Linear dependencies in a vector space

Definition

Let each element x of a vector space Λ be assigned a unique element y of a vector space Λ^*. Then we say that in Λ is given an operator acting Λ in and having values in Λ^*, the action of which is denoted as $y = \hat{A}x$.

In this case, the element y is called the *image* of the element x, and the element x is called the *preimage* of the element y.

Operators are usually divided into *mappings*, if $\Lambda^* \not\subset \Lambda$, and *transformations*, if $\Lambda^* \subseteq \Lambda$.

In what follows, except in specially stipulated cases, it will be assumed that it is clear from the context whether we are talking about a mapping or a transformation.

Definition

An operator $y = \hat{A}x$ is called linear if for any $x, x_1, x_2 \in \Lambda$ and any λ number the equalities

1°. $\hat{A}(x_1 + x_2) = \hat{A}x_1 + \hat{A}x_2$ and

2°. $\hat{A}(\lambda x) = \lambda \hat{A}x$

are valid.

Examples

1°. In the space of 2-dimensional vectors, a linear operator is the rule

$$\left\| \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right\| = \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\| \left\| \begin{matrix} \xi_1 \\ \xi_2 \end{matrix} \right\|,$$

connecting the vector-preimage $x = \left\| \begin{matrix} \xi_1 \\ \xi_2 \end{matrix} \right\|$ with the vector-image $y = \left\| \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right\|$.

2°. In the space of infinitely differentiable functions, a linear operator is the operation of *differentiation*, which associates each element of this space