Systems of *m* linear equations with *n* unknowns

Our main task is to find a formula that describes all solutions of a system of linear equations with unknowns of the form

$$\begin{cases} \alpha_{11}\xi_1 + \alpha_{12}\xi_2 + \ldots + \alpha_{1n}\xi_n = \beta_1, \\ \alpha_{21}\xi_1 + \alpha_{22}\xi_2 + \ldots + \alpha_{2n}\xi_n = \beta_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{m1}\xi_1 + \alpha_{m2}\xi_2 + \ldots + \alpha_{mn}\xi_n = \beta_m, \end{cases} \qquad (1.1)$$

which in unexpanded form is written as

$$\sum_{j=1}^{n} \alpha_{ij}\xi_j = \beta_i \ , \quad i = [1, m] \ , \qquad \text{or in matrix form} \qquad \| A \| \, \| x \| = \| b \|,$$

where the matrix $\| A \|$ (of size $m \times n$) has arbitrary numerical components α_{ij}, and the columns $\| x \|$ and $\| b \|$ have, respectively, the components $\xi_j \quad j = [1, n]$ and $\beta_i \quad i = [1, m]$.

Let us recall the basic definitions needed to solve systems of linear equations.

Definition An ordered set of numbers $\{\xi_1^0, \xi_2^0, \ldots, \xi_n^0\}$ will be called a *particular solution* of the system of linear equations (1.1) if we obtain true equalities when substituting these numbers into the system. A particular solution of the system of linear equations will be written in the form of a column $\|x^0\| = \left\| \begin{matrix} \xi_1^0 \\ \xi_2^0 \\ \ldots \\ \xi_n^0 \end{matrix} \right\|$.

The set of all particular solutions of the system of linear equations (1.1) will be called the general solution of system (1.1).

If system (1.1) has at least one particular solution, then it is called *compatible*, otherwise - an *incompatible* system of equations.

Definition System (1.1) is called *homogeneous* if $\beta_i = 0 \ \forall i = [1, m]$, otherwise - an *inhomogeneous* system of equations.

A homogeneous system $\|A\| \|x\| = \|o\|$ is always compatible, since it has an obvious *zero* (*trivial*) solution.

Definition The matrix $\|A\|$ is called the *main* matrix of system (1.1), and the matrix $\|A \mid b\| = \left\| \begin{matrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{matrix} \right| \left. \begin{matrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_m \end{matrix} \right\|$ is called the *augmented* matrix of this system.

Let us now formulate the main theoretical statements used in solving systems of linear equations.

We will need:

Theorem (Kronecker-Capelli). **In order for system (1.1) to be compatible, it is necessary and sufficient that the rank of its main matrix be equal to the rank of the extended one.**

For a system of linear equations n with n unknowns $\sum_{j=1}^{n} \alpha_{ij} \xi_j = \beta_j \; ; \; i = [1, n]$, we have

Theorem (Cramer's rule). **In order for the system of linear equations (1.1) to have a unique solution for , it is necessary and sufficient that , and in this case the solution of this system will have the form**

$$\xi_j = \frac{\Delta_j}{\Delta} \; ; \; j = 1, 2, \dots, n \, ,$$

where Δ_j is the determinant of the matrix obtained from the matrix $\| A \|$ by replacing its j-th column with a column of free terms $\| b \|$:

$$\Delta_j = \det \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \dots & \beta_1 & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \beta_2 & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \beta_n & \dots & \alpha_{nn} \end{Vmatrix} .$$

$\uparrow$

j-th column.

Rank of a matrix

Let k a number be such that $1 \le k \le \min\{m,n\}$. Let us choose in $\|A\|$ some way k columns and rows at the intersection of which there are elements that form a square matrix of a minor of order . Note: the choice of columns and the choice of rows are performed *independently* of each other.

Let all minors of order k be zero, then all minors of order higher than k will also be zero, since each minor of order $k+1$ is representable as a linear combination of minors of order k.

