

## Tracing of lines defined parametrically or by unsolvable equations

### Tracing of lines defined parametrically

We will say that a line is defined *parametrically* in a rectangular Cartesian coordinate system  $Oxy$ , if the relationship between the coordinates of the points of the line is given by the system

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases} \quad (8.1)$$

where  $t$  belongs to some interval of  $\mathbb{R}$ .

To plot a lines defined parametrically, you can use, for example, the following sequence of actions.

- 1°. Study the functions  $x(t)$  and  $y(t)$ . Plot sketches of their graphs.
- 2°. Choose for the functions  $x(t)$  and  $y(t)$  values of  $t$ , such as  $\left\| \begin{matrix} x(t) \\ y(t) \end{matrix} \right\| \rightarrow \left\| \begin{matrix} p \\ q \end{matrix} \right\|$ , where at least one of the symbols  $p$  or  $q$  is either 0 or  $\infty$ .  
Such values of  $t$  (for brevity) will be called *support or reference values*.
- 3°. Find the relationship between  $x$  and  $y$  in a small neighborhood of the support values of  $t$ .
- 4°. Use information from 1°, 2° and 3° to construct an approximate sketch of the line.
- 5°. Calculate the derivatives of  $y'_x(t)$  and  $y''_{xx}(t)$ . Determine their support points. When calculating the derivatives you can use the following formulas

$$y'_x(t) = \frac{y'_t(t)}{x'_t(t)} \quad \text{and} \quad y''_{xx}(t) = \frac{y''_{tt}(t)x'_t(t) - y'_t(t)x''_{tt}(t)}{(x'_t(t))^3}.$$

- 6°. Using the information from 5°, find intervals of monotonicity, points of local extrema, as well as directions of convexity and points of inflection of the line.
- 7°. Reduce the obtained information about the line behavior to the final table.
- 8°. Draw a refined sketch of the line.

Example 8.1. Draw a line specified parametrically

$$x(t) = \frac{t^2}{1 - 2t}, \quad y(t) = \frac{t^3}{1 - 2t}.$$

Solution. 1°. The graphs of the functions  $x(t)$  and  $y(t)$  are shown in Fig. 1. Moreover, the left graph also shows the oblique asymptote  $u(t) = -\frac{1}{2}t - \frac{1}{4}$  (highlighted in blue).

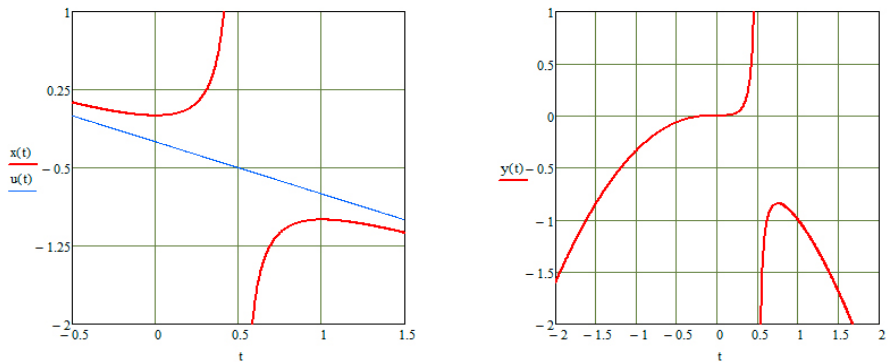


Fig. 1. Graphs of  $x(t)$  and  $y(t)$

2°. We find the support points for the system of functions  $x(t)$  and  $y(t)$

$t \rightarrow$	$-\infty$	0	$\frac{1}{2}$	$+\infty$
$x(t) \rightarrow$	$+\infty$	0	$\pm\infty$	$+\infty$
$y(t) \rightarrow$	$-\infty$	0	$\pm\infty$	$+\infty$

and define relationship between  $x$  and  $y$  for them.

