Study of properties of functions and construction of graphs

Reference theoretical information

Let us consider methods of describing properties of a function $y = f(x)$, both *local*, and related to some *segment* of the real axis. We will describe these properties, using the operations of calculating limits and differentiation. We will also visually demonstrate the properties of functions using their *graphs*.

Definition 7.1	*The graph of the function* $y = f(x)$ in the Cartesian rectangular coordinate system $\{Oxy\}$ is the set of points in the coordinate plane that have a coordinate column of the form $\left\| \begin{array}{c} x \\ f(x) \end{array} \right\| \quad \forall x \in D$, where D is the domain of definition for $y = f(x)$.

When studying a function, it is necessary to describe its domain of definition and domain of values.

It is also necessary to find out whether the function is *continuous*, by indicating the discontinuity points and calculating the one-sided limits at them. Establish whether this function has the properties of *evenness, oddness* and *periodicity.*

Next, we need to find

— *vertical asymptotes* of the graph, that is, lines of the form $x = x_0 \ \forall y$, for which $\lim\limits_{x\to x_0} f(x) = \pm\infty$;

— *inclined and horizontal asymptotes* of the graph, i.e. straight lines of the form $y = ax + b$, for which $a = \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x}$ and $b = \lim\limits_{x\to\pm\infty} \big(f(x) - ax\big)$;

— *intervals of sign constancy* and the points of intersection of the graph with the coordinate axes.

Using the collected information about the properties of the function, it is advisable to construct a preliminary sketch of the function graph.

At this stage, you can also use the *selection of the main part* method, the essence of which will be explained below using specific examples.

For a more precise description of the properties of the function $y = f(x)$ it is necessary to find out at what points it has *extremum*, that is, *maximum or minimum.*

For this, we first give

Definition 7.2	We will say that the function $y = f(x)$ has at the point x_0 *a strict local minimum* (*a maximum*), if there exists $\dot{U}_\rho(x_0)$ — a punctured ρ-neighborhood of the point x_0 such that $$f(x) > f(x_0) \quad \big(f(x) < f(x_0)\big) \quad \forall x \in \dot{U}_\rho(x_0).$$ If this inequality is true in the entire domain of definition, then we will talk about a *global* extremum.

In the future, for brevity, point x_0 with $f'(x_0) = 0$, will be called *stationary.*

When using Definition 7.2, we need to be able to estimate the sign of the difference between the values of $f(x)$ and $f(x_0)$.

In the case of a function differentiable at a point x_0, this can be done using the Taylor formula of the *second* order with the remainder term in Peano form

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o\big((x - x_0)^2\big) \ .$$

In the case of a non-stationary point x_0, where $f'(x_0) \neq 0$, this formula can be written as

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0) \, .$$

From the last formula it follows that there can be no extremum of a differentiable function $y = f(x)$ at a non-stationary point.

Indeed, if we take a neighborhood $\dot{U}_\rho(x_0)$ so small that we can neglect $o(x - x_0)$, and choose $x = x_0 + t$ so that the term

$$f'(x_0)(x - x_0) = f'(x_0) \cdot t > 0,$$

then we get $f(x_0 + t) > f(x_0)$.

It will obviously also be true $f(x_0 - t) < f(x_0)$ because of the linearity of $f(x) - f(x_0)$ in t . Consequently, for $f'(x_0) \neq 0$ there is no extremum.

We conclude that the equality $f'(x_0) = 0$ is *a necessary condition for the presence of an extremum* of the function $y = f(x)$ at the point $x = x_0$.

This condition *is not sufficient.*

Example: for function $f(x) = x^3$ at $x_0 = 0$ the derivative is equal to zero, but there is no extremum at $x_0 = 0$.

Now let $f'(x_0) = 0$. This means that the increment of the function value $y = f(x)$ in a small neighborhood of *stationary* point x_0 is determined by the formula

$$f(x) - f(x_0) = \frac{f''(x_0)}{2}(x - x_0)^2 + o\big((x - x_0)^2\big) \; .$$

Since the factor $(x - x_0)^2$ is non-negative, then *for any* x in a small neighborhood of point x_0 $f(x) > f(x_0)$ with $f''(x_0) > 0$. Likewise $f(x) < f(x_0)$ with $f''(x_0) < 0$.

From above we conclude that $f''(x_0) > 0$ is *a sufficient condition for a local minimum* at x_0. Similarly $f''(x_0) < 0$ is *a sufficient condition for a local maximum.* In the case of $f''(x_0) = 0$ nothing can be said about the extremum at $x_0 = x_0$. Additional research is required there.

Important: these conditions *are not necessary.*

For example, for the functions $f(x) = x^3$, $f(x) = -x^4$ and $f(x) = x^4$ at $x = 0$ the first and second order derivatives are zero. Wherein the second of $f(x)$ has a strict maximum, the third $f(x)$ has a strict minimum, and the first $f(x)$ has no extremum.

It should also be borne in mind that the extrema of the function can exist at points where the derivative does not exist.

For example, for the functions $f(x) = |x|$ there is a minimum at $x_0 = 0$.

By analogy with extremal points for the function , we can define *points of increase* and *points of decrease.*

`Definition 7.3`	A point x_0 is called a point *of increase* of a continuous function $f(x)$, if there exists a δ-neighborhood of it such that $$\forall x \in (x_0 - \delta, x_0) \;\; f(x) < f(x_0) \quad \text{and}$$ $$\forall x \in (x_0, x_0 + \delta) \;\; f(x) > f(x_0).$$

The point of decrease can be defined similarly.

Here, we can note an interesting fact: if a continuous function $f(x)$ is defined on an segment, then not necessarily each point of this segment is either decreasing, or increasing,or extremal.

For example, the point 0 for a continuous on $\mathbb{R}$ function

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0, \end{cases}$$

whose graph is shown in Fig. 1, does not belong to any of the three types indicated.

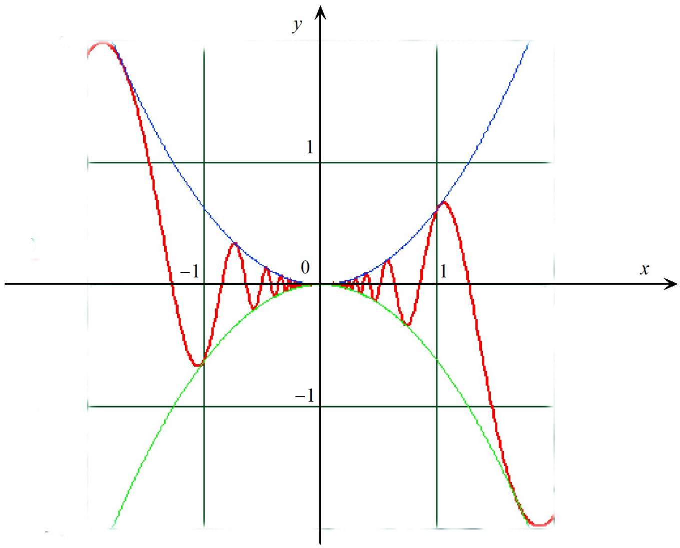

Fig. 1

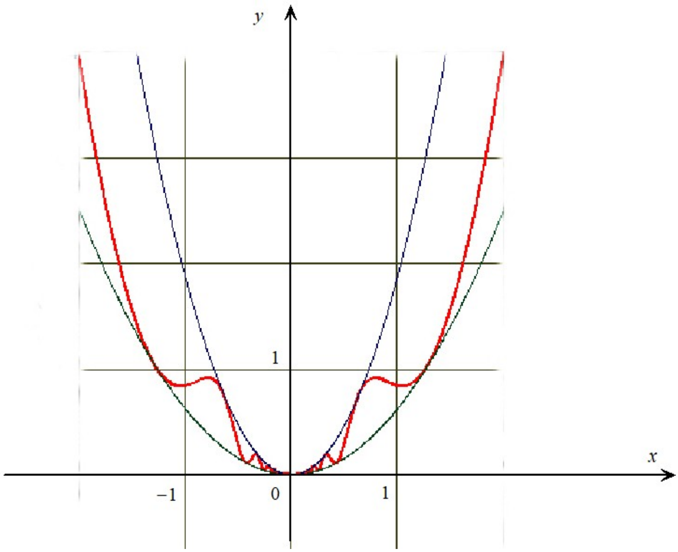

Fig. 2

Let's consider an example of another function

$$f(x) = \begin{cases} x^2\left(2+\cos\dfrac{1}{x}\right) & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Its graph is given in Fig. 2.

