VECTOR SPACE

Definition of a vector space

Definition A set Λ consisting of elements $x, y, z, \dots$ for which the comparison operation is defined is called a *vector space* if

1°. Each pair of elements x, y of this set is assigned a third element of the same set Λ, called their "*sum*" and denoted by $x + y$, such that the axioms are satisfied

a) $x + y = y + x$;
b) $x + (y + z) = (x + y) + z$;
c) there is a *zero* element o such that for any $x \in \Lambda$ we have $x + o = x$;
d) for each x there is an *opposite* element $-x$ such that $-x + x = o$.

2°. For any element x and any number λ there is a belonging Λ element, denoted λx and called the "*product of a number and an element*", such that the axioms are satisfied:

a) $1x = x$;
b) $(\lambda\mu)x = \lambda(\mu x)$.

3°. For the operations of addition of elements and multiplication of an element by a number, the *distributivity* axioms are satisfied:

a) $(\lambda + \mu)x = \lambda x + \mu x$;
b) $\lambda(x + y) = \lambda x + \lambda y \quad \forall x, y \in \Lambda$ and for any numbers λ, μ.

Notes: 1°. By “numbers” in the axioms of the second and third groups are meant real or complex numbers

2°. The comparison operation makes it possible to establish the facts of “equality of x and y” $(x = y)$ or “inequality of and y” $(x \neq y)$ for any pair of two elements belonging to the set Λ.

3°. The first group of axioms is equivalent to the requirement that Λ is an Abelian group with respect to the addition operation.

Examples A vector space is (it is assumed that the operations of addition and multiplication by a number are performed in accordance with the previously given definitions):

1°. The set of all vectors on the plane.

2°. The set of all vectors in space.

3°. The set of all n-component columns.

4°. The set of all polynomials of degree no higher than n.

5°. The set of all matrices of size $m \times n$.

6°. $C[\alpha,\beta]$ – the set of all functions continuous on $[\alpha,\beta]$.

7°. The set of all particular solutions of a homogeneous system of m linear equations with n unknowns.

Task 2.01 *Show that in the general case the set of radius vectors of points belonging to the plane* $(\vec{n},\vec{r})=\delta$ *is not a vector space. Find out for what values of* δ *the parameter this set will be a vector space.*

Task 2.02 *Show that a set consisting of one zero element is a vector space.*

Task 2.03 *Will the set of all positive numbers* R^+ *be a vector space*?

Solution. The answer depends on the way the operations of addition and multiplication by the number of elements of the set under consideration are introduced.

1°. Let the operations be introduced in a "natural" way. In this case, the set of positive numbers does not form a vector space, since it does not have a zero element.

2°. If the operation of "addition" is defined as the usual product of two numbers, and "multiplication of a number λ by x" is defined as raising a positive number x to a power $\lambda \in R$:

$$\text{«addition } x+y\text{»} \quad := \quad x \cdot y\,; \;\; x>0,\, y>0 \;,$$
$$\text{«product } \lambda x\text{»} \quad := \quad x^{\lambda}\,; \quad x>0,$$

then the set of positive numbers will be a vector space in which the role of the zero element is played by the number "1".

Solution is found

These theorems follow from the axiomatics of vector space:

Theorem 01 **A vector space has a unique zero element.**

Theorem 02 **For each element x of a vector space, the equality holds $0x = o$.**

Theorem 03 **For each element of a vector space, there is a unique opposite element.**

Theorem 04 **For $x \in \Lambda$ each, the opposite element is $-x = (-1)x$.**

Proof of Theorem 02

From the axiomatics of vector space, we have

$$x = 1x = (0+1)x = 0x + 1x = 0x + x.$$

Adding the element opposite to the element to both parts of the equality $x = 0x + x$, we obtain that $0x = o$.

The theorem is proved.

Linear dependence, dimension and basis in vector space

Definition

1°. The expression $\sum_{i=1}^{n} \lambda_i x_i$ is called a linear combination of elements $x_1, x_2, ..., x_n$ of the vector space Λ.