- 1) For the reference point  $\pm\infty$  we have  $x(t) \sim -\frac{t}{2}$  and, since  $y(t) = tx(t)$ , then  $y \sim -2x^2$ .
- 2) For the reference point 0 from  $x(t) \sim t^2$  and  $y(t) \sim t^3$  we get  $y \sim x\sqrt{x}$ .
- 3) Finally, for the reference point  $\frac{1}{2}$ , we note that

$$a = \lim_{t \rightarrow \frac{1}{2}} \frac{y(t)}{x(t)} = \lim_{t \rightarrow \frac{1}{2}} t = \frac{1}{2}.$$

In this case

$$\begin{aligned} b &= \lim_{t \rightarrow \frac{1}{2}} (y(t) - ax(t)) = \lim_{t \rightarrow \frac{1}{2}} \left( \frac{t^3}{1-2t} - \frac{1}{2} \frac{t^2}{1-2t} \right) = \\ &= \lim_{t \rightarrow \frac{1}{2}} \left( -\frac{1}{2} t^2 \right) = -\frac{1}{8}. \end{aligned}$$

That is, at this reference point there is an oblique asymptote given by the equation  $y = \frac{1}{2}x - \frac{1}{8}$ .

3°. Using the formulas

$$x'_t(t) = -\frac{2t(t-1)}{(2t-1)^2} \quad \text{and} \quad y'_t(t) = \frac{t^2(3-4t)}{(2t-1)^2}$$

we obtain

$$y'_x(t) = \frac{t(4t-3)}{2(t-1)} \quad \text{and} \quad y''_{xx}(t) = \frac{(2t-1)^3(2t-3)}{4t(1-t)^3}.$$

Note that the last equality can also be obtained using

$$y''_{xx}(t) = \frac{y''_{xt}(t)}{x'_t(t)}.$$

4°. Let us now find the reference points for the derivatives  $y'_x(t)$  and  $y''_{xx}(t)$ . Taking into account the signs of the derivatives, we make the following table:

$t \rightarrow$	$-\infty$		0		$\frac{1}{2}$		$\frac{3}{4}$		1		$\frac{3}{2}$		$+\infty$
$y'_x(t) \rightarrow$	$-\infty$	$\searrow$	0	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\nearrow$	$\nearrow$	$+\infty$
$y''_{xx}(t) \rightarrow$	-2	$\cap$	$\mp\infty$	$\cup$	0	$\cap$	$\cap$	$\cap$	$\mp\infty$	$\cup$	0	$\cap$	-2

5°. We place the received information in a summary table.

$t \rightarrow$	$-\infty$		0		$\frac{1}{2}$		$\frac{3}{4}$		1		$\frac{3}{2}$		$+\infty$
$x(t)$	$+\infty$	-	0	+	$\pm\infty$	-	-	-	-	-	-	-	$-\infty$
$y(t)$	$-\infty$	-	0	+	$\pm\infty$	-	-	-	-	-	-	-	$-\infty$
$y'_x(t)$	$-\infty$	$\searrow$	0	$\nearrow$	$\nearrow$	$\nearrow$	0	$\searrow$	$\mp\infty$	$\nearrow$	$\nearrow$	$\nearrow$	$+\infty$
$y''_{xx}(t)$	-2	$\cap$	$\mp\infty$	$\cup$	0	$\cap$	$\cap$	$\cap$	$\mp\infty$	$\cup$	0	$\cap$	-2
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6°. A sketch of the line is shown in Fig. 2. Fragments of the asymptote  $y = \frac{1}{2}x - \frac{1}{8}$  are highlighted in blue here.