Definition .	The *maximum* of the orders of minors of matrix $\|A\|$ different from zero is called the *rank* of the matrix and is denoted by $\mathrm{rg}\|A\|$ (or $\mathrm{rank}\|A\|$).
	Any nonzero minor of the matrix whose order is equal to its rank is called a *basic minor*.
	The columns (rows) of the matrix that are part of the matrix of a basic minor are called *basic*.

Linear dependence of columns

Consider n m-component columns of the form

$$\| a_1 \| = \left\| \begin{matrix} \alpha_{11} \\ \alpha_{21} \\ \dots \\ \alpha_{m1} \end{matrix} \right\| ; \; \| a_2 \| = \left\| \begin{matrix} \alpha_{12} \\ \alpha_{22} \\ \dots \\ \alpha_{m2} \end{matrix} \right\| ; \dots ; \; \| a_n \| = \left\| \begin{matrix} \alpha_{1n} \\ \alpha_{2n} \\ \dots \\ \alpha_{mn} \end{matrix} \right\| \quad \text{and columns} \quad \| b \| = \left\| \begin{matrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_m \end{matrix} \right\| ; \; \| o \| = \left\| \begin{matrix} 0 \\ 0 \\ \dots \\ 0 \end{matrix} \right\| .$$

Since for columns (as a special case of matrices) the operations of comparison, addition and multiplication by a number are defined, then we will say that a column $\|b\|$ is a linear combination of columns $\|a_1\|,\|a_2\|,\dots,\|a_n\|$ if $\exists$ numbers $\|a_1\|,\|a_2\|,\dots,\|a_n\|$, such that $\| b \| = \sum\limits_{j=1}^{n} \lambda_j \| a_j \|$.

Definition Columns $\|a_1\|,\|a_2\|,\dots,\|a_n\|$ will be called *linearly dependent* if there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ that are not simultaneously equal to zero , such that

$$\sum_{j=1}^{n} \lambda_j \| a_j \| = \| o \| , \quad (\sum_{j=1}^{n} | \lambda_j | > 0) .$$

The definition of *linear independence* is also given similarly to the definition for vectors.

Theorems using the concept of matrix rank

Let us first recall that the following statements are true:

Lemma

For the columns (rows) of a matrix to be linearly dependent, it is necessary and sufficient that one of them be a linear combination of the others.

If among the columns of a matrix there is a linearly dependent subset, then the set of all columns of this matrix is also linearly dependent.

Theorem (on the basic minor).

Every column (row) of a matrix is a linear combination of the basic columns (rows) of this matrix.

Corollary

For the determinant to be equal to zero, it is necessary and sufficient that the columns (rows) of its matrix be linearly dependent.

Theorem (on the rank of a matrix).

The maximum number of linearly independent columns of a matrix is equal to the maximum number of linearly independent rows and is equal to the rank of this matrix.

Formula for solving a system of m linear equations with n unknowns

When constructing a general solution to system (1.1), we use the following easily verifiable statements.

Lemma

1. Any linear combination of particular solutions of the homogeneous system (1.1) is also its particular solution.

2. The sum of some particular solution of the homogeneous system (1.1) and some particular solution of the inhomogeneous system is a particular solution of the inhomogeneous system (1.1)

3. The difference of two particular solutions of the inhomogeneous system (1.1) is a particular solution of the homogeneous system (1.1).

It follows that the following important

Theorem A **The general solution of a non-homogeneous system of equations is the general solution of a homogeneous system plus some particular solution of a non-homogeneous system,**

GENERAL solution of a NON-homogeneous *SLE*	=	GENERAL solution of a homogeneous *SLE*	+	Particular solution of a NON-homogeneous *SLE*

It should be noted that in this theorem the particular solution of a non-homogeneous system of linear equations can be anything.

First, let us consider the problem of finding a general solution to a homogeneous system of linear equations.

Note that since each particular solution of the system of linear equations (1.1) can be represented as an n-component column, we can use the concepts of *linearly dependent* and *linearly independent* particular solutions of this system.