This example shows that for a function $f(x)$ continuous in the δ-neighborhood of a point x_0, the condition of decreasing on $(x_0-\delta, x_0)$ and increasing on $(x_0, x_0+\delta)$, is a *sufficient*, but not a necessary for the existence of a minimum at x_0.

A similar scheme can be used to study their other local properties.

To do this, we first give

Definition 7.4a	We will say that the function $y = f(x)$ on a nonzero interval $[a, b]$ *is convex downwards* (*is convex*), if $\forall x_1, x_2$ such that $a \le x_1 < x_2 \le b$ the inequality holds $$\frac{f(x_1) + f(x_2)}{2} \ge f\left(\frac{x_1 + x_2}{2}\right) .$$

And also similar to it

Definition 7.4b	We will say that the function $y = f(x)$ on a nonzero segment $[a, b]$ *is convex upward* (*is concave*), if $\forall x_1, x_2$ such that $a \le x_1 < x_2 \le b$ the inequality holds $$\frac{f(x_1) + f(x_2)}{2} \le f\left(\frac{x_1 + x_2}{2}\right) .$$

It is easy to see that geometrically the condition of convex downwards means that the chord (whose endpoints have coordinates $\left\| \begin{matrix} a \\ f(a) \end{matrix} \right\|$ and $\left\| \begin{matrix} b \\ f(b) \end{matrix} \right\|$) is located on the coordinate plane not lower than the graph of the function $y = f(x)$. While in the case of upward convexity, the points of this chord are located not above the graph, which is illustrated by the following Fig. 3.

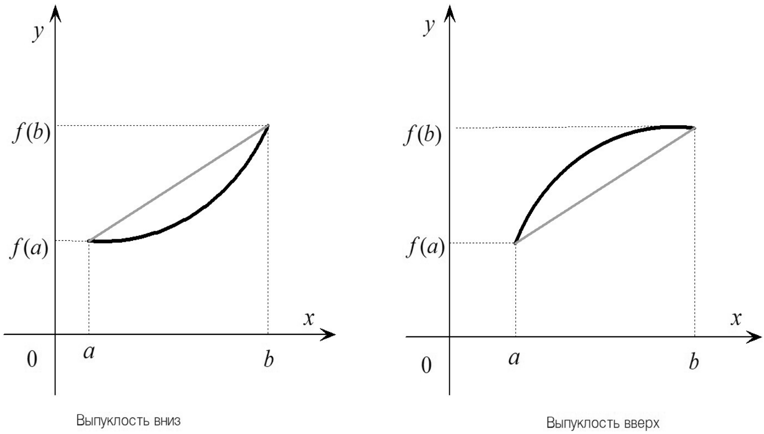

Fig. 3

Let us determine the conditions for the convexity of a twice differentiable function. For this, we use the formula for an approximation of the values of the second derivative of $f(x)$ at small absolute values of t.

$$f''(x_0) \approx \frac{f'(x_0+t)-f'(x_0)}{t} \approx$$

$$\approx \frac{\dfrac{f(x_0+2t)-f(x_0+t)}{t} - \dfrac{f(x_0+t)-f(x_0)}{t}}{t} =$$

$$= \frac{f(x_0+2t)-2f(x_0+t)+f(x_0)}{t^2}\,.$$

We take into account that

1) (by definition 7.4) the inequality

$$f(x_0+t) \le \frac{f(x_0+2t)+f(x_0)}{2}$$

means *downward convexity* of the function $y = f(x)$ in a small neighborhood of the point $x_0 + t$,

2) non-strict inequalities are preserved under limit passages.

Then we come to the conclusion that the condition $f''(x_0) \ge 0$ is sufficient for downward convexity.

Similarly, we obtain that the fulfillment of the inequality $f''(x_0) \le 0$ is sufficient to guarantee *upward convexity* of the function $y = f(x)$ in a small neighborhood of the point x_0.

Thus, a conclusion about the direction of local convexity of the function under study can be made based on the sign of its second derivative.

In the process of studying the function $y = f(x)$, it is also desirable to find

— intervals of monotonicity and points of local extrema (maximums and minimums),

— intervals of convexity (up or down) and inflection points.

In addition, summarizing the obtained information in a special summary table facilitates the construction of the function graph.

Approximate sketch of the graph

In practice, it is often sufficient to construct an approximate sketch of the graph, reflecting only some of the main properties of the function under study.

A tool for constructing such sketches can be the *method of selecting the principal part*. This method consists of finding relatively simple functions that locally approximate the function under study in the neighborhoods of its zeros, vertical asymptotes and at infinity.

Recall that functions $f(x)$ and $g(x)$ are called *equivalent at a point* x_0, if there exists some punctured neighborhood of this point at which $g(x) \neq 0$ and $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1$.

Equivalence is usually denoted as follows: $f(x) \sim g(x)$ at $x \to x_0$.

The method of extracting the principal part is based on the assumption that the function under study $y(x)$ can be approximeted as

$$y(x) \approx (x - x_0)^{\frac{p}{q}}\, g(x)\,, \quad \text{where} \quad g(x_0) \neq 0\,.$$

It is also assumed that p and $q \neq 0$ are integers, and the function $g(x)$ is *continuous* in some neighborhood of the point x_0, that is, it can be represented as

$$g(x) = g(x_0) + \alpha(x)\,, \quad \text{where} \quad \lim_{x \to x_0} \alpha(x) = 0\,.$$

In this case, the principal part of the function $y(x)$ in a small neighborhood of the point x_0 is taken to be the function $g(x_0)\,(x - x_0)^{\frac{p}{q}}$.

Let us explain the method of extracting the principal part with the following example. Let it be required to construct an approximate sketch of the graph of function $y(x) = \dfrac{x^3}{(x-1)^2}$.

Note that $y(x) \sim x^3$ at point $x_0 = 0$, $y(x) \sim \dfrac{1}{(x-1)^2}$ at point $x_0 = 1$ and, finally, $y(x) \sim x$ for $x_0 \to \infty$.

We construct fragments of functions that are locally equivalent to $y(x)$. In Fig. 4, these fragments, which are the corresponding principal parts for the function $y(x)$, are shown in red.

Then we connect the ends of the fragments smoothly with blue lines. The resulting Fig. 4 is the required sketch.