2°. Elements $x_1, x_2, ..., x_n$ of the vector space Λ are called *linearly dependent* if there exist numbers $\lambda_1, \lambda_2, ..., \lambda_n$ that are not equal to zero simultaneously, such that $\sum_{i=1}^{n} \lambda_i x_i = o$.

3°. Elements $x_1, x_2, ..., x_n$ of the vector space Λ are called linearly independent if it follows from the equality $\sum_{i=1}^{n} \lambda_i x_i = o$ that $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

For elements of a vector space, the following holds:

Lemma 01 **For some set of elements of a vector space to be linearly dependent, it is necessary and sufficient that one of these elements is a linear combination of the others.**

Lemma 02 **If some subset of a set of elements $x_1, x_2, \ldots, x_n$ is linearly dependent, then the elements $x_1, x_2, \ldots, x_n$ themselves are also linearly dependent.**

Definition — A *basis* in a vector space Λ is any ordered set of its elements if

1) these elements are *linearly independent*;
2) *any subset* of Λ containing $n+1$ elements, including these elements, is linearly dependent.

Definition — A vector space Λ is called *n*-dimensional and is denoted by Λ^n if it has a basis consisting of n elements. The number n is called the dimension of the vector space and is denoted by $\dim(\Lambda^n)$.

Theorem 05 — **For each element of a vector space Λ^n, there is a *unique* representation as a linear combination of basis elements.**

In general, a vector space may not have a basis. For example, a vector space consisting of one zero element has this property.

Examples of bases in vector spaces

Vector space	**Dimen-sionality**	**Example of a basis**
The set of all vectors in the plane	2	An ordered pair of non-collinear vectors on the plane.
The set of all vectors in the space	3	An ordered triple of normalized, pairwise orthogonal vectors.
The set of all n-component columns	n	n of columns of the form $\left\Vert\begin{matrix}1\\0\\\dots\\0\end{matrix}\right\Vert;\left\Vert\begin{matrix}0\\1\\\dots\\0\end{matrix}\right\Vert;\dots;\left\Vert\begin{matrix}0\\0\\\dots\\1\end{matrix}\right\Vert.$
The set of all algebraic polynomials of degree no higher than n	$n+1$	$n+1$ monomial of the form $P_1(\tau)=1;\ P_2(\tau)=\tau;$ $P_3(\tau)=\tau^2;\ P_4(\tau)=\tau^3;$ $\dots;$ $P_n(\tau)=\tau^{n-1};\ P_{n+1}(\tau)=\tau^n.$
The set of all matrices of size $m\times n$	$n\cdot m$	$n\cdot m$ all possible different matrices of size $m\times n$, all elements of which are equal to zero, except for one, equal to 1.
The set of all functions continuous on $[\alpha,\beta]$	∞	The basis does not exist, since for any natural n one can construct a linearly independent set consisting of $n+1$ elements. For example, the set of functions of the form $\{1,\ \tau,\ \tau^2,\ \dots,\ \tau^n\}$..
*The set of solutions of m a **homogeneous** system of linear equations with n unknowns and the rank of the fundamental matrix equal to r*	$n-r$	Normal fundamental system of solutions.

Subsets of a vector space

Subspace

Definition	A non-empty set Ω formed from elements of a vector space Λ is called a subspace of this vector space if for any $x, y \in \Omega$ and any number λ 1) $x + y \in \Omega$, 2) $\lambda x \in \Omega$.

Note: from this definition it follows that the set Ω itself is a vector space, since all axioms of operations in a vector space are obviously satisfied for it.

Examples

1°. The set of radius vectors of all points lying on some plane passing through the *origin* is a subspace in the set of radius vectors of all points of a three-dimensional geometric space.

2°. The set of all polynomials of degree no higher than n is a subspace in the vector space of continuous functions on $[\alpha, \beta]$.

3°. In the space of n-dimensional columns, the set of solutions of a homogeneous system of linear equations with n unknowns and with a fundamental matrix of rank r forms a subspace of dimension $n-r$.