The basis for constructing a formula for a general solution to a homogeneous system is

Theorem B **The maximum number of linearly independent particular solutions of a homogeneous system (1.1) is $n - \mathrm{rg}\|A\|$.**

Definition A *fundamental set* of solutions for a system of linear equations (1.1) is a set of any $n - \mathrm{rg}\|A\|$ linearly independent, particular solutions of a homogeneous system (1.1), where n is the number of unknowns in the system (1.1), and $\|A\|$ is its matrix.

A matrix whose columns (or rows) are the columns of fundamental solutions is called *fundamental*.

By virtue of the first assertion of the lemma, any linear combination of fundamental solutions is a particular solution of the homogeneous system (1.1).

On the other hand, the following is also true:

Theorem C **_Each particular_ solution of the homogeneous system (1.1) can be represented as a linear combination of particular solutions that make up the fundamental set of solutions.**

Based on the last two assertions and Theorem A, we conclude that the following is true:

Theorem D **The _general_ solution of the inhomogeneous system (1.1) can be represented as the sum of an arbitrary linear combination of fundamental particular solutions of the homogeneous system and some particular solution of the inhomogeneous system (1.1).**

Let us illustrate the application of the presented theory in detail by solving

Task 1.01. *Find the general solution of a system of linear equations*

$$\begin{cases} x_1 + x_2 - 3x_3 - 6x_4 = -8 , \\ x_1 + 2x_2 - 9x_3 - 6x_4 = -12 , \\ x_1 + 3x_3 - 6x_4 = -4 , \\ x_2 - 6x_3 = -4 . \end{cases} \tag{A}$$

Solution: 1. Replace the original system with an equivalent one, such that finding the particular solutions we need is not a difficult task.

To do this, subtract the first equation successively from the second and third, without changing the first and fourth equations. We obtain the system

$$\begin{cases} x_1 + x_2 - 3x_3 - 6x_4 = -8 , \\ x_2 - 6x_3 = -4 , \\ -x_2 + 6x_3 = 4 , \\ x_2 - 6x_3 = -4 . \end{cases}$$

From which we obtain that the original system will be equivalent to a system of the form

$$\begin{cases} x_1 + x_2 - 3x_3 - 6x_4 = -8 , \\ x_2 - 6x_3 = -4 . \end{cases} \tag{B}$$

2. If we transform the last system to the form

$$\begin{cases} x_1 + x_2 = -8 + 3x_3 + 6x_4 \,, \\ \quad\quad\; x_2 = -4 + 6x_3 \,, \end{cases} \tag{C}$$

then, based on Cramer's theorem, we can assert that the unknowns and have uniquely determined values for any predetermined values of the unknowns x_3 and x_4.

3. We will use this property of system (C) to find the necessary particular solutions.

First, we need some particular solution of system (A). We will find it by putting, for example, in system (C) $x_3 = x_4 = 0$. This will give $x_1 = x_2 = -4$.

4. Second, we need fundamental solutions of the homogeneous system (A).

By definition, the right-hand sides of the equations of a homogeneous system (A) are zero. Therefore, having performed the same transformations for a homogeneous system as for a non-homogeneous one, we obtain a system of the form

$$\begin{cases} x_1 + x_2 = 3x_3 + 6x_4 , \\ x_2 = 6x_3 . \end{cases} \qquad \text{(D)}$$

For the fundamentality of particular solutions, it is required that

- first, they are linearly independent and,
- second, their number is equal to $n - \operatorname{rg}\|A\|$.

In our case $n = 4$, and $\operatorname{rg}\|A\| = 2$.

Indeed, the transformations we have performed on the equations of system (A) *do not change the ranks* of the matrices, and, therefore, the ranks of the fundamental matrices of systems (A) and (B) are the same. Then it follows from the formula $n - \operatorname{rg}\|A\|$ that we need to find only two linearly independent particular solutions of system (D).

Since the unknowns x_3 and x_4 in system (D) are arbitrary, then (*prove it yourself*) the linear independence of a pair of particular solutions of this system is guaranteed by using $x_3 = 1, x_4 = 0$ for the first solution and $x_3 = 0, x_4 = 1$ - for the second.