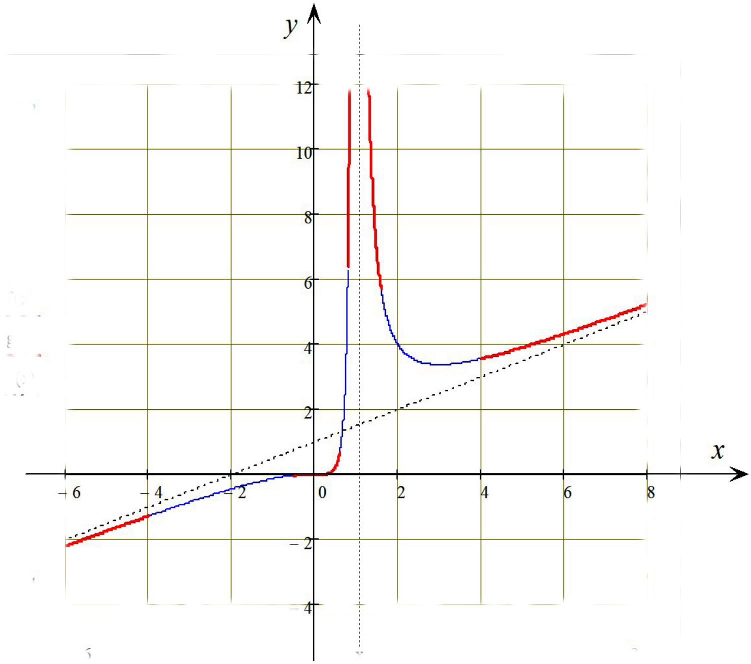

Fig. 4

When constructing a sketch of the graph, you must also look at the sign of the number $g(x_0)$. This characteristic can change only at the zeros of the function or at break points (for example, on vertical asymptotes).

Table 1 shows examples of the main parts of the function under study. For simplicity, this table assumes that the sign of the principal part is positive and that $x_0 = 0$.

Table 1.

VIEW OF THE MAIN PART	EXAMPLE
Simple intersection $\left(\frac{p}{q} = 1\right)$	$y(x) \sim x$
Tangency without intersection $\left(\frac{p}{q} = 2k \quad k \in \mathbb{N}\right)$	$y(x) \sim x^2$
Tangency with intersection $\left(\frac{p}{q} = 2k + 1 \quad k \in \mathbb{N}\right)$	$y(x) \sim x^3$
One-sided vertical tangency $\left(\frac{p}{q} = \frac{1}{2k} \quad k \in \mathbb{N}\right)$	$y(x) \sim \sqrt{x}$
Vertical tangency with intersection $\left(\frac{p}{q} = \frac{1}{2k+1} \quad k \in \mathbb{N}\right)$	$y(x) \sim \sqrt[3]{x}$
Vertical tangency with return $\left(\frac{p}{q} = \frac{2m}{2k+1} \quad k \geq m \in \mathbb{N}\right)$	$y(x) \sim \sqrt[3]{x^2}$
Two-sided vertical asymptote $\left(\frac{p}{q} = -2k + 1 \quad k \in \mathbb{N}\right)$	$y(x) \sim \frac{1}{x}$
One-sided vertical asymptote $\left(\frac{p}{q} = -2k \quad k \in \mathbb{N}\right)$	$y(x) \sim \frac{1}{x^2}$

This table could be significantly expanded, but the given cases are quite sufficient to clarify the method for extracting the principal part.

Figure 5 shows the graphs of the principal parts included in the following table.

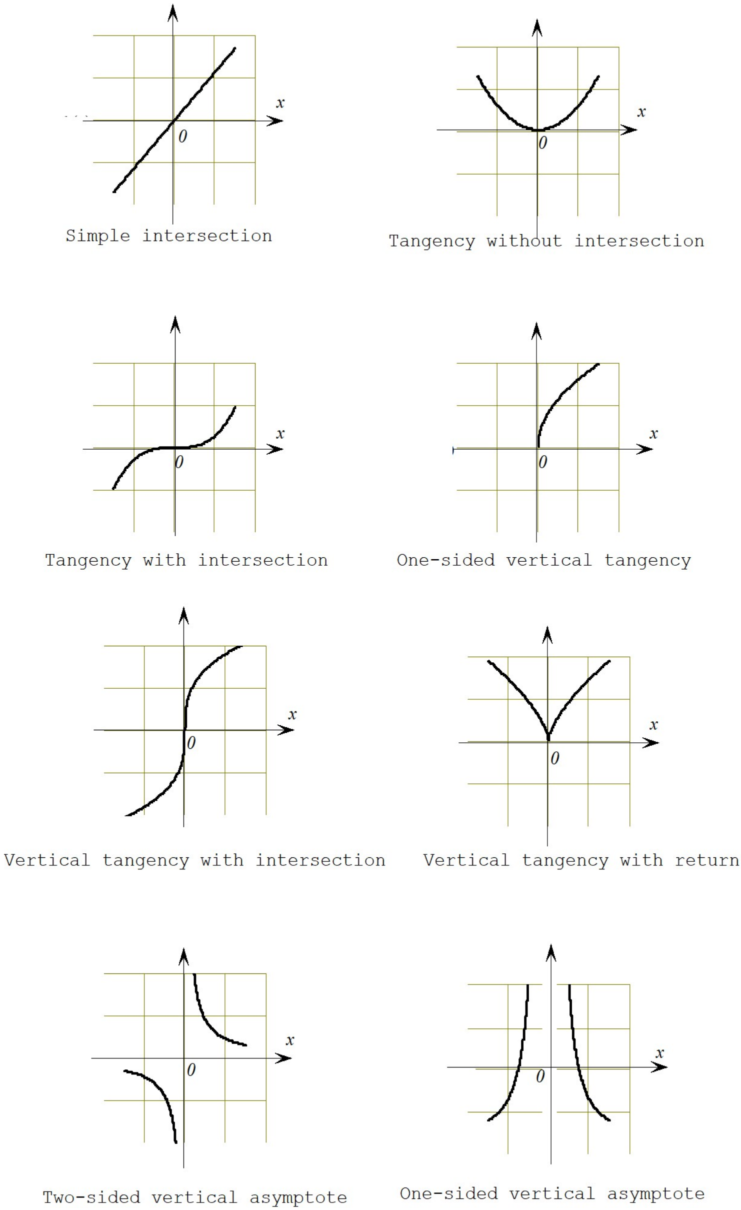
x
0
Simple intersection
x
0
Tangency without intersection
x
0
Tangency with intersection
x
0
One-sided vertical tangency
x
0
Vertical tangency with intersection
x
0
Vertical tangency with return
x
0
Two-sided vertical asymptote
x
0
One-sided vertical asymptote

Fig. 5

Let us consider two more examples illustrating the application of the principal part extraction method.

Example 7.1. Sketch the graph of the function

$$y(x) = \frac{(x+2)^2 x^3 (x-3)}{(x+1)^2 (x-2)}.$$

Solution. Let us construct local approximations of this function for each of the points at which the principal part extraction method is applicable. We obtain the following set of main parts:

x_0	MAIN PART at x_0	MAIN PART VIEW
$-\infty$	$y(x) \sim x^3$	Cubic parabola
-2	$y(x) \sim -10(x+2)^2$	Tangency without intersection
-1	$y(x) \sim -\frac{4}{3}\frac{1}{(x+1)^2}$	One-sided vertical asymptote
0	$y(x) \sim 6x^3$	Tangency with intersection
2	$y(x) \sim -\frac{128}{9}\frac{1}{x-2}$	Two-sided vertical asymptote
3	$y(x) \sim \frac{675}{16}(x-3)$	Simple intersection
$+\infty$	$y(x) \sim x^3$	Cubic parabola

We construct the principal parts locally equivalent to $y(x)$ for the selected points x_0 . In Fig. 6, these principal parts for the function $y(x)$, are shown in blue. Then we smoothly connect the ends of the principal parts with black lines. The resulting Fig. 6 is the required sketch.

