Tracing of lines defined parametrically or by unsolvable equations

Tracing of lines defined parametrically

We will say that a line is defined *parametrically* in a rectangular Cartesian coordinate system Oxy, if the relationship between the coordinates of the points of the line is given by the system

$$\left\{ \begin{array}{l} x = x(t), \\ y = y(t), \end{array} \right. \tag{8.1}$$

where t belongs to some interval of $\mathbb{R}$.

To plot a lines defined parametrically, you can use, for example, the following sequence of actions.

1°. Study the functions $x(t)$ and $y(t)$. Plot sketches of their graphs.

2°. Choose for the functions $x(t)$ and $y(t)$ values of t, such as $\left\| \begin{matrix} x(t) \\ y(t) \end{matrix} \right\| \to \left\| \begin{matrix} p \\ q \end{matrix} \right\|$, where at least one of the symbols p or q is either 0 or ∞.

Such values of t (for brevity) will be called *support or reference values.*

3°. Find the relationship between x and y in a small neighborhood of the support values of t.

4°. Use information from 1°, 2° and 3° to construct an approximate sketch of the line.

5°. Calculate the derivatives of $y'_x(t)$ and $y''_{xx}(t)$. Determine their support points. When calculating the derivatives you can use the following formulas

$$y'_x(t) = \frac{y'_t(t)}{x'_t(t)} \qquad \text{and} \qquad y''_{xx}(t) = \frac{y''_{tt}(t)x'_t(t) - y'_t(t)x''_{tt}(t)}{\big(x'_t(t)\big)^3}\,.$$

6°. Using the information from 5°, find intervals of monotonicity, points of local extrema, as well as directions of convexity and points of inflection of the line.

7°. Reduce the obtained information about the line behavior to the final table.

8°. Draw a refined sketch of the line.

`Example 8.1.` Draw a line specified parametrically

$$x(t) = \frac{t^2}{1-2t}, \qquad y(t) = \frac{t^3}{1-2t}.$$

`Solution.` 1°. The graphs of the functions $x(t)$ and $y(t)$ are shown in Fig. 1. Moreover, the left graph also shows the oblique asymptote $u(t) = -\frac{1}{2}t - \frac{1}{4}$ (highlighted in blue).

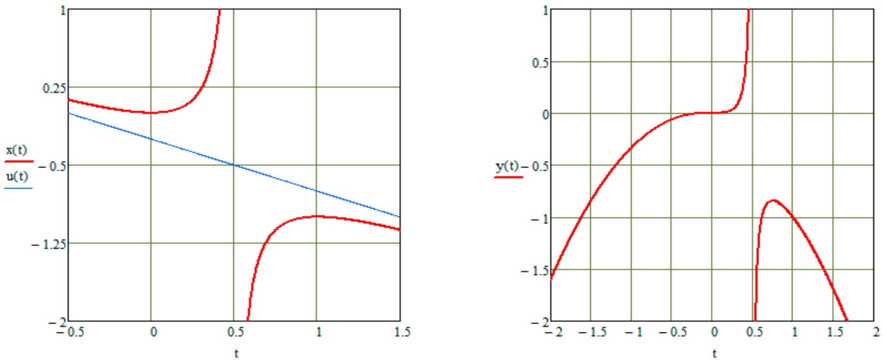

Fig. 1. Graphs of $x(t)$ and $y(t)$

2°. We find the support points for the system of functions $x(t)$ and $y(t)$

$t \to$	$-\infty$	0	$\frac{1}{2}$	$+\infty$
$x(t) \to$	$+\infty$	0	$\pm\infty$	$+\infty$
$y(t) \to$	$-\infty$	0	$\pm\infty$	$+\infty$

and define relationship between x and y for them.

1) For the reference point $\pm\infty$ we have $x(t) \sim -\dfrac{t}{2}$ and, since $y(t) = tx(t)$, then $y \sim -2x^2$.

2) For the reference point 0 from $x(t) \sim t^2$ and $y(t) \sim t^3$ we get $y \sim x\sqrt{x}$.