4°. A subspace of any vector space will be:

a) the vector space itself;

b) a set consisting of one zero element.

Definition

Let two subspaces Ω_1 and Ω_2 of a vector space Λ be given. Then

1°. The *sum* of subspaces Ω_1 and Ω_2 is the set of all elements $x_1 + x_2 \in \Lambda$ provided that $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. The sum of subspaces Ω_1 and Ω_2 is denoted by $\Omega_1 + \Omega_2$.

2°. The *direct sum* of subspaces Ω_1 and Ω_2 is the set of all elements $x_1 + x_2 \in \Lambda$ provided that $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ and intersection $\Omega_1 \cap \Omega_2 = \{o\}$. The direct sum is denoted by $\Omega_1 \oplus \Omega_2$.

The following theorems hold

Theorem 06 **Both the sum Ω_1 and Ω_2 (as well as the intersection) of the subspaces are also subspaces of Λ.**

Theorem 07 **The dimension of the sum of subspaces Ω_1 and Ω_2 is equal to**

$$\dim(\Omega_1 + \Omega_2) = \dim(\Omega_1) + \dim(\Omega_2) - \dim(\Omega_1 \cap \Omega_2).$$

Linear span of a system of elements

Definition	The set of all possible linear combinations of some set of elements $\{x_1, x_2, ..., x_k\}$ of a vector space Λ is called the *linear span* of this set and is denoted by $L\{x_1, x_2, ..., x_k\}$.

Example The set of polynomials of degree no higher than n is the linear span of a set of monomials $\{1, \tau, \tau^2, ..., \tau^n\}$ in the vector space of continuous functions on $[\alpha, \beta]$.

Let a set of elements $\{x_1, x_2, ..., x_k\} \in \Lambda$ generating the linear span $L\{x_1, x_2, ..., x_k\}$ be given, then any element of this linear span has the form $x = \sum_{i=1}^{k} \lambda_i x_i$ and the following holds:

Theorem 08 **The set of all elements belonging to the linear span $L\{x_1, x_2, ..., x_k\}$ is a subspace of Λ dimension m, where m is the maximum number of linearly independent elements in the set $\{x_1, x_2, ..., x_k\}$.**

Hyperplane

Definition A set Γ formed from elements of the form $x + x_0$, where x_0 is an arbitrary fixed element of the vector space Λ, and x is any element of some subspace $\Omega \subseteq \Lambda$, is called a *hyperplane* (or linear manifold) in the vector space Λ.

Notes.

1°. In general, a hyperplane is not a subspace.
2°. If $\dim(\Omega) = k$, then we speak of an *k*-dimensional hyperplane.

For example, the general solution of a joint ***inhomogeneous*** system of linear equations with n unknowns is a hyperplane in the vector space of *n*-component columns.

Coordinate representation of elements of a vector space

Definition The coefficients $\xi_1, \xi_2, \ldots, \xi_n$ of the expansion in a basis $x = \sum_{i=1}^{n} \xi_i g_i$ are called the coordinates (or components) of an element $x \in \Lambda^n$ in the basis $\{g_1, g_2, \ldots, g_n\}$.

Note that by Theorem 05, an element x of a vector space Λ^n in a basis $\{g_1, g_2, \ldots, g_n\}$ is uniquely represented by a *n*-component column, called the coordinate representation of the element x in the basis $\{g_1, g_2, \ldots, g_n\}$:

$$\| x \|_g = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \ldots \\ \xi_n \end{matrix} \right\| .$$

Let us find out how operations with elements of a vector space are performed in coordinate form.

Let in a specific basis $x=\sum_{i=1}^{n}\xi_i g_i$ and $y=\sum_{i=1}^{n}\eta_i g_i$, then by virtue of the definition of the basis and axiomatic properties of a vector space the following relations will be valid:

1°. For the comparison operation: two elements in Λ^n are equal if and only if

$$\sum_{i=1}^{n}\xi_i g_i = x = y = \sum_{i=1}^{n}\eta_i g_i \ ,$$

or in coordinate form $x = y \quad \Leftrightarrow \quad \| x \|_g = \| y \|_g$.

