Elements of differential geometry

The line Γ in space can be defined in the Cartesian coordinate system by an ordered triple of *continuous* functions

$$\Big\{ x = x(t)\,,\ y = y(t)\,,\ z = z(t) \Big\},$$

where t belongs to the interval of the real axis $\{a, b\}$.

It is also possible to use a vector method of definition in the form $\vec{r} = \vec{r}(t)$, where

$$\| \vec{r}(t) \| = \left\| \begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array} \right\| .$$

The line can be viewed as the set of values of the map $\{a, b\} \longrightarrow \mathbb{R}^3$ as well. Note that in this interpretation, the lines

$$x(t) = R\cos t\,,\ y(t) = R\sin t\,,\ z(t) = 0 \quad t \in [0, 2\pi]$$

and

$$x(t) = R\cos t\,,\ y(t) = R\sin t\,,\ z(t) = 0 \quad t \in [0, 4\pi]$$

formally are *different* lines, although they have the same geometrical appearance.

The use of differential calculus allows to describe the properties of lines more fully.

For example, a point t_0 is called *non-singular* for a piecewise continuously differentiable line $\Gamma : \quad \vec{r} = \vec{r}(t)$, if at this point

$$x'^2(t_0) + y'^2(t_0) + z'^2(t_0) > 0\,.$$

Otherwise (including cases of non-differentiability), the point t_0 will be called *singular*.

A line is called *piecewise smooth* if it consists of parts separated by singular points and if each part contains only non-singular points.

It is known that the direction of a line tangent to the line Γ at a non-singular point t_0, is given by the vector $\vec{T} = \vec{r}\,'(t_0) = \dfrac{d\,\vec{r}}{d\,t}(t_0)$, which is called *tangent* and where

$$\|\,\vec{r}\,'_t(t)\,\| = \left\| \begin{array}{c} x'_t(t) \\ y'_t(t) \\ z'_t(t) \end{array} \right\| .$$

In this case, the equation of the tangent in vector form will be

$$\vec{r} = \vec{r}(t_0) + \tau\,\vec{T} \quad \forall \tau \in \mathbb{R}\,,$$

and in coordinate form, for example,

$$\frac{x - x(t_0)}{x'_t(t_0)} = \frac{y - y(t_0)}{y'_t(t_0)} = \frac{z - z(t_0)}{z'_t(t_0)}\,.$$

On the other hand, the point $\vec{r}(t_0)$ and the tangent vector $\vec{T}$ uniquely determine the plane with the equation $\big(\vec{T}, \vec{r} - \vec{r}(t_0)\big) = 0$, called the *normal plane.*

We define *the length of a line* Γ as *the supremum of the lengths* of all possible broken lines inscribed in Γ.

In this case, for any line without singular points, there exists its representation $\vec{r} = \vec{r}(s)$. Here the variable s is the arc length of the line Γ, measured from some point of the line (for example, from the beginning of the line). This method of parametrically defining a line is usually called *natural parameterization.*

Note that with such a representation of the line we have $\left|\vec{r}\,'_s\right| = 1$. That is, the tangent vector $\vec{T}$ has unit length at each point of the line. Its components are called *direction cosines.* They are equal to the cosines of the angles between $\vec{T}$ and the coordinate unit vectors.

When the value of t changes, the point $\vec{r}(t)$ moves along the line Γ. The tangent vector also changes.

Recall that changing a vector of fixed length consists of rotating it by a certain angle. The derivative of the *unit tangent vector* (that is, the vector of instantaneous angular velocity of rotation of the vector $\dfrac{\vec{T}}{|\vec{T}|}$) is called the *curvature vector of the line* Γ.

This vector is orthogonal to $\vec{T}$, is denoted by $\vec{N}$ and is called the *principal normal*. Its length $k = |\vec{N}|$ is called the *curvature* of the line Γ. In the case of natural parametrization, we have $k = \vec{r}\,''_{ss}(s)$.

It is clear that both the curvature vector and the curvature itself are some functions of t.

Let us introduce another vector $\vec{B} = [\vec{T}, \vec{N}]$ which is usually called *binormal* for the line Γ.

In this case, the vectors $\vec{r}(t_0)$ and $\vec{N}$ are uniquely determined by the equation $\left(\vec{N}, \vec{r} - \vec{r}(t_0)\right) = 0$ plane, called

the rectifying plane.