3) Finally, for the reference point $\frac{1}{2}$, we note that

$$a = \lim_{t\to\frac{1}{2}} \frac{y(t)}{x(t)} = \lim_{t\to\frac{1}{2}} t = \frac{1}{2}.$$

In this case

$$b = \lim_{t\to\frac{1}{2}} \big(y(t) - ax(t)\big) = \lim_{t\to\frac{1}{2}} \left(\frac{t^3}{1-2t} - \frac{1}{2}\frac{t^2}{1-2t} \right) =$$

$$= \lim_{t\to\frac{1}{2}} \left(-\frac{1}{2}t^2 \right) = -\frac{1}{8}.$$

That is, at this reference point there is an oblique asymptote given by the equation $y = \dfrac{1}{2}x - \dfrac{1}{8}$.

3°. Using the formulas

$$x'_t(t) = -\frac{2t(t-1)}{(2t-1)^2} \qquad \text{and} \qquad y'_t(t) = \frac{t^2(3-4t)}{(2t-1)^2}$$

we obtain

$$y'_x(t) = \frac{t(4t-3)}{2(t-1)} \qquad \text{and} \qquad y''_{xx}(t) = \frac{(2t-1)^3(2t-3)}{4t(1-t)^3}.$$

Note that the last equality can also be obtained using

$$y''_{xx}(t) = \frac{y''_{xt}(t)}{x'_t(t)}.$$

4°. Let us now find the reference points for the derivatives $y'_x(t)$ and $y''_{xx}(t)$. Taking into account the signs of the derivatives, we make the following table:

$t \to$	$-\infty$		0		$\frac{1}{2}$		$\frac{3}{4}$		1		$\frac{3}{2}$		$+\infty$
$y'_x(t) \to$	$-\infty$	$\searrow$	0	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\nearrow$	$\nearrow$	$+\infty$
$y''_{xx}(t) \to$	-2	$\bigcap$	$\mp\infty$	$\bigcup$	0	$\bigcap$	$\bigcap$	$\bigcap$	$\mp\infty$	$\bigcup$	0	$\bigcap$	-2

5°. We place the received information in a summary table.

$t\to$	$-\infty$		0		$\frac{1}{2}$		$\frac{3}{4}$		1		$\frac{3}{2}$		$+\infty$
$x(t)$	$+\infty$	$-$	0	$+$	$\pm\infty$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-\infty$
$y(t)$	$-\infty$	$-$	0	$+$	$\pm\infty$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-\infty$
$y'_x(t)$	$-\infty$	$\searrow$	0	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\nearrow$	$\nearrow$	$+\infty$
$y''_{xx}(t)$	-2	$\bigcap$	$\mp\infty$	$\bigcup$	0	$\bigcap$	$\bigcap$	$\bigcap$	$\mp\infty$	$\bigcup$	0	$\bigcap$	-2
comm.	asim.						max		v. tang.		infl.		asim.

6°. A sketch of the line is shown in Fig. 2. Fragments of the asymptote $y = \frac{1}{2}x - \frac{1}{8}$ are highlighted in blue here.