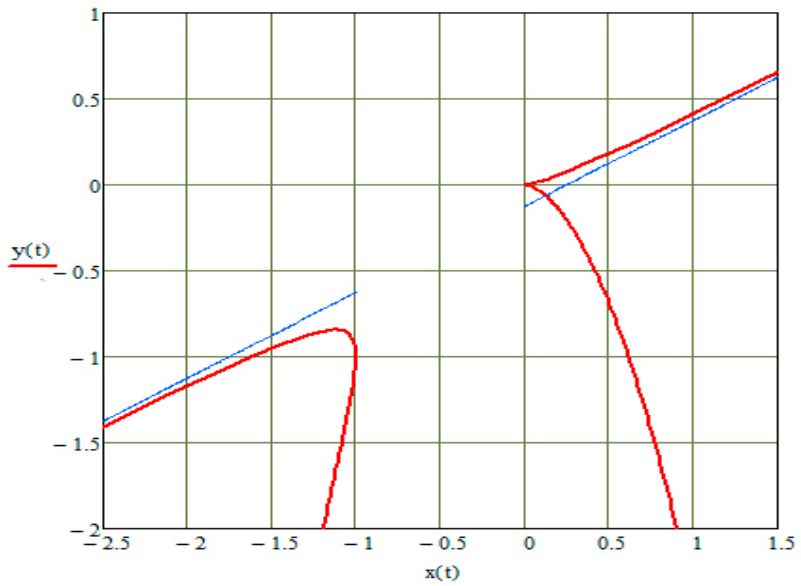


Fig. 2.

## Plotting lines defined by unsolvable equations

We will consider a line, which is defined by the equation  $F(x, y) = 0$ . And yet this equation turns out to be unsolvable (or, perhaps, difficult to solve) both with respect to  $y$ , and with respect to  $x$ .

The method of constructing the graphical form of such a line consists in parametrizing its description. That is, it is reduced to replacing the equation  $F(x, y) = 0$  with an equivalent system of the form (8.1).

**Example 8.2.** Plot the line given by the equation

$$x^3 + y^3 - 3xy = 0.$$

**Solution.** 1°. It is convenient to perform parameterization in this problem by putting  $y = tx$ . This gives

$$x(t) = \frac{3t}{t^3 + 1}, \quad y(t) = \frac{3t^2}{t^3 + 1}.$$

The graphs of the functions  $x(t)$  and  $y(t)$  are shown in Fig. 3.

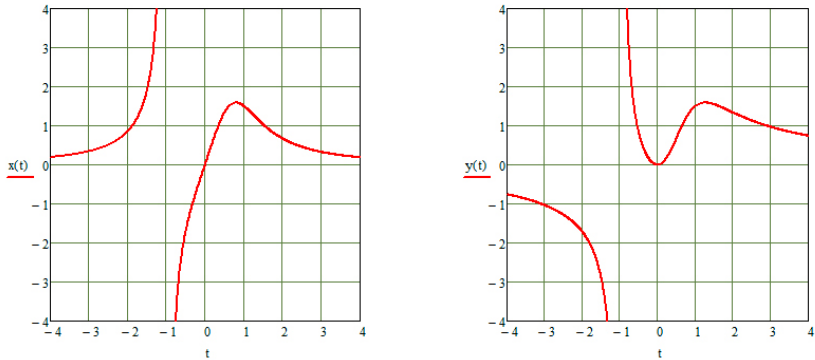


Fig. 3. Graphs of functions  $x(t)$  and  $y(t)$

From the original equation of the line, we can conclude that it has symmetry with respect to the straight line  $y = x$ .

From the parametric representation of the line it also follows that

$$x + y = 3 \frac{t + t^2}{t^3 + 1} = 3 \frac{t}{t^2 - t + 1} = \frac{3}{t + \frac{1}{t} - 1} \leq 3 \quad \forall t \geq 0.$$

This means that each coordinate of any point of the line in the first quadrant is bounded from above by the value 3.

2°. Find the reference points for the system of functions  $x(t)$  and  $y(t)$  :

$t \rightarrow$	$-\infty$	$-1$	$0$	$+\infty$
$x(t) \rightarrow$	$+0$	$\pm\infty$	$0$	$+0$
$y(t) \rightarrow$	$-0$	$\mp\infty$	$0$	$-0$

and define the type of relationship between  $x$  and  $y$  for them.

- 1) For the reference point  $\pm\infty$  we have  $x(t) \sim \frac{3}{t^2}$  and  $y(t) \sim \frac{3}{t}$ , therefore  $x \sim \frac{y^2}{3}$ .
- 2) For the reference point  $0$  from  $x(t) \sim 3t$  and  $y(t) \sim 3t^2$  we get  $y \sim \frac{x^2}{3}$ .
- 3) For the reference point  $-1$ , we have that

$$a = \lim_{t \rightarrow -1} \frac{y(t)}{x(t)} = \lim_{t \rightarrow -1} t = -1.$$

Similarly

$$\begin{aligned} b &= \lim_{t \rightarrow -1} (y(t) - ax(t)) = \lim_{t \rightarrow -1} \left( \frac{3t^2}{t^3 + 1} + \frac{3t}{t^3 + 1} \right) = \\ &= \lim_{t \rightarrow -1} \frac{3t}{t^2 - t + 1} = -1. \end{aligned}$$

This means that at this reference point the line has an oblique asymptote  $y = -x - 1$ .