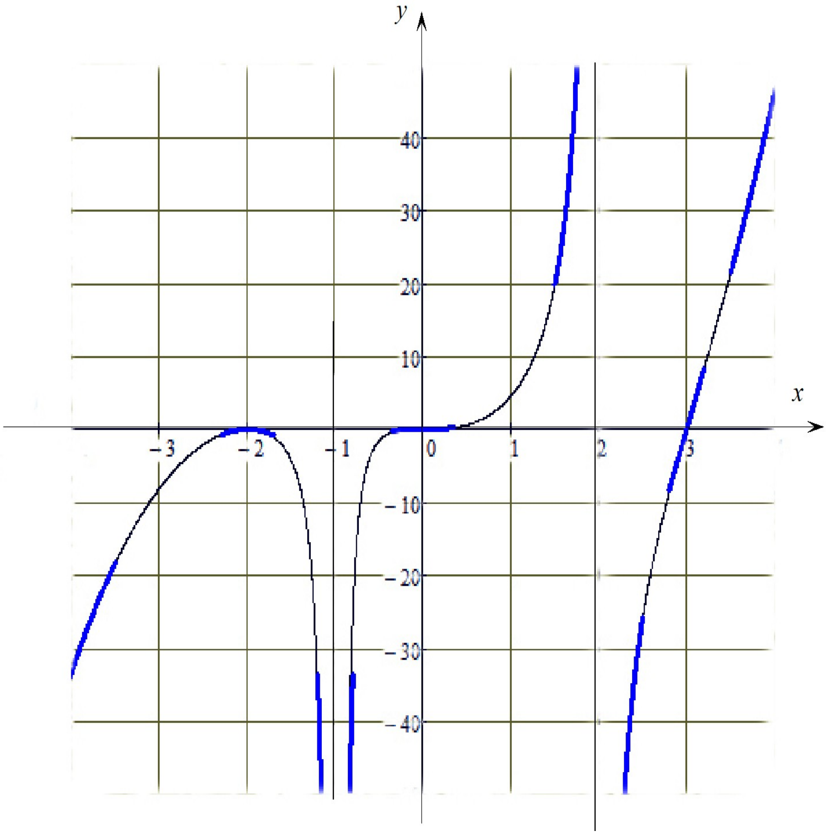

Fig. 6

Example 7.2. Sketch the graph of the function

$$y(x) = \sqrt[3]{\sin^2 x}\,.$$

Solution. This function $y(x) = p\Big(q(x)\Big)$ is a composite function, that is, a superposition of other functions.
The inner function $q(x) = \sin^2 x$ is even, π-periodic and continuous. It has obvious local extrema: zero minima at the points $\pi k\,,\ k \in \mathbb{Z}$ and maxima equal to 1 at the points $\dfrac{\pi}{2} + \pi k\,.$

The outer function $p(x) = \sqrt[3]{x}$ is defined, continuous and monotone on the entire real axis. Therefore, the function $y(x)$ will also be even, continuous, π-periodic and having local extrema at the same points as $q(x)\,.$

It is also easy to verify that $y(x)$ has finite derivatives of the first and second orders at all points, except $\pi k\,,\ k \in \mathbb{Z}\,.$

A description of the behavior of the studied function in the neighborhoods of extreme points is obtained by applying the method of extracting the principal part .

From the Maclaurin formula for the function $\sin x$ it follows that $\sin x \sim x$ for $x_0 = 0$. Therefore, $y(x) \sim \sqrt[3]{(x-x_0)^2}$ in the neighborhood of the points $x_0 = \pi k$, $k \in \mathbb{Z}$. According to the sixth row of Table 1 these will be cusp points with a vertical tangent.

At points $x_0 = \dfrac{\pi}{2} + \pi k$ we have $y(x_0) = 1$, $y'(x_0) = 0$ and $y''(x_0) = -\dfrac{2}{3}$. Then by Taylor's formula

$$y(x) = 1 - \frac{1}{3}(x-x_0)^2 + o\Big((x-x_0)^3\Big),$$

that is, at points x_0 the main part of the function $y(x)$ is equal to $1 - \dfrac{1}{3}(x-x_0)^2$.

As a result, the required sketch of the graph of the function $y(x) = \sqrt[3]{\sin^2 x}$ will have the form shown in Fig. 7.

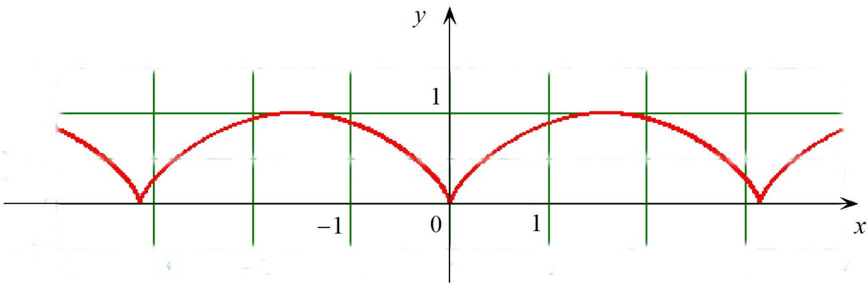

Fig. 7

Examples of detailed study of functions

Let us now consider examples of detailed study of functions and constructing their graphs.

`Example 7.3.` Investigate the function $y = x^3 - 2x + 1$ and construct its graph.

1°. *Domain of definition*: all operations used to write the function formula are feasible for any real x. So $X : x \in (-\infty, +\infty)$.

2°. *Domain of values*: it is obvious that the equation $x^3 = -2x - 1 + p$ has real solutions for any real value of the parameter p. That is, the domain of values is the set of all real numbers.

3°. This function $y = f(x)$ does not have the properties *of being even, odd, and periodic.*

4°. Since
$$\lim_{x\to\infty} \frac{x^3 - 2x + 1}{x^3} = \lim_{x\to\infty} \left(1 - \frac{2}{x^2} + \frac{1}{x^3}\right) = 1 \ ,$$
then for large absolute values of x this function is equivalent to the function $y = x^3$.

5°. *Intervals of constancy of sign*:
we transform the formula of the function

$$y = x^3 - 2x + 1 = (x^3 - x) - (x - 1) = (x - 1)(x^2 + x - 1) =$$

$$= (x - x_1)(x - x_2)(x - 1) \ ,$$

where

$$x_1 = -\frac{1}{2} - \frac{\sqrt{5}}{2} \approx -1.6 \qquad x_2 = -\frac{1}{2} + \frac{\sqrt{5}}{2} \approx 0.6 \ .$$

Using the «interval method», we conclude that

$$y \geq 0, \text{ if } \left[\begin{array}{r} x_1 \leq x \leq x_2 \\ x \geq 1 \end{array} \right. \qquad \text{and} \qquad y < 0, \text{ if } \left[\begin{array}{r} x < x_1 \\ x_2 < x < 1 \ . \end{array} \right.$$

6°. *Intervals of monotonicity*:

according to definition 7.3 the value of the differentiable function increases in the neighborhood of the point, where the derivative is positive, and decreases where the derivative is negative.

Therefore, to determine the intervals of increasing or decreasing values of the function, it is necessary to find its derivative function and examine it «for the sign».