The vectors $\vec{r}(t_0)$ and $\vec{B}$, in turn, are uniquely determined by the equation $\left(\vec{B}, \vec{r} - \vec{r}(t_0)\right) = 0$ plane, called

the osculating plane.

It is worth paying attention to the similarity of the geometric definitions of a tangent line to a smooth line on a plane (or in space) and an osculating plane for a smooth line in space.

Recall that at a point M a tangent line to a smooth line on a plane is called the limit position of the secant MM_1, when $M_1 \to M$. Obviously, a secant passing through two non-coincident points M and M_1 lying on a line is unique.

For a smooth spatial line at a point M the osculating plane can be defined as the limit position of the secant plane passing through points M, M_1 and M_2. These points belong to the line Γ but not lying on the same straight line, when both $M_1 \to M$, and $M_2 \to M$. Here it is also obvious that such a cutting plane is unique for each triple of points M, M_1 and M_2.

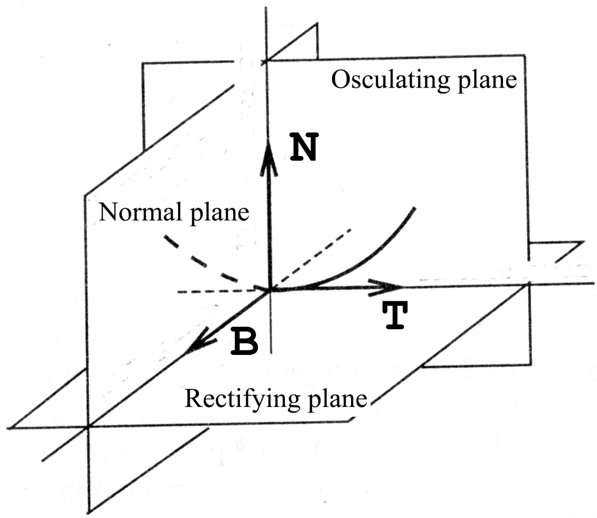

Fig. 1. The accompanying Frenet trihedron

The set of vectors $\{\vec{T},\ \vec{N},\ \vec{B}\}$ is usually called *the accompanying Frenet trihedron.* These vectors are shown in Fig. 1.

The osculating plane rotates when the point is displaced along the line. The instantaneous axis of this rotation is the tangent vector. The modulus of the angular velocity vector of this rotation is denoted by κ and is called *torsion*. The normalized vector of this angular velocity is collinear with the vector $\vec{N}$.

In the case of natural parametrization the equality

$$\frac{d\vec{B}}{d\,s} = -\kappa\vec{N}$$

is valid. It can also be shown that

$$\frac{d\vec{N}}{d\,s} = -k\vec{T} + \kappa\vec{B}\,.$$

Finally, a circle for which the tangent is also tangent for Γ is called a *osculating circle.* Its center lies on the principal normal and is called the *center of curvature of the line* Γ for the point $\vec{r}(t_0)$

The radius of the osculating circle R is called the *radius of curvature.* For it the equality $R = \dfrac{1}{k}$ is true.

The set of all centers of curvature is called the *evolute* of the line. If the line Δ is an evolute for Γ, then Γ is called the *involute* for Δ.

Let the line Γ be given in the form $\|\vec{r}\| = \left\| \begin{matrix} x(t) \\ y(t) \\ z(t) \end{matrix} \right\|$, where functions of t $x = x(t)$, $y = y(t)$, $z = z(t)$ have continuous derivatives up to and including the third order. Then the following formulas for curvature and torsion are valid

$$k = \frac{|\,[\vec{r}\,', \vec{r}\,'']\,|}{|\,\vec{r}\,'|^3} \qquad\qquad \kappa = \frac{(\,\vec{r}\,', \vec{r}\,'', \vec{r}\,''')}{|\,[\vec{r}\,', \vec{r}\,'']\,|^3}\,.$$

For example, in the right orthonormal coordinate system, the curvature formula takes the form

$$k = \sqrt{\frac{(y'z'' - y''z')^2 + (x'z'' - x''z')^2 + (x'y'' - x''y')^2}{\left(x'^2 + y'^2 + z'^2\right)^3}}\,.$$

If the line is flat (for example, it lies in the plane Oxy, which gives $z(t) = 0$) , then

$$k = \frac{|\,x'y'' - x''y'|}{\sqrt{\left(x'^2 + y'^2\right)^3}}\,.$$

If the flat line is also a graph for $y = f(x)$, then, taking x as a parameter, by virtue of $x' = 1$, $x'' = 0$, $y' = f'$, $y'' = f''$, we get

$$k = \frac{|\,f''|}{\sqrt{\left(1 + f'^2\right)^3}}\,.$$

Example 9.1 Find the radius of curvature for each point of the parabola $y^2 = 2px$, if $p > 0$.