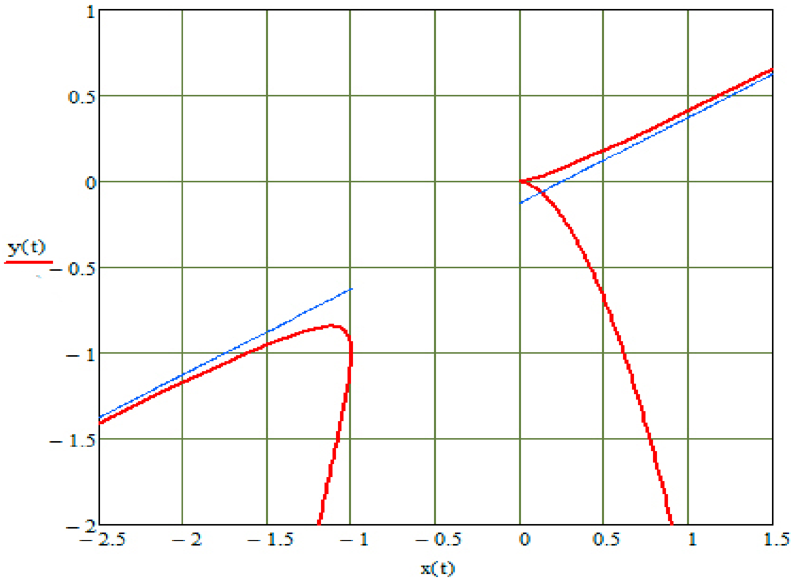

Fig. 2.

Plotting lines defined by unsolvable equations

We will consider a line, which is defined by the equation $F(x, y) = 0$. And yet this equation turns out to be unsolvable (or, perhaps, difficult to solve) both with respect to y, and with respect to x.

The method of constructing the graphical form of such a line consists in parametrizing its description. That is, it is reduced to replacing the equation $F(x, y) = 0$ with an equivalent system of the form (8.1).

Example 8.2. Plot the line given by the equation

$$x^3 + y^3 - 3xy = 0\,.$$

Solution. 1°. It is convenient to perform parameterization in this problem by putting $y = tx$. This gives

$$x(t) = \frac{3t}{t^3+1}\,, \qquad y(t) = \frac{3t^2}{t^3+1}\,.$$

The graphs of the functions $x(t)$ and $y(t)$ are shown in Fig. 3.

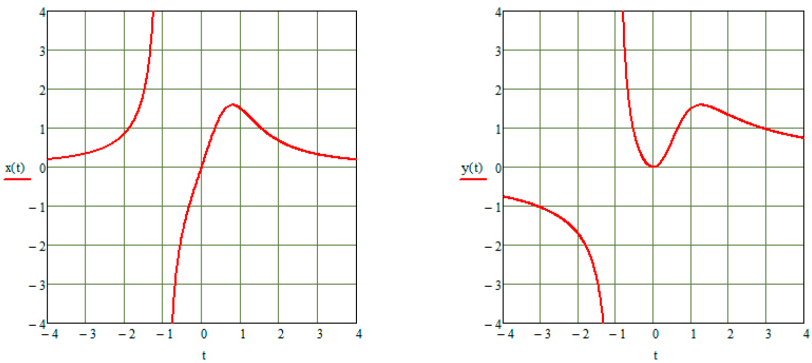

Fig. 3. Graphs of functions $x(t)$ and $y(t)$

From the original equation of the line, we can conclude that it has symmetry with respect to the straight line $y = x$.

From the parametric representation of the line it also follows that

$$x + y = 3\,\frac{t+t^2}{t^3+1} = 3\,\frac{t}{t^2-t+1} = \frac{3}{t + \dfrac{1}{t} - 1} \le 3 \qquad \forall t \ge 0\,.$$

This means that each coordinate of any point of the line in the first quadrant is bounded from above by the value 3.

2°. Find the reference points for the system of functions $x(t)$ and $y(t)$:

$t \to$	$-\infty$	-1	0	$+\infty$
$x(t) \to$	$+0$	$\pm\infty$	0	$+0$
$y(t) \to$	-0	$\mp\infty$	0	-0

and define the type of relationship between x and y for them.

1) For the reference point $\pm\infty$ we have $x(t) \sim \dfrac{3}{t^2}$ and $y(t) \sim \dfrac{3}{t}$, therefore $x \sim \dfrac{y^2}{3}$.
2) For the reference point 0 from $x(t) \sim 3t$ and $y(t) \sim 3t^2$ we get $y \sim \dfrac{x^2}{3}$.
3) For the reference point -1, we have that

$$a = \lim_{t\to -1} \frac{y(t)}{x(t)} = \lim_{t\to -1} t = -1\,.$$

Similarly

$$b = \lim_{t\to -1} \big(y(t) - ax(t)\big) = \lim_{t\to -1} \left(\frac{3t^2}{t^3+1} + \frac{3t}{t^3+1} \right) =$$

$$= \lim_{t\to -1} \frac{3t}{t^2 - t + 1} = -1\,.$$

This means that at this reference point the line has an oblique asymptote $y = -x - 1$.