2°. For the addition operation: $x+y=\sum_{i=1}^{n}(\xi_i+\eta_i)g_i$, or in coordinate form

$$\| x+y \|_g = \| x \|_g + \| y \|_g \, .$$

3°. For the operation of multiplication by a number: $\lambda x = \lambda\sum_{i=1}^{n}\xi_i g_i = \sum_{i=1}^{n}(\lambda\xi_i)g_i$, or in coordinate form $\| \lambda x \|_g = \lambda \| x \|_g$.

It follows that the elements of a finite-dimensional vector space can not only be represented by matrices (columns), but also the rules for performing operations with these elements coincide with the definition of the corresponding matrix operations.

Let us prove the rule of addition of vectors in coordinate form.

$$\| x+y \|_g = \left\| \sum_{i=1}^{n} \xi_i g_i + \sum_{i=1}^{n} \eta_i g_i \right\|_g =$$

$$= \left\| \sum_{i=1}^{n} (\xi_i + \eta_i)\, g_i \right\|_g =$$

$$= \left\| \begin{matrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \cdots \\ \xi_n + \eta_n \end{matrix} \right\| = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{matrix} \right\| + \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_n \end{matrix} \right\| = \| x \|_g + \| y \|_g .$$

In Λ^n the basis can be chosen in more than one way. Therefore it is necessary to establish a rule for changing the coordinates of an element of a vector space Λ^n when passing from one basis to another.

Let in Λ^n be given two bases: "old" $\{g_1, g_2, \ldots, g_n\}$ and "new" $\{g'_1, g'_2, \ldots, g'_n\}$ with corresponding coordinate decompositions of the element x:

$$x = \sum_{i=1}^{n} \xi'_i g_i \qquad \text{and} \qquad x = \sum_{i=1}^{n} \xi'_i g'_i .$$

And let, in addition, the decompositions of the elements of the "new" basis by the elements of the "old" one be known: $g'_j = \sum_{i=1}^{n} \sigma_{ij} g_i ;\ j = [1, n]$.

Definition The matrix $\|S\|$, the j-th $(\forall j = [1, n])$ column of which consists of the coefficients of the coordinate decompositions of the elements of the "new" basis by the elements of the "old" one, is called the *matrix of transition* from basis $\{g_1, g_2, \ldots, g_n\}$ to basis $\{g'_1, g'_2, \ldots, g'_n\}$.

In this case, the following will be true

Theorem **The coordinates** $\xi_1, \xi_2, \ldots, \xi_n$ **and** $\xi'_1, \xi'_2, \ldots, \xi'_n$ **are linked by relations** $\xi_i = \sum_{j=1}^{n} \sigma_{ij} \xi'_j \quad \forall i = [1, n]$**, called *transition formulas*, where the coefficients** σ_{ij} **are the elements of the transition matrix** $\|S\|$**.**

Note that if a column of elements of the "new" basis is expressed through a column of elements of the "old" basis using left-hand multiplication by the transposed transition matrix $\| S \|^{\mathrm{T}}$, then the coordinate column in the "old" basis is equal to the product of the transition matrix and the coordinate column in the "new" basis.

Indeed, considering the columns $\| x \|_g$ and $\| x \|_{g'}$ in the transition formulas as *two-dimensional* matrices, we obtain the formula $\xi_{i1} = \sum_{j=1}^{n} \sigma_{ij} \xi'_{j1} \quad \forall i = [1, n]$, which can be written in matrix form $\| x \|_g = \| S \| \| x \|_{g'}$.

