

Therefore, $m = 3$.

2) We obtain the expansion of the base as follows. We have

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^3)\,,$$

$$\sqrt{1+2x} = 1 + \tfrac{1}{2}\,2x + \tfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\,4x^2 + \tfrac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\,8x^3 + o(x^3) =$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + o(x^3)$$

Then

$$\sqrt{1+2x}\cdot\cos x =$$

$$= \left(1 - \frac{1}{2}x^2 + o(x^3)\right)\cdot\left(+x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + o(x^3)\right) =$$

$$= 1 + x - x^2 + o(x^3)\,.$$

Similarly, taking into account table formulas 13) and 6), from

$$\arcsin x = 1 + \frac{1}{6}x^3 + o(x^3)$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3)\,,$$

we get

$$\frac{\arcsin x}{1+x} =$$

$$= \left(1 + \frac{1}{6}x^3 + o(x^3)\right) \cdot \big(1 - x + x^2 - x^3 + o(x^3)\big) =$$

$$= x - x^2 - \frac{5}{6}x^3 + o(x^3)\,.$$

As a result, for the base we have $\quad 1 + \dfrac{5}{6}x^3 + o(x^3)\,.$

3) Finally, by item 2) of Theorem 6.1 we obtain

$$A = \lim_{x\to 0}\Bigl(1 + \frac{5}{6}x^3 + o(x^3)\Bigr)^{\dfrac{1}{\frac{1}{4}\,x^3 + o(x^3))}} = e^{\frac{5}{6}\cdot 4} = e^{\frac{10}{3}}\,.$$

Note that under the conditions of Theorem 6.1 $A = e^B$, where

$$B = \lim_{x\to 0}\frac{\ln(1 + f(x))}{g(x)} = \lim_{x\to 0}\frac{f(x)}{g(x)} = \ln A\,. \qquad (6.3)$$

Let's consider another example of uncertainty disclosure «1^{∞}» .

`Example` 6.7. Find $A = \lim\limits_{x\to 0}\left(\dfrac{2x}{\sin 2x} - \dfrac{2}{3}x^2\right)^{\dfrac{x^2}{x^2 - \operatorname{arctg} x^2}}$.

`Solution.` 1) Let's start with constructing the expansion for the exponent. We have

$$\frac{x^2}{x^2 - \operatorname{arctg} x^2} = \frac{x^2}{x^2 - \left(x^2 - \dfrac{x^6}{3} + o(x^9)\right)} = \frac{1}{\dfrac{x^4}{3} + o(x^7)} .$$

That is, $m = 4$.

2) To obtain the Maclaurin formula for the base, we first expand the fraction to $o(x^4)$

$$\frac{2x}{\sin 2x} = \frac{2x}{2x - \frac{1}{6}8x^3 + \frac{1}{120}32x^5 + o(x^6)} =$$

$$= \frac{1}{1 - \left(\frac{2}{3}x^2 - \frac{2}{15}x^4 + o(x^5)\right)} =$$

$$= 1 + \left(\frac{2}{3}x^2 - \frac{2}{15}x^4 + o(x^5)\right) + \left(\frac{2}{3}x^2 + o(x^3)\right)^2 + o(x^5) =$$

As a result, the Maclaurin representation of the base will be

$$\frac{2x}{\sin 2x} - \frac{2}{3}x^2 \quad = \quad 1 + \frac{14}{45}x^4 + o(x^5)\,.$$

Finally, applying the second assertion of Theorem 6.1, we obtain $A = e^{\frac{14}{45}\cdot 3} = e^{\frac{14}{15}}$.