Consequently, we find the first fundamental solution by solving the system $\begin{cases} x_1 + x_2 = 3, \\ x_2 = 6. \end{cases}$ This gives $x_1 = -3, x_2 = 6$. $\begin{cases} x_1 + x_2 = 6, \\ x_2 = 0. \end{cases}$ That is, $x_1 = 6, x_2 = 0$.

5. We write the found solutions in matrix form: the set of fundamental solutions will consist of $\left\|\begin{matrix}-3\\6\\1\\0\end{matrix}\right\|$ and $\left\|\begin{matrix}6\\0\\0\\1\end{matrix}\right\|$, and the particular solution of the inhomogeneous system $\left\|\begin{matrix}-4\\-4\\0\\0\end{matrix}\right\|$.

In this case, the general solution of the homogeneous system is described by the formula ,

$$\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \lambda_1\left\|\begin{matrix}-3\\6\\1\\0\end{matrix}\right\| + \lambda_2\left\|\begin{matrix}6\\0\\0\\1\end{matrix}\right\| \quad \forall\lambda_1,\lambda_2,$$

a and the general solution of the inhomogeneous

$$\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \lambda_1\left\|\begin{matrix}-3\\6\\1\\0\end{matrix}\right\| + \lambda_2\left\|\begin{matrix}6\\0\\0\\1\end{matrix}\right\| + \left\|\begin{matrix}-4\\-4\\0\\0\end{matrix}\right\| \quad \forall\lambda_1,\lambda_2.$$

Note that in the last formula we can exclude λ_1 and λ_2. Then we will get a more compact, but less visual form of writing the general solution:

$$\begin{cases} x_1 = -3x_3 + 6x_4 - 4, \\ x_2 = \quad 6x_3 \qquad\quad - 4. \end{cases}$$

Solution is found

Another condition for compatibility of a system of linear equations, useful for applications, is given by

Theorem (Fredholm). **In order for the system (1.1) to be compatible, it is necessary and sufficient that each solution $\| y \| = \| \eta_1 \quad \eta_2 \quad \dots \quad \eta_m \|^{\mathrm{T}}$ of the *adjoint* system**

$$\begin{cases} \alpha_{11}\eta_1 + \alpha_{21}\eta_2 + \dots + \alpha_{m1}\eta_m = 0, \\ \alpha_{12}\eta_1 + \alpha_{22}\eta_2 + \dots + \alpha_{m2}\eta_m = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{1n}\eta_1 + \alpha_{2n}\eta_2 + \dots + \alpha_{mn}\eta_m = 0 \end{cases}$$

(or in matrix form $\| A \|^{\mathrm{T}} \| y \| = \| o \|$) satisfies the condition $\sum_{i=1}^{m} \beta_i \eta_i = 0$ (or in matrix form $\| b \|^{\mathrm{T}} \| y \| = 0$).

Elementary transformations. The Gauss method

The practical application of the theory of solving systems of linear equations is complicated by the fact that, as a rule, it is not known in advance whether the system being solved is compatible. A more effective computational algorithm that allows either finding the general solution of system (1.1) or establishing the fact of its incompatibility is the *Gauss method*.

The essence of this method lies in transforming the augmented matrix of a system of linear equations to the *simplest* form by a sequence of so-called *elementary* transformations, each of which does not change the general solution of the system of equations.

By "*the simplest*" form of the augmented matrix we mean the *upper triangular* form (i.e. the case when $\alpha_{ij} = 0$ for $i > j$), for which it is possible to recursively find unknowns by only solving a linear equation with one unknown at each step of the procedure.

Below is an example of a matrix of size $m \times n$ $(n > m)$, having an upper triangular form

$$\left\|\begin{matrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,m-2} & a_{1,m-1} & a_{1,m} & a_{1,m+1} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,m-2} & a_{2,m-1} & a_{2,m} & a_{2,m+1} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3,m-2} & a_{3,m-1} & a_{3,m} & a_{3,m+1} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & a_{m-1,m-1} & a_{m-1,m} & a_{m-1,m+1} & \dots & a_{m-1,n} \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{mm} & a_{m,m+1} & \dots & a_{mn} \end{matrix}\right\|.$$

Elementary matrix transformations include:

- row permutation (renumbering of equations);
- permutation of columns of the main matrix (renumbering of unknowns);
- deletion of the zero row (exclusion of equations identically satisfied by any values of unknowns);
- multiplication of a row by a non-zero number (normalization of equations);
- addition of a row with a linear combination of the remaining rows with the result written in place of the original row (replacement of one of the equations of the system with a consequence of its equations obtained using linear operations).