3°. Using the formulas specified in paragraph 5° of the research plan, as well as the equalities

$$x'_t(t) = \frac{3(2t^3 - 1)}{(t^3 + 1)^2} \qquad \text{and} \qquad y'_t(t) = \frac{3t(t^3 - 2)}{(t^3 + 1)^2}$$

we obtain

$$y'_x(t) = \frac{t(t^3 - 2)}{2t^3 - 1} \qquad \text{and} \qquad y''_{xx}(t) = -\frac{2}{3}\,\frac{(t^3 + 1)^4}{(2t^3 - 1)^3}\,.$$

4°. Let us now find the reference points for the derivatives $y'_x(t)$ and $y''_{xx}(t)$. Taking into account *the sign of the derivatives*, we make the following table:

$t \to$	$-\infty$		-1		0		$\frac{1}{\sqrt[3]{2}}$		$\sqrt[3]{2}$		$+\infty$
$y'_x(t) \to$	$-\infty$	$\searrow$	-1	$\searrow$	0	$\nearrow$	$\pm\infty$	$\searrow$	0	$\nearrow$	$+\infty$
$y''_{xx}(t) \to$	$+\infty$	$\bigcup$	0	$\bigcup$	$\bigcup$	$\bigcup$	$\pm\infty$	$\bigcap$	0	$\bigcap$	$-\infty$

5°. Let's collect the obtained information in a summary table of line properties.

t	$-\infty$		-1		0		$\frac{1}{\sqrt[3]{2}}$		$\sqrt[3]{2}$		$+\infty$
$x(t)$	0	$+$	$\pm\infty$	$-$	0	$+$	$\sqrt[3]{4}$	$+$	$\sqrt[3]{2}$	$+$	0
$y(t)$	0	$-$	$\mp\infty$	$+$	0	$+$	$\sqrt[3]{2}$	$+$	$\sqrt[3]{4}$	$+$	0
$y'_x(t)$	$-\infty$	$\searrow$	-1	$\searrow$	0	$\nearrow$	$\pm\infty$	$\searrow$	0	$\nearrow$	$+\infty$
comment.			asim.				v. tang.		max		

6°. A sketch of the line is shown in Fig. 4. Fragments of the asymptote $y = -x - 1$ are highlighted in the figure in blue.

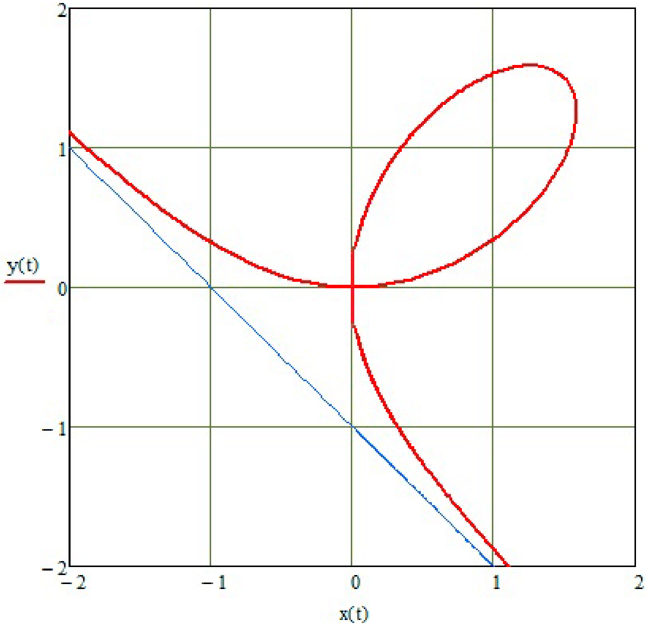

Fig. 4.