Inverse transition formulas (from the "new" basis to the "old") $\| x \|_{g'} = \| S \|^{-1} \| x \|_g$ or, in co-coordinates $\xi'_{i1} = \sum_{j=1}^{n} \tau_{ij} \xi_{j1} \quad \forall i = [1, n]$, where τ_{ij} are the elements of the matrix $\| T \| = \| S \|^{-1}$, called the *inverse transition matrix*. Note that the matrix $\| T \|$, exists for any transition matrix $\| S \|$, since the latter is *not singular*.

AN IMPORTANT NOTE

Problem conditions often contain initial information, for example, in the following coordinate form:

$$\textit{Let an element of space be given in } \Lambda^4 \quad \begin{Vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Vmatrix} \ \ldots$$

without explicitly indicating in which basis this coordinate representation is used.

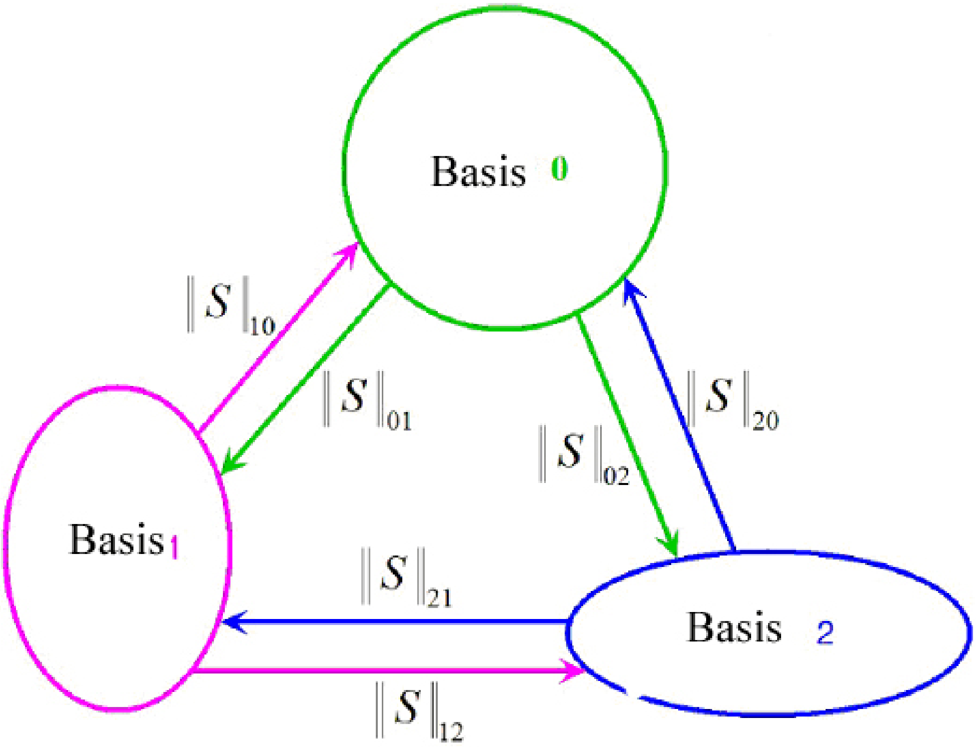
In this case, it is assumed by default that this basis is "standard", that is, formed by a set of elements of the form $\left\{ \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Vmatrix} \right\}$,. since in any basis $\{g_1, g_2, g_3, g_4\}$ the equalities are true

$$\begin{cases} g_1 = 1 \cdot g_1 + 0 \cdot g_2 + 0 \cdot g_3 + 0 \cdot g_4\,, \\ g_2 = 0 \cdot g_1 + 1 \cdot g_2 + 0 \cdot g_3 + 0 \cdot g_4 \\ g_3 = 0 \cdot g_1 + 0 \cdot g_2 + 1 \cdot g_3 + 0 \cdot g_4 \\ g_4 = 0 \cdot g_1 + 0 \cdot g_2 + 0 \cdot g_3 + 1 \cdot g_4 \end{cases}$$

and there is no need to specify the basis $\{g_1, g_2, g_3, g_4\}$.