The solution of a non-homogeneous system of equations (as well as the rank of its matrix) will not change even when using any combination of elementary operations.

A direct check can show that elementary transformations of any matrix can be performed by multiplying it by matrices of the following special type. For example:

- permutation of *columns* with numbers i and j of a matrix $\|A\|$ of size $m\times n$ is performed by multiplying it from the *right* by a matrix $\|S\|_1$ of size $n\times n$, which in turn is obtained from the identity matrix of order n by permuting the *i*-th and *j*-th columns in the last one;

- multiplication of the *i*-th *row* of a matrix $\|A\|$ by some number $\lambda \neq 0$ is performed by multiplying $\|A\|$ from the *left* by a matrix $\|S\|_2$, which is obtained from the identity matrix $\|E\|$ of size $m\times m$ by replacing the *i*-th diagonal element (equal to one) in the last one by λ;

- addition of *rows* with numbers i and j of a matrix $\|A\|$ is performed by multiplying it from the left by a matrix $\|S\|_3$ of size $m\times n$, which is obtained from the identity matrix $\|E\|$ of order m by replacing the zero element in the *i*-th row and *j*-th column in the last one by one (in this case, the result of the summation will be in the place of the i-th row of the original matrix $\|A\|$).

It can be shown that if the matrix $\|S\|$ is square and *non-singular* and matrix multiplication $\|S\|$ by matrix $\|A\|$ is possible, then the equality is true

$$\mathrm{rg}\,(\| S \| \| A \|) = \mathrm{rg}\, \| A \| \,.$$

Since $\det\|S\|_1 = -1$, $\det\| S \|_2 = \lambda \neq 0$, and $\det\|S\|_3 = 1$, the rank $\|A\|$ does not change under the transformations considered above.

Check for yourself that the following theorems will also be true

Theorem **Successive application of several elementary transformations is a new transformation that has a matrix that is the *product* of the matrices of these elementary transformations.**

Theorem **If the multiplication of a matrix $\|A\|$ from the left by a square matrix $\|S\|$ implements some transformation over the rows of $\|A\|$, then the multiplication $\|A\|$ from the right by $\| S \|^{\mathrm{T}}$ implements the same transformation of the matrix $\|A\|$, but performed over its columns.**

The noted properties of elementary transformations allow in a number of cases to simplify computational procedures with matrix expressions. Let, for example, be the matrix $\|S\|^*$ of the transformation that transforms a non-singular matrix $\|A\|$ into the identity matrix. Then the transformation with the matrix $\|S\|^*$ will transform the identity matrix $\|E\|$ into the matrix $\|A\|^{-1}$, since by virtue of $\|E\|=\|S\|^*\|A\|$ and non-singularity $\|A\|$ the equalities are true

$$\|E\|\|A\|^{-1}=\|S\|^*\|A\|\|A\|^{-1} \quad \text{or} \quad \|A\|^{-1}=\|S\|^*\|E\|.$$

From these relations it follows that the calculation of the product of square matrices $\|A\|^{-1}\|B\|$ can be reduced to a sequence of elementary transformations of the matrix $\|A\,|\,B\|$ (that is, the matrix formed by adding the columns of the matrix $\|B\|$ to the matrix $\|A\|$), reducing the submatrix to the identity matrix. As a result, the desired product is in the place of the submatrix $\|B\|$.

The Gauss method for solving systems of linear equations, based on elementary transformations of the augmented matrix, is a universal procedure that does not require the use of any specific properties of this matrix. Let us illustrate this statement with the following example.