In our case $y' = 3x^2 - 2$, so we can say that

$$y' \geq 0, \text{ if } x \leq -\sqrt{\frac{2}{3}}\,, \text{ or } x \geq \sqrt{\frac{2}{3}}$$

and

$$y' < 0, \text{ if } -\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}\,.$$

7°. *Direction of convexity*:

points $x_3 = -\sqrt{\dfrac{2}{3}}$ and $x_4 = \sqrt{\dfrac{2}{3}}$ – stationary, at them the first derivative vanishes.

The fact that there is an extremum at these points can be established by the sign of the second derivative, which is $y'' = 6x$. This means that for positive x the function graph has a downward convexity, for negative values it has an upward convexity, and for $x = 0$ the direction of convexity changes to the opposite, that is, $x = 0$ is *an inflection point* of the function graph.

It follows that at the point $x_3 = -\sqrt{\dfrac{2}{3}}$ the function has a local maximum, and at the point $x_4 = \sqrt{\dfrac{2}{3}}$ – a local minimum.

8°. *Summary table of function properties*

x	$-\infty$	$-$	x_1	$-$	x_3	$-$	0	$+$	x_2	$+$	x_4	$+$	1	$+$	$+\infty$
y	$-\infty$	$-$	0	$+$	$+$	$+$	1	$+$	0	$-$	$-$	$-$	0	$+$	$+\infty$
y'	$+\infty$	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	0	$\nearrow$	$\nearrow$	$\nearrow$	$+\infty$
y''	$-\infty$	$\bigcap$	$\bigcap$	$\bigcap$	$\bigcap$	$\bigcap$	0	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$+\infty$
					max		infl.				min				

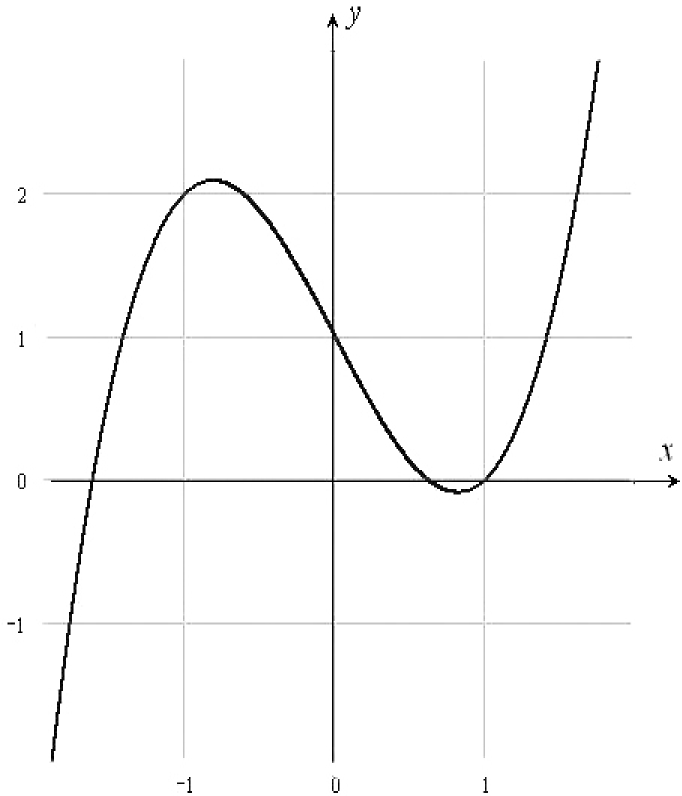

Fig. 8. Graph of $y = x^3 - 2x + 1$

Example 7.4. Investigate the function $y = \dfrac{x^3 + 6x - 2}{x^2}$ and plot its graph.

1°. *Domain of definition*: any real $x \neq 0$.

2°. *Range of value*: any real y.

3°. This function does not have the properties *even, odd, or periodic.*

4°. The graph of the function has a one-sided vertical asymptote at $x \to 0$, since $\lim\limits_{x\to 0} \frac{x^3+6x-2}{x^2} = -\infty$.
In addition, rewriting the formula defining the function under study as $y = x + \frac{6}{x} - \frac{2}{x^2}$, we can see that this function has *an oblique asymptote* of the form $y = x$.
This linear function can be considered as *the principal part* on $\pm\infty$.

5°. *Intervals of sign constancy*:

from the formula $y = x + \frac{6}{x} - \frac{2}{x^2}$, it is obvious that $y < 0$ for any $x < 0$. In the region $x > 0$ the sign of y coincides with the sign of the trinomial $x^3 + 6x - 2$, which is negative for $x < \sqrt[3]{4} - \sqrt[3]{2}$ and positive for $x > \sqrt[3]{4} - \sqrt[3]{2}$.

This statement is clearly not obvious and therefore we will provide its justification.

We need to find the real roots of the equation $x^3 + 6x - 2 = 0$. We will look for them as the sum of two new unknowns u and v, that is, $x = u + v$. Substituting them into the equation, we get

$$(u+v)^3+6(u+v)-2 = 0 \quad \Longrightarrow \quad \big(u^3 + v^3 + 3uv(u + v)\big)+6(u+v)-2 = 0$$

$$\Longrightarrow \quad u^3 + v^3 - 2) + (u + v)(3uv + 6) = 0 \ .$$

For x to be a solution to this equation, the values of the unknowns u and v must satisfy the system of equations

$$\begin{cases} u^3 + v^3 = 2, \\ uv = -2, \end{cases}$$

which in turn is equivalent to the system

$$\begin{cases} v = -\frac{2}{u}, \\ u^3 - \frac{8}{u^3} = 2. \end{cases}$$

We introduce a new unknown $t = u^3$. Then the second equation of the last system is reduced to a quadratic equation:

$$t - \frac{8}{t} - 2 = 0 \quad \Longrightarrow \quad t^2 - 2t - 8 = 0 \quad \Longrightarrow \quad \left[\begin{array}{l} t_1 = 4, \\ t_2 = -2 \end{array} \right. \quad \Longrightarrow$$

$$\Longrightarrow \quad \left[\begin{array}{l} u_1 = \sqrt[3]{4}, \\ u_2 = -\sqrt[3]{2} \end{array} \right. \quad \Longrightarrow \quad \left[\begin{array}{l} v_1 = -\sqrt[3]{2}, \\ v_2 = \sqrt[3]{4}. \end{array} \right.$$

It follows from the original system that $u^3 + v^3 = 2$. Therefore, the real solution of the equation $x^3 + 6x - 2 = 0$ will be the number $x_3 = u_1 + v_1 = \sqrt[3]{4} - \sqrt[3]{2}$.

6°. *Monotonicity intervals*:

for this function we have

$$y' = \frac{x^3 - 6x + 4}{x^3} = \frac{(x-2)(x^2+2x-2)}{x^3} =$$

$$= \frac{(x-2)(x+1-\sqrt{3})(x+1+\sqrt{3})}{x^3} .$$

Therefore it can be stated that

$$y' = 0, \text{ if } \left[\begin{array}{l} x = 2, \\ x = -1+\sqrt{3}, \\ x = -1-\sqrt{3} \end{array} \right.$$

and, according to the «interval method»,

$$y' > 0, \text{ if } x \in (-\infty, -1-\sqrt{3}) \bigcup (0, -1+\sqrt{3}) \bigcup (2, +\infty)$$

$$\text{and} \quad y' < 0, \text{ if } (-1-\sqrt{3}, 0) \bigcup (-1+\sqrt{3}, 2) .$$

7°. *Direction of convexity*:

the second derivative is $y'' = \frac{12(x-1)}{x^4}$.