3°. Using the formulas specified in paragraph 5° of the research plan, as well as the equalities

$$x'_t(t) = \frac{3(2t^3 - 1)}{(t^3 + 1)^2} \quad \text{and} \quad y'_t(t) = \frac{3t(t^3 - 2)}{(t^3 + 1)^2}$$

we obtain

$$y'_x(t) = \frac{t(t^3 - 2)}{2t^3 - 1} \quad \text{and} \quad y''_{xx}(t) = -\frac{2(t^3 + 1)^4}{3(2t^3 - 1)^3}.$$

4°. Let us now find the reference points for the derivatives  $y'_x(t)$  and  $y''_{xx}(t)$ . Taking into account *the sign of the derivatives*, we make the following table:

$t \rightarrow$	$-\infty$		-1		0		$\frac{1}{\sqrt[3]{2}}$		$\sqrt[3]{2}$		$+\infty$
$y'_x(t) \rightarrow$	$-\infty$	$\searrow$	-1	$\searrow$	0	$\nearrow$	$\pm\infty$	$\searrow$	0	$\nearrow$	$+\infty$
$y''_{xx}(t) \rightarrow$	$+\infty$	$\cup$	0	$\cup$	$\cup$	$\cup$	$\pm\infty$	$\cap$	0	$\cap$	$-\infty$

5°. Let's collect the obtained information in a summary table of line properties.

$t$	$-\infty$		$-1$		$0$		$\frac{1}{\sqrt[3]{2}}$		$\sqrt[3]{2}$		$+\infty$
$x(t)$	$0$	$+$	$\pm\infty$	$-$	$0$	$+$	$\sqrt[3]{4}$	$+$	$\sqrt[3]{2}$	$+$	$0$
$y(t)$	$0$	$-$	$\mp\infty$	$+$	$0$	$+$	$\sqrt[3]{2}$	$+$	$\sqrt[3]{4}$	$+$	$0$
$y'_x(t)$	$-\infty$	$\searrow$	$-1$	$\searrow$	$0$	$\nearrow$	$\pm\infty$	$\searrow$	$0$	$\nearrow$	$+\infty$
comment.			asim.				v. tang.		max		

6°. A sketch of the line is shown in Fig. 4. Fragments of the asymptote  $y = -x - 1$  are highlighted in blue.

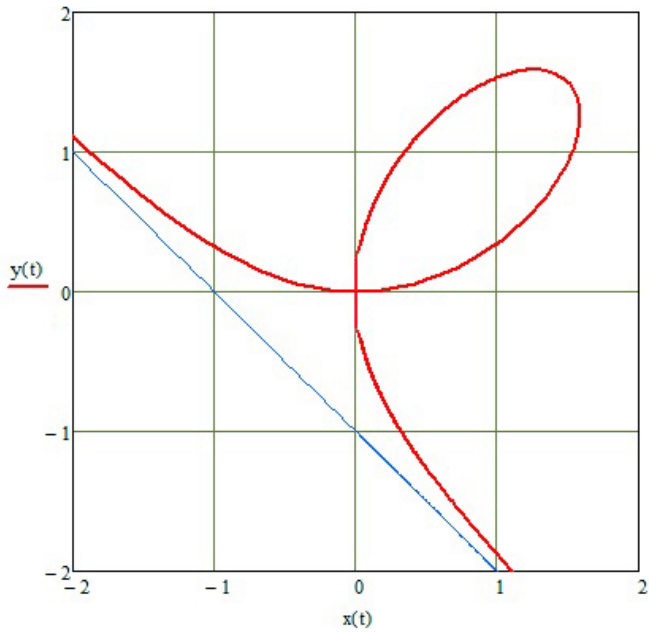


Fig. 4.