Task 1.02. *Solve the system of equations*

$$\begin{cases} \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 7, \\ 3\xi_1 + 2\xi_2 + \xi_3 + \xi_4 - 3\xi_5 = -2, \\ \xi_2 + 2\xi_3 + 2\xi_4 + 6\xi_5 = 23, \\ 5\xi_1 + 4\xi_2 + 3\xi_3 + 3\xi_4 - \xi_5 = 12. \end{cases}$$

Solution. 1°. We compose the augmented matrix of the system

$$\left\| \begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 5 & 4 & 3 & 3 & -1 & 12 \end{array} \right\| .$$

2°. We bring it to the upper triangular form. To do this,

a) we transform all the elements of the first column to zeros, except for the element in the first row. For example, to zero the element in the second row of the first column, we replace the second row of the matrix with a row that is the sum of the first row multiplied by (-3) and the second row. We do the same with the fourth row: we replace it with a linear combination of the first and fourth rows with coefficients (-5) and 1, respectively. Naturally, we do not change the third row: it already contains the zero necessary for the upper triangular form. As a result, the matrix takes the form

$$\left\|\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & -2 & -2 & -6 & -23 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & -1 & -2 & -2 & -6 & -23 \end{array}\right\|;$$

b) now we perform the operation of zeroing the elements of the second column, located in its third and fourth rows. To do this, we replace the third row of the matrix with the sum of the second and third, and the fourth with the difference of the second and fourth. We get

$$\left\|\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & -2 & -2 & -6 & -23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right\|;$$

c) since in this particular case the element located in the fourth row of the third column turned out to be equal to zero, then the reduction of the augmented matrix to upper triangular form is complete.

3°. The resulting matrix is the augmented matrix of a system of linear equations equivalent to the original system. The rank of this matrix coincides with the rank of the original. Therefore, we conclude that

1) the system is compatible, since the rank of the main matrix is equal to the rank of the augmented one and is 2 (by the Kronecker–Capelli theorem);
2) the homogeneous system of equations will have $n - \mathrm{rg}\|A\| = 5 - 2 = 3$ linearly independent solutions.

4°. Since the general solution of the inhomogeneous system is the general solution of the homogeneous system plus a particular solution of the inhomogeneous system, it is sufficient for us to find any three linearly independent solutions of the homogeneous system and any one solution of the inhomogeneous system.
Let us rewrite the original system in a transformed form, taking the first and second unknowns as the main ones, and the third, fourth and fifth as free:

$$\begin{cases} \xi_1 + \xi_2 = 7 - \xi_3 - \xi_4 - \xi_5, \\ \xi_2 = 23 - 2\xi_3 - 2\xi_4 - 6\xi_5. \end{cases} \tag{1.2}$$

For convenience of calculations, we multiply the second row by (-1), and discard the third and fourth rows, since the equations corresponding to them are satisfied identically.

By setting the free unknowns in system (1.2) equal to zero, we find a particular solution of the inhomogeneous system $\begin{Vmatrix} -16 \\ 23 \\ 0 \\ 0 \\ 0 \end{Vmatrix}$. The values of the main unknowns are determined from an easily solvable system of linear equations

$$\begin{cases} \xi_1 + \xi_2 = 7, \\ \qquad\ \ \xi_2 = 23. \end{cases}$$

For the homogeneous system

$$\begin{cases} \xi_1 + \xi_2 = 0 - \xi_3 - \xi_4 - \xi_5, \\ \qquad\ \ \xi_2 = 0 - 2\xi_3 - 2\xi_4 - 6\xi_5 \end{cases}$$

we construct a fundamental set of solutions using the standard scheme. The first linearly independent solution $\begin{Vmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{Vmatrix}$ is found from the system

$$\begin{cases} \xi_1 + \xi_2 = -1, \\ \qquad\ \ \xi_2 = -2. \end{cases}$$

Similarly, we get the second $\left\| \begin{matrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{matrix} \right\|$ and third $\left\| \begin{matrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{matrix} \right\|$ solutions.