Task 2.04 *Find the transition matrix from basis 1 to basis 2 if*

$$\|g_1'\| = \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix}, \ \|g_2'\| = \begin{Vmatrix} -1 \\ -1 \\ 1 \end{Vmatrix}, \ \|g_3'\| = \begin{Vmatrix} -1 \\ 0 \\ 1 \end{Vmatrix}, \qquad \|g_1''\| = \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}, \ \|g_2''\| = \begin{Vmatrix} -1 \\ 0 \\ 1 \end{Vmatrix} \quad \|g_3''\| = \begin{Vmatrix} 0 \\ 1 \\ 1 \end{Vmatrix}$$



Solution 1°. Let $\| x \|$, $\| x' \|$ and $\| x'' \|$ denote the coordinate columns of the element x in three bases: the *original*, $\{ g'_1 , g'_2 , g'_3 \}$ and $\{ g''_1 , g''_2 , g''_3 \}$. Then, according to the definition of the transition matrix, the equalities hold

$$\| x \| = \| S_{01} \| \| x' \| \quad \text{and} \quad \| x \| = \| S_{02} \| \| x'' \| ,$$

where the columns in the matrices $\| S_{01} \|$ and $\| S_{02} \|$ are the coordinate columns of the basis elements $\{ g'_1 , g'_2 , g'_3 \}$ and $\{ g''_1 , g''_2 , g''_3 \}$, that is,

$$\| S_{01} \| = \begin{Vmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{Vmatrix} \quad \text{and} \quad \| S_{02} \| = \begin{Vmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{Vmatrix} ,$$

Let us denote by $\| S_{12} \|$ the desired transition matrix from basis $\{ g'_1 , g'_2 , g'_3 \}$ to basis $\{ g''_1 , g''_2 , g''_3 \}$, for which $\| x' \| = \| S_{12} \| \| x'' \|$.

2°. But from the conditions $\| x \| = \| S_{01} \| \| x' \|$ and $\| x \| = \| S_{02} \| \| x'' \|$ it follows that $\| x' \| = \| S_{01} \|^{-1} \| S_{02} \| \| x'' \|$, since the matrix $\| S_{01} \|$ is obviously non-singular.

Then $\| S_{12} \| \| x'' \| = \| S_{01} \|^{-1} \| S_{02} \| \| x'' \|$ for any element $\| x'' \|$. This means that the desired transition matrix is $\| S_{12} \| = \| S_{01} \|^{-1} \| S_{02} \|$.

To calculate the product

$$\left\| \begin{matrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix} \right\|^{-1} \left\| \begin{matrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \right\|$$

we use the scheme for calculating an expression of the form $\| A \|^{-1} \| B \|$, which consists of the following.

We construct an extended matrix $\| A \mid B \|$, the left-hand side of which we reduce to the identity by some elementary transformations. In this case, applying the same transformations to the right-hand side of the extended matrix yields the desired product $\| A \|^{-1} \| B \|$.

3°. Let us describe the possible course of calculations performed simultaneously in the right and left parts of the augmented matrix.

From the first row of the augmented matrix $\| S_{01} \,|\, S_{02} \|$ we subtract the second and place the result in the place of the second:

$$\left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right\| \Rightarrow \left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right\|$$

Then we add the second and third rows, writing the result in the second

$$\left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right\| \Rightarrow \left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right\|.$$

After that we subtract the second from the third row and write the result in the place of the third

$$\left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right\| \Rightarrow \left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right\|$$

Finally, we write the sum of all three rows in the place of the first

$$\left\| \begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right\| \Rightarrow \left\| \begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right\|$$

In the right part we have obtained the desired matrix $\| S_{12} \| = \left\| \begin{array}{ccc} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array} \right\|$. This is the answer!

The solution is found

Isomorphism of Vector spaces

As an example, let us consider two vector spaces:

the set of polynomials $P_2(\tau)$ of degree no higher than 2, and
the set of vectors of a three-dimensional geometric space.