Therefore, for $x < 1$ the graph of the function has convexity upward, and for $x > 1$ it has convexity downward.

It follows that at the point $x = 2$ the function has a local minimum, at the points $x_1 = -1-\sqrt{3}$ and $x_2 = -1+\sqrt{3}$ it has a local minimum, and at the point $x = 1$ it has an inflection point of the graph of the function.

8°. *Summary table of function properties*

x	$-\infty$	$-$	x_1	$-$	0	$+$	x_3	$+$	x_2	$+$	1	$+$	2	$+$	$+\infty$
y	$-\infty$	$-$	$-$	$-$	$-\infty$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+\infty$
y'	1	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\searrow$	$\searrow$	0	$\nearrow$	1
y''	-0	$\bigcap$	$\bigcap$	$\bigcap$	$-\infty$	$\bigcap$	$\bigcap$	$\bigcap$	$\bigcap$	$\bigcap$	0	$\bigcup$	$\bigcup$	$\bigcup$	$+0$
	asim.		max		asim.				max		infl.		min		asim.

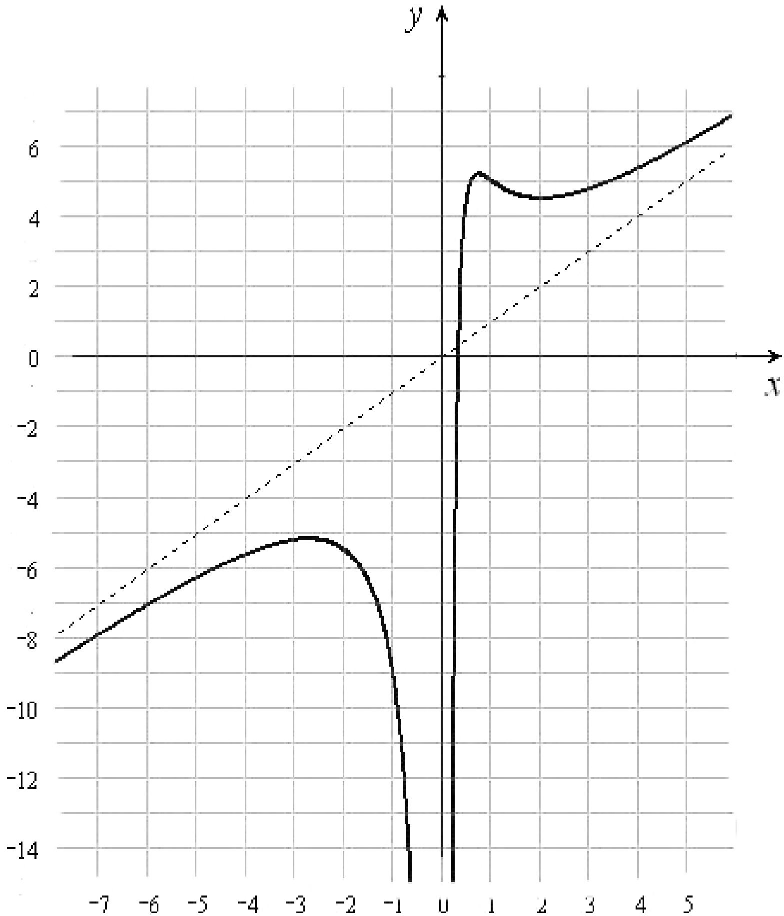

Fig. 9. Graph of $y = \frac{x^3+6x-2}{x^2}$

`Example 7.5.` Investigate the function $y = \sqrt[3]{x^3 - 2x^2}$ and plot its graph.

1°. *Domain*: $y(x)$ is defined and continuous for any real x, that is, $\forall x \in (-\infty, +\infty)$.

2°. *Range*: obviously, the range is the set of all real numbers.

3°. The function does not have the properties *even, odd or periodic.*

4°. We have

$$\sqrt[3]{x^3 - 2x^2} = x\sqrt[3]{1 - \frac{2}{x}} = x\left(1 - \frac{2}{3x} + o\left(\frac{1}{x}\right)\right) = x - \frac{2}{3} + O\left(\frac{1}{x}\right) \ ,$$

by Taylor's formula. Therefore, the graph of this function asymptotically approaches to the line $y = x - \dfrac{2}{3}$ at $x \to \infty$ as the principal part.

Note that the equation of the oblique asymptote $y = ax + b$ in this example could also be obtained using the limits

$$a = \lim_{x\to\pm\infty} \frac{y(x)}{x} = \lim_{x\to\infty} \frac{\sqrt[3]{x^3 - 2x}}{x} = 1\,,$$

$$b = \lim_{x\to\pm\infty} \big(y(x) - ax\big) = \lim_{x\to\infty} \big(\sqrt[3]{x^3 - 2x^2} - x\big) =$$

$$= \lim_{x\to\infty} \frac{x^3 - 2x^2 - x^3}{\sqrt[3]{(x^3 - 2x^2)^2} + x\sqrt[3]{x^3 - 2x^2} + x^2} =$$

$$= \lim_{x\to\infty} \frac{-\,2}{\sqrt[3]{\left(1 - \dfrac{2}{x}\right)^2} + \sqrt[3]{1 - \dfrac{2}{x}} + 1} = -\frac{2}{3}\,.$$

5°. *Intervals of sign constancy*: The expression under the root in $\sqrt[3]{x^3 - 2x^2}$ can be represented as $x^2(x-2)$, whence it follows that $y(x)$ is negative on $(-\infty, 0) \bigcup (0, 2)$, is positive on $(2, +\infty)$ and is equal to 0 at $x_1 = 0$ and $x_2 = 2$.

6°. *Monotonicity intervals and extrema*: In the case under consideration, we find using

$$y'(x) = \frac{3x - 4}{3\sqrt[3]{x(x-2)^2}} \qquad \text{for} \quad x \neq 0 \quad \text{and} \quad x \neq 2\,.$$

At the points $x_1 = 0$ and $x_2 = 2$, the absolute value of $y'(x)$ tends to infinity. Moreover, in any sufficiently small neighborhood of the point $x_2 = 2$, the value of $y'(x)$ is positive. When passing through the point $x_1 = 0$, the derivative changes its sign from plus to minus.

Geometrically, this means that the tangents to the graph $y(x)$ at these points are vertical.

Finally, it is obvious that when passing through the point $x_1 = 0$, the derivative changes its sign from plus to minus and at the same time $y(0) = 0$. This means that at zero the function has a local *maximum*.

At the point $x_3 = \frac{4}{3}$ the derivative is zero and when passing through it changes sign from minus to plus, we have a local *minimum* with the value $y\left(\dfrac{4}{3}\right) = -\dfrac{2\sqrt[3]{4}}{3}$.

7°. *The direction of convexity* can be determined by the sign of the second derivative. In this case, it is equal to

$$y''(x) = -\frac{8}{3x(x-2)\sqrt[3]{x(x-2)^2}} .$$

Check it yourself.
From this formula it follows that $\forall x \in (-\infty, 0) \bigcup (0, 2)$ the second derivative is positive and, therefore, the graph of the function has a downward convexity, and for $2 < x < +\infty$ – an upward convexity.

Thus, at the point $x_2 = 2$ the function has an inflection. At the point $x_1 = 0$ we have a singularity called the *cusp*.