Finally, the general solution of the original inhomogeneous system in matrix form can be written as:

$$\left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{matrix} \right\| = \lambda_1 \left\| \begin{matrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{matrix} \right\| + \lambda_2 \left\| \begin{matrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{matrix} \right\| + \lambda_3 \left\| \begin{matrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{matrix} \right\| + \left\| \begin{matrix} -16 \\ 23 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \quad \forall \lambda_1, \lambda_2, \lambda_3 .$$

Solution is found

Note: since there is freedom of choice of both a particular solution of the heterogeneous system and linearly independent solutions of the homogeneous system, the general solution of the heterogeneous system can be written in different, but naturally equivalent forms.

As will be seen from what follows, a large number of problems whose solution is necessary are (or are reduced to) problems of solving systems of linear equations. One of these problems is the problem of constructing a homogeneous system of linear equations that has a general solution of a given type. In essence, we are talking about a problem 'inverse' to the problem of solving a system of linear equations. Let us consider

Task 1.03. *Determine the possible type of a homogeneous system of linear equations that has particular solutions of the form*

$$\begin{Vmatrix} -1 \\ -5 \\ 3 \\ 2 \end{Vmatrix}; \quad \begin{Vmatrix} 0 \\ 2 \\ -1 \\ -1 \end{Vmatrix}; \quad \begin{Vmatrix} 1 \\ -1 \\ 0 \\ 1 \end{Vmatrix}.$$

Solution: 1) For any constants λ_1, λ_2 and λ_3, the solution to the desired system will be a linear combination of the form

$$\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \lambda_1 \begin{Vmatrix} -1 \\ -5 \\ 3 \\ 2 \end{Vmatrix} + \lambda_2 \begin{Vmatrix} 0 \\ 2 \\ -1 \\ -1 \end{Vmatrix} + \lambda_3 \begin{Vmatrix} 1 \\ -1 \\ 0 \\ 1 \end{Vmatrix}. \qquad \text{(A)}$$

Let us assume that some equation of the desired system has the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0 \ ,$$

or in matrix form $$\left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \right\| \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \right\| = 0 . \qquad \text{(B)}$$

2) Let us find out for what values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ this equality will be true for any λ_1, λ_2 and λ_3. To do this, we substitute (A) into (B) and regroup the terms on the left-hand side:

$$\left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \right\| \left(\lambda_1 \left\| \begin{matrix} -1 \\ -5 \\ 3 \\ 2 \end{matrix} \right\| + \lambda_2 \left\| \begin{matrix} 0 \\ 2 \\ -1 \\ -1 \end{matrix} \right\| + \lambda_3 \left\| \begin{matrix} 1 \\ -1 \\ 0 \\ 1 \end{matrix} \right\| \right) = 0$$

$$\lambda_1 \left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \right\| \left\| \begin{matrix} -1 \\ -5 \\ 3 \\ 2 \end{matrix} \right\| + \lambda_2 \left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \right\| \left\| \begin{matrix} 0 \\ 2 \\ -1 \\ -1 \end{matrix} \right\| + \lambda_3 \left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \right\| \left\| \begin{matrix} 1 \\ -1 \\ 0 \\ 1 \end{matrix} \right\| = 0$$

3) This equality will obviously be true for any λ_1, λ_2 and λ_3, if the numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the solution of the following homogeneous system of linear equations:

$$\begin{cases} -\alpha_1 - 5\alpha_2 + 3\alpha_3 + 2\alpha_4 = 0, \\ \qquad 2\alpha_2 - \alpha_3 - \alpha_4 = 0, \\ \alpha_1 - \alpha_2 + \alpha_4 = 0. \end{cases} \tag{C}$$

4) We solve this system using the Gauss method, transforming its main matrix:

$$\begin{Vmatrix} -1 & -5 & 3 & 2 \\ 0 & 2 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{Vmatrix} \sim \begin{Vmatrix} -1 & -5 & 3 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & -6 & 3 & 3 \end{Vmatrix} \sim \begin{Vmatrix} -1 & -5 & 3 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{D}$$

We take α_1 and α_4 as the main unknowns, and α_2 and α_3 as free ones. We obtain a simplified system: $\begin{cases} \alpha_1 - 2\alpha_4 = -5\alpha_2 + 3\alpha_3, \\ \qquad \alpha_4 = 2\alpha_2 - \alpha_3. \end{cases}$

5) The rank of the matrix (D) is 2, therefore the system (C) has $4-2=2$ fundamental solutions of the form:

$$\begin{Vmatrix} -1 \\ 1 \\ 0 \\ 2 \end{Vmatrix}; \quad \begin{Vmatrix} 1 \\ 0 \\ 1 \\ -1 \end{Vmatrix}$$

Check this yourself.