The operations of addition of polynomials and their multiplication by a number look as follows:

$$(\xi_1 + \xi_2\tau + \xi_3\tau^2) + (\eta_1 + \eta_2\tau + \eta_3\tau^2) =$$
$$= (\xi_1 + \eta_1) + (\xi_2 + \eta_2)\tau + (\xi_3 + \eta_3)\tau^2,$$
$$\lambda(\xi_1 + \xi_2\tau + \xi_3\tau^2) = (\lambda\xi_1) + (\lambda\xi_2)\tau + (\lambda\xi_3)\tau^2.$$

The same operations with three-dimensional vectors in coordinate form are in turn written as follows:

$$\left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| + \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\| = \left\| \begin{matrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \xi_3 + \eta_3 \end{matrix} \right\| ; \quad \lambda \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| = \left\| \begin{matrix} \lambda\xi_1 \\ \lambda\xi_2 \\ \lambda\xi_3 \end{matrix} \right\| .$$

Comparing these entries, we can conclude that the nature of these sets does not play a role when their characteristics are studied, related only to the operations of comparison, addition and multiplication by a number.

The noted property of vector spaces is called *isomorphism*. It is described more precisely by

Definition

Two vector spaces Λ_1 and Λ_2 are called *isomorphic* if there exists a one-to-one mapping $\hat{F}: \Lambda_1 \to \Lambda_2$, such that for $\forall\lambda$ and $\forall x, y \in \Lambda_1$,

1°. $\hat{F}(x+y) = \hat{F}x + \hat{F}y$;

2°. $\hat{F}(\lambda x) = \lambda \hat{F}x.$

The mapping $\hat{F}$ is called an *isomorphism* of the vector spaces $\hat{F}$ and Λ_2.

Theorem (on isomorphism)

Two finite-dimensional vector spaces Λ_1 and Λ_2 are isomorphic if and only if their *dimensions are equal*.

Example The isomorphism of one-dimensional spaces R of real numbers and all positive numbers R^{+} (with the operations defined in the conditions of Task 2.03) is defined using the functions $x = \ln(y)$ *and* $y = e^{x}; x \in R; y \in R^{+}$.

An obvious consequence of the previous theorem is the isomorphism of any linear n-dimensional space and the vector space of n-component columns.

For example, we have

Theorem **The maximum number of linearly independent elements in any finite set of elements from Λ^{n} is equal to the rank of the matrix whose columns contain the coordinates of the elements of this set in some basis.**

Corollary **k elements in Λ^{n} are linearly dependent if and only if the rank of the matrix whose columns contain the coordinates of these elements in some basis is less than $\min\{n,k\}$.**

Let in Λ^{n} be given a basis $\{g_1, g_2, \ldots, g_n\}$ in which the coordinate representation of the elements is represented as $x = \sum_{i=1}^{n} \xi_i g_i$. Then we have

Corollary **Every *homogeneous* linear system of m linear equations with n unknowns $\sum_{i=1}^{n} \alpha_{ji}\xi_i = 0$, $j = [1,m]$ determines some subspace Ω in Λ^{n}.**

Thus, each subspace in Λ^{n} can be defined either by a homogeneous system of linear equations or as the linear span of a basis of the subspace – the fundamental system of its solutions.

Task 2.05 *In the vector space of polynomials of degree no higher than 3, find the basis and dimension of the intersection of two linear spans of elements:*

$$x_1(\tau) = 1 + 2\tau + \tau^2 + 3\tau^3,$$
$$x_2(\tau) = -1 + 8\tau - 6\tau^2 + 5\tau^3,$$
$$x_3(\tau) = 10\tau - 5\tau^2 + 8\tau^3$$

and

$$y_1(\tau) = 1 + 4\tau - \tau^2 + 5\tau^3,$$
$$u \quad y_2(\tau) = 3 - 2\tau + 6\tau^2 + 3\tau^3,$$
$$y_3(\tau) = 4 + 2\tau + 5\tau^2 + 8\tau^3.$$