Solution. Since $R = \dfrac{1}{k}$, then $R = \dfrac{\sqrt{\left(1+y'^2\right)^3}}{|\,y''|}$. In our case, the parabola is symmetrical to the Ox axis, so it suffices to consider the case $y \geq 0$.

We have

$$y = \sqrt{2px} \quad \Longrightarrow \quad y' = \sqrt{\frac{p}{2x}} \quad \Longrightarrow \quad y'' = -\frac{1}{2x}\sqrt{\frac{p}{2x}}\,.$$

Where we get $R(x) = \sqrt{\dfrac{\left(2x+p\right)^3}{p}}$.

Example 9.2 On the graph of the function $y = e^x$ find the point with maximum curvature.

Solution. For $y = e^x$ we have $y' = y'' = e^x$. Then

$$k(x) = \frac{|\,y''|}{\sqrt{\left(1+y'^2\right)^3}} = \frac{e^x}{\sqrt{\left(1+e^{2x}\right)^3}}\,.$$

We find stationary points for the function $k(x)$. Since

$$k'(x) = \frac{e^x\left(1+e^{2x}\right)^{3/2} - e^x\frac{3}{2}\left(1+e^{2x}\right)^{1/2}e^{2x}2}{\left(1+e^{2x}\right)^3}\,,$$

then $k'(x) = 0,$ if

$$\left(1+e^{2x}\right) - 3e^{2x} = 0 \qquad \Longrightarrow \qquad e^{2x} = \frac{1}{2},$$

which gives $x = -\ln\sqrt{2} \approx -0.347$. The graph of the function $k(x)$ is shown in Fig. 2.

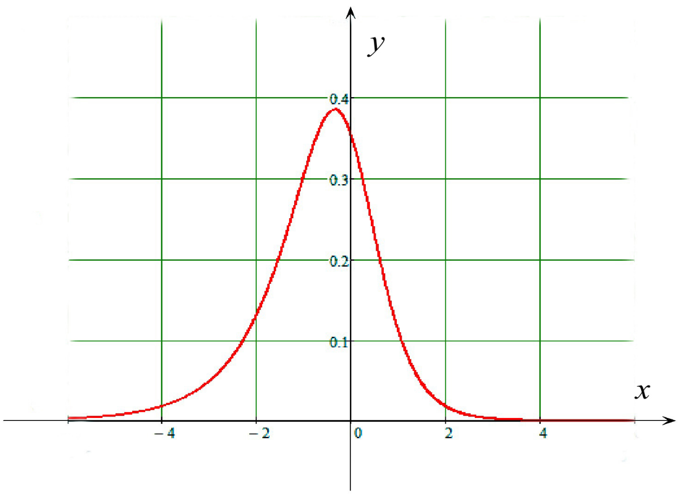

Fig. 2.

`Example 9.3` Find the radius of curvature of the line

$$x(t) = \operatorname{tg} t, \quad y(t) = \cos 2t \qquad \text{for} \;\; t = \frac{\pi}{4}.$$

`Solution.` Let's use the formula

$$R(t) = \frac{1}{k(t)} = \frac{\sqrt{\left(x'^2 + y'^2\right)^3}}{|\, x'y'' - x''y' |}.$$

In our case, at $t = \dfrac{\pi}{4}$ we get

$$\begin{array}{lllllllll} x' & = & \dfrac{1}{\cos^2 t} & = & 2, & y' & = & -2\sin 2t & = & -2, \\ x'' & = & \dfrac{2\sin t}{cos^3 t} & = & 4, & y'' & = & -4\cos 2t & = & 0. \end{array}$$

Therefore,

$$R(t) = \frac{\sqrt{\left(2^2 + (-2)^2\right)^3}}{|\, 2 \cdot 0 - 4(-2)|} = 2\sqrt{2}\,.$$