6) From which it follows that the desired system will have a maximum of two independent equations, for example, of the form $\begin{cases} -x_1 + x_2 + 2x_4 = 0, \\ x_1 + x_3 - x_4 = 0. \end{cases}$

This is the answer to the task.

Solution is found

For the sake of completeness, we note that there is an alternative method for solving the problem under consideration. We will describe it.

.

Solution: 1) Again, for any constants λ_1, λ_2 and λ_3 the solution to the desired system will be a linear combination of the form

$$\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \lambda_1 \begin{Vmatrix} -1 \\ -5 \\ 3 \\ 2 \end{Vmatrix} + \lambda_2 \begin{Vmatrix} 0 \\ 2 \\ -1 \\ -1 \end{Vmatrix} + \lambda_3 \begin{Vmatrix} 1 \\ -1 \\ 0 \\ 1 \end{Vmatrix} . \qquad \text{(A)}$$

2) According to the rules of operation with matrices, it follows from equality (A) that the numbers λ_1, λ_2 and λ_3 must be a solution to a non-homogeneous system of linear equations of the form

$$\begin{cases} -\lambda_1 & & + & \lambda_3 & = & x_1, \\ -5\lambda_1 & + & 2\lambda_2 & - & \lambda_3 & = & x_2, \\ 3\lambda_1 & - & \lambda_2 & & & = & x_3, \\ 2\lambda_1 & - & \lambda_2 & + & \lambda_3 & = & x_4. \end{cases} \qquad \text{(E)}$$

3) System (E) must be true for any λ_1, λ_2 and λ_3, and, therefore, its right-hand sides must be such that this system has a solution.

The Kronecker-Capelli theorem states that for the compatibility of a non-homogeneous system it is necessary and sufficient that the rank of its fundamental matrix be equal to the rank of the extended one.

Let us find these ranks for the system (E) using the Gauss method. We have

$$\left\|\begin{array}{rrrcc} -1 & 0 & 1 & | & x_1 \\ -5 & 2 & -1 & | & x_2 \\ 3 & -1 & 0 & | & x_3 \\ 2 & -1 & 1 & | & x_4 \end{array}\right\| \sim \left\|\begin{array}{rrrcr} -1 & 0 & 1 & | & x_1 \\ 0 & 2 & -6 & | & -5x_1+x_2 \\ 0 & -1 & 3 & | & 3x_1+x_3 \\ 0 & -1 & 3 & | & 2x_1+x_4 \end{array}\right\| \sim$$

$$\sim \left\|\begin{array}{rrrcr} -1 & 0 & 1 & | & x_1 \\ 0 & 0 & 0 & | & x_1+x_2+2x_3 \\ 0 & -1 & 3 & | & 3x_1+x_3 \\ 0 & 0 & 0 & | & x_1+x_3-x_4 \end{array}\right\|$$

4) The rank of the main matrix is 2. The rank of the augmented matrix for arbitrary values x_1, x_2, x_3 and x_4 may well be greater than 2. However, if we require that these values satisfy a homogeneous system of equations,

$$\begin{cases} x_1 + x_2 + 2x_3 = 0, \\ x_1 + x_3 - x_4 = 0. \end{cases} \qquad \text{(F)}$$

then the ranks will coincide. Therefore, the system (F) is the answer to the task.

5) Finally, we note that if we add the first equation of the system (F) to the doubled second and replace the first equation with this sum, we will obtain an equivalent system that coincides with the answer in the first method.

Solutioon is found