8°. *Summary table of function properties*

x	$-\infty$	$-$	0	$+$	$\frac{4}{3}$	$+$	2	$+$	$+\infty$
y	$-\infty$	$-$	0	$-$	$-\frac{2\sqrt[3]{4}}{3}$	$-$	0	$+$	$+\infty$
y'	$\to 1$	$\nearrow$	$\pm\infty$	$\searrow$	0	$\nearrow$	$+\infty$	$\nearrow$	$\to 1$
y''	$to + 0$	$\bigcup$	$+\infty$	$\bigcup$	$\bigcup$	$\bigcup$	$\pm\infty$	$\bigcap$	$\to -0$
	asim.		max		min.		infl.		asim.

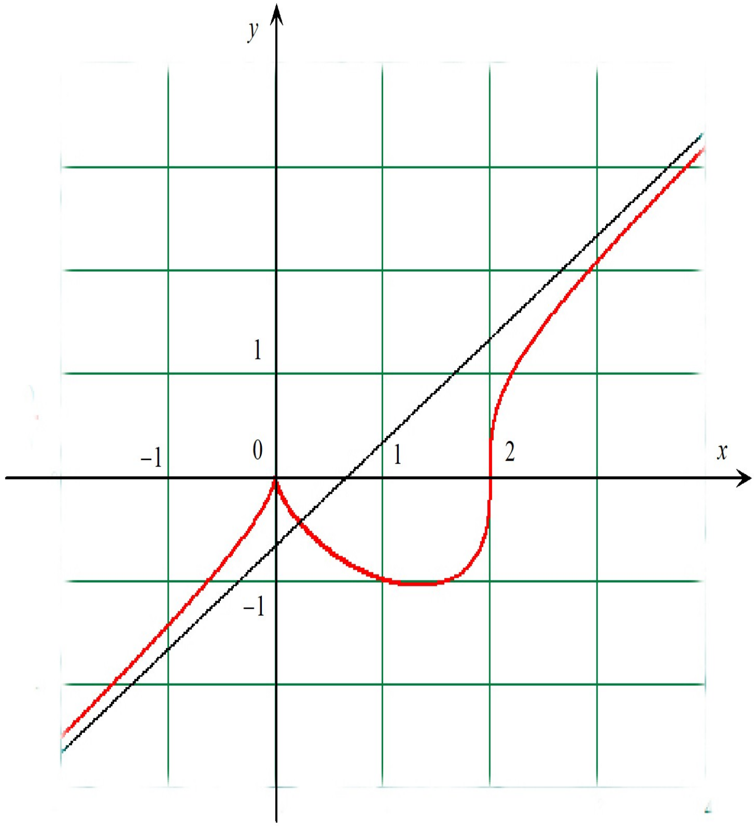

Fig. 10. Graph of $y = \sqrt[3]{x^3 - 2x^2}$

`Example 7.6.` Study the function $y(x) = 2x + \sqrt{|x^2 - 1|}$ and plot its graph.

1°. *Domain*: $y(x)$ is defined and continuous $\forall x \in (-\infty, +\infty)$. Note that the function under study can be written as follows

$$y(x) = \begin{cases} 2x + \sqrt{x^2 - 1} & \text{for} \quad |x| \geq 1\,, \\ 2x + \sqrt{1 - x^2} & \text{for} \quad |x| < 1\,. \end{cases}$$

2°. *Range*: Obviously, the domain is the set of all real numbers.

3°. The function does not have the properties *even, odd, or periodic.*

4°. The equation of the oblique asymptote $y = ax + b$ in this example is obtained separately for $+\infty$ and $-\infty$ using the limits for the coefficient a

$$a = \lim_{x\to\pm\infty} \frac{y(x)}{x} = \lim_{x\to\pm\infty} \frac{2x + \sqrt{x^2-1}}{x} =$$

$$= \lim_{x\to\pm\infty} \left(2 + \frac{|x|}{x} \cdot \frac{\sqrt{x^2-1}}{|x|} \right) = \begin{cases} 3 & \text{for} \quad x \to +\infty\,, \\ 1 & \text{at} \quad x \to -\infty \end{cases}$$

and, accordingly, for the coefficient b

$$b = \lim_{x\to\pm\infty} \big(y(x) - ax\big) = \lim_{x\to\pm\infty} \big(2x + \sqrt{x^2-1} - ax\big)\,,$$

where we have two cases:

$$b_+ = \lim_{x\to+\infty} \big(\sqrt{x^2-1} - x\big) = \lim_{x\to+\infty} \frac{-\,1}{\sqrt{x^2-1}+x} = 0$$

and

$$b_- = \lim_{x\to-\infty} \big(\sqrt{x^2-1} + x\big) = \lim_{x\to-\infty} \frac{-\,1}{\sqrt{x^2-1}-x} = 0$$

So, we have two asymptotes: $y = 3x$ at $x \to +\infty$ and $y = x$ at $x \to -\infty$.

5°. *Intervals of constant sign*: From the definition of the arithmetic square root it follows that solutions of the equation

$$2x + \sqrt{|x^2 - 1|} = 0$$

(if such exist) can only be non-positive numbers.
After squaring we arrive at the equation

$$4x^2 = |x^2 - 1|,$$

which has no roots for $x \leq -1$, and on the half-interval $-1 < x \leq 0$ there is a single root $x_1 = -\frac{1}{\sqrt{5}}$.
The function under study is continuous on the entire real axis and the point x_1 is its only zero. It gives that $y < 0$ for $x < x_1$ and $y > 0$ for $x > x_1$.

6°. *Monotonicity intervals and extrema* we will search by calculating the derivative function

$$y'(x) = \begin{cases} 2 + \dfrac{x}{\sqrt{x^2-1}} & \text{for} \quad |x| > 1\,, \\[2ex] 2 - \dfrac{x}{\sqrt{1-x^2}} & \text{for} \quad |x| < 1\,. \end{cases}$$

At the points $x_2 = -1$ and $x_3 = 1$ the absolute value of $y'(x)$ tends to infinity, and when passing these points the derivative changes its sign from minus to plus. These points are local *minima* of the function, which are cusp points, where the tangent to the graph is vertical.

On the other hand, at the points $x_4 = -\frac{2}{\sqrt{3}}$ and $x_5 = \frac{2}{\sqrt{5}}$ the derivative is zero, and when passing through these points it changes sign from plus to minus. These points for the function under study will be points of strict local *maximum* with a horizontal tangent at them.

7°. *The direction of convexity* is determined by the sign of the second derivative, which is equal to

$$y''(x) = \begin{cases} -\dfrac{1}{(x^2-1)^{\frac{3}{2}}} & \text{for} \quad |x| > 1\,, \\ -\dfrac{1}{(1-x^2)^{\frac{3}{2}}} & \text{for} \quad |x| < 1\,. \end{cases}$$

It follows from this formula that $\forall x \in \mathbb{R}$ such that $|x| \neq 1$, the second derivative is negative and, therefore, the graph of the function has convexity upward. At the points $|x| = 1$ the second derivative tends to $-\infty$.

8°. *Summary table of function properties*

x	$-\infty$	$-$	$-\frac{2}{\sqrt{3}}$	$-$	-1	$-$	$-\frac{1}{\sqrt{5}}$	$\mp$	$\frac{2}{\sqrt{5}}$	$+$	1	$+$	$+\infty$
y	$-\infty$	$-$	$-\frac{1}{\sqrt{3}}$	$-$	-2	$-$	0	$+$	$\sqrt{5}$	$+$	2	$+$	$+\infty$
y'	$\to 1$	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\to 3$
y''	$\to 0$	$\bigcap$	$\bigcap$	$\bigcap$	$-\infty$	$\bigcap$	$\bigcap$	$\bigcap$	$\bigcap$	$\bigcap$	$-\infty$	$\bigcap$	$\to 0$
	asim.		max		min				max		min		asip.

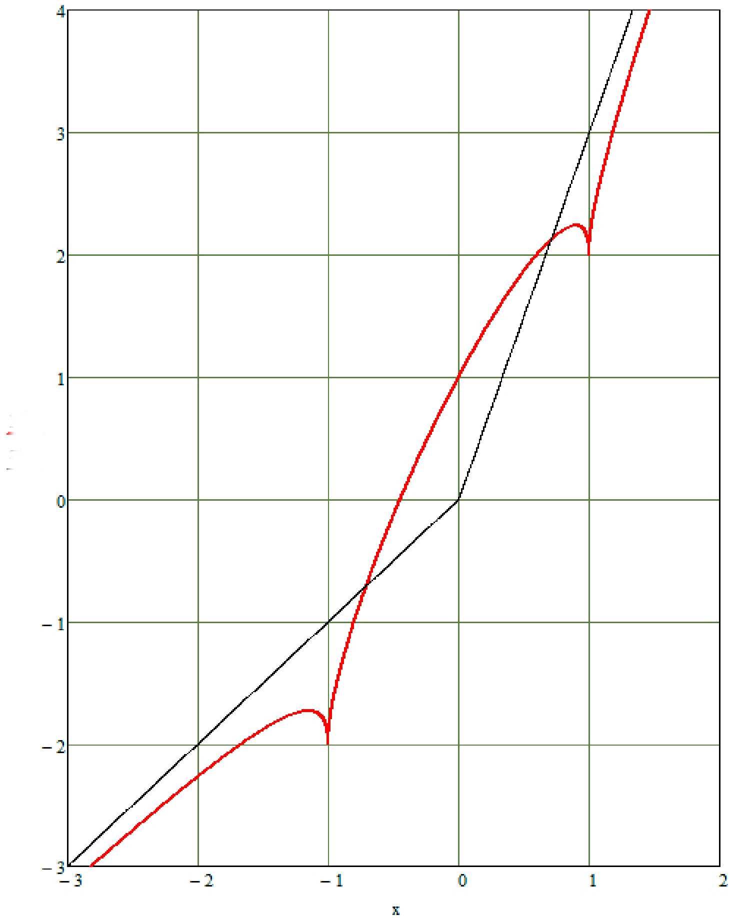

Fig. 11 Graph of $y = 2x + \sqrt{|x^2 - 1|}$

`Example 7.7.` Investigate the function $y = (2x+5)e^x$ and plot its graph.

1°. *Domain*: since the operations used to write the function formula are feasible for any real x, then $X : x \in (-\infty, +\infty)$.

2°. *Range*: obviously, for $x \geq 0$ the value of y is infinitely large. If $x \leq 0$, then by virtue of $\lim\limits_{x\to-\infty} (2x+5)e^x = 0$ and the continuity of this function for any x, we come to the conclusion about its «boundedness from below».
In this case, it will be more convenient to find the lower boundary of the range of values from the stationarity condition, since the function under study is not only continuous, but also has a derivative for any value of x.

3°. The function $y = (2x+5)e^x$ does not have the properties *of being even, odd, or periodic.*

4°. Since $\lim\limits_{x\to-\infty}(2x+5)e^x = 0$, the graph of this function asymptotically approaches the Ox axis as $x \to -\infty$.

5°. *Intervals of sign constancy*: Since $e^x > 0\ \forall x$, then $y = 0$ for $x = -\frac{5}{2}$ and

$$y < 0,\ \text{if}\ -\infty < x < -\frac{5}{2} \quad \text{and} \quad y > 0,\ \text{if}\ -\frac{5}{2} < x < +\infty$$

6°. *Monotonicity intervals*: In the case under consideration

$$y' = 2e^x + (2x+5)e^x = (2x+7)e^x\ ,$$

whence it follows that

$$y' = 0,\ \text{if}\ x = -\frac{7}{2}\ ,$$

that is,

$$y' < 0,\ \text{if}\ -\infty < x \le -\frac{7}{2} \quad \text{and} \quad y' > 0,\ \text{if}\ -\frac{7}{2} < x < +\infty\ .$$

7°. *The direction of convexity* can be determined by the sign of the second derivative, which in this case is equal to

$$y'' = (2x+9)e^x.$$

That is, for $-\infty < x < -\frac{9}{2}$ the graph of the function has an upward convexity, and for $-\dfrac{9}{2} < x < +\infty$ – a downward convexity.

It follows that at the point $x = -\dfrac{7}{2}$ the function has a *local minimum*, and at the point $x = -\dfrac{9}{2}$ it has an *inflection point* of the graph of the function.

8°. *Summary table of function properties*

x	$-\infty$	$-$	$-\frac{9}{2}$	$-$	$-\frac{7}{2}$	$-$	$-\frac{5}{2}$	$+$	0	$+$	$+\infty$
y	$\to -0$	$-$	$-$	$-$	$-$	$-$	0	$+$	5	$+$	$+\infty$
y'	$\to -0$	$\searrow$	$\searrow$	$\searrow$	0	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$+\infty$
y''	$\to -0$	$\bigcap$	0	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$	$+\infty$
	asim.		infl.		min						

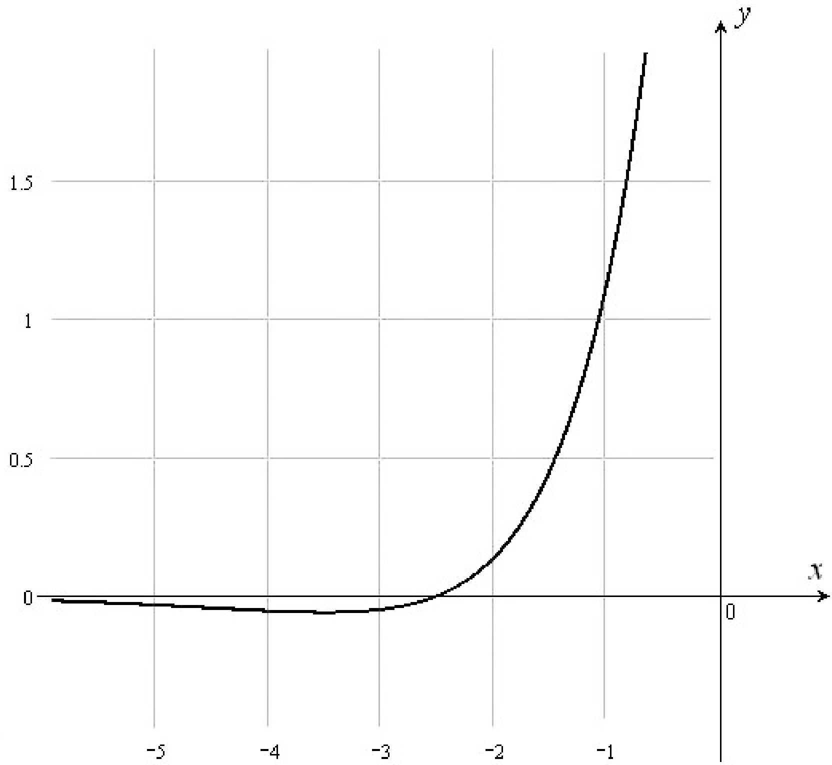

Fig. 12. Graph of $y = (2x + 5)e^x$