Solution

1°. Each of the linear spans is a subspace. The first of them Π_1 is formed by elements of the form $x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, and the second – by elements of Π_2, respectively

$$y = \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 .$$

Let us compose homogeneous systems of linear equations defining these subspaces. Let each of the equations of these systems have the form

$$\left\| \alpha_1 \alpha_2 \alpha_3 \alpha_4 \right\| \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{matrix} \right\| = 0 .$$

Then, using the isomorphism between *the vector space of polynomials of degree no higher than 3* and the space of four-component columns of the form, we have for elements of Π_1

$$\left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{matrix} \right\| = \lambda_1 \left\| \begin{matrix} 1 \\ 2 \\ 1 \\ 3 \end{matrix} \right\| + \lambda_2 \left\| \begin{matrix} -1 \\ 8 \\ -6 \\ 5 \end{matrix} \right\| + \lambda_3 \left\| \begin{matrix} 0 \\ 10 \\ -5 \\ 8 \end{matrix} \right\| ,$$

where $\lambda_1, \lambda_2, \lambda_3$ are any numbers.

It gives us the condition

$$\left\| \alpha_1 \alpha_2 \alpha_3 \alpha_4 \right\| \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{matrix} \right\| = \left\| \alpha_1 \alpha_2 \alpha_3 \alpha_4 \right\| (\lambda_1 \left\| \begin{matrix} 1 \\ 2 \\ 1 \\ 3 \end{matrix} \right\| + \lambda_2 \left\| \begin{matrix} -1 \\ 8 \\ -6 \\ 5 \end{matrix} \right\| + \lambda_3 \left\| \begin{matrix} 0 \\ 10 \\ -5 \\ 8 \end{matrix} \right\|) = 0 ,$$

which will be satisfied for any $\lambda_1,\lambda_2,\lambda_3$, if the numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ form a solution to the following system of linear equations:

$$\begin{cases} \alpha_1+2\alpha_2+\alpha_3+3\alpha_4=0,\\ -\alpha_1+8\alpha_2-6\alpha_3+5\alpha_4=0,\\ 10\alpha_2-5\alpha_3+8\alpha_4=0. \end{cases}$$

Having solved this system, we obtain a general solution in the form

$$\begin{Vmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{Vmatrix}=\kappa_1\begin{Vmatrix}-4\\ 1\\ 2\\ 0\end{Vmatrix}+\kappa_2\begin{Vmatrix}-7\\ -4\\ 0\\ 5\end{Vmatrix};\ \forall\kappa_1,\kappa_2\ ,$$

from which we conclude that there are two independent sets of the desired numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, and, therefore, the homogeneous system of linear equations defining the subspace Π_1 has the form

$$\begin{cases} -4\xi_1+\xi_2+2\xi_3=0,\\ -7\xi_1-4\xi_2+5\xi_4=0. \end{cases}$$

Similarly, we construct a homogeneous system of linear equations defining Π_2:

$$\begin{cases} -22\xi_1+9\xi_2+14\xi_3=0,\\ -11\xi_1-6\xi_2+7\xi_4=0. \end{cases}$$

Finally, the subspace $\Pi_1\cap\Pi_2$ will be defined by the system

$$\begin{cases} -4\xi_1 + \xi_2 + 2\xi_3 = 0, \\ -7\xi_1 - 4\xi_2 + 5\xi_4 = 0, \\ -22\xi_1 + 9\xi_2 + 14\xi_3 = 0, \\ -11\xi_1 - 6\xi_2 + 7\xi_4 = 0, \end{cases}$$

the general solution of which is

$$\left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{matrix} \right\| = \sigma_1 \left\| \begin{matrix} 2 \\ -6 \\ 7 \\ -2 \end{matrix} \right\|, \quad \forall \sigma_1 \in \mathbf{R}$$

and, therefore, for $\Pi_1 \cap \Pi_2$ we have $\dim(\Pi_1 \cap \Pi_2) = 1$ and a basis consisting of one element $\left\| \begin{matrix} 2 \\ -6 \\ 7 \\ -2 \end{matrix} \right\|$.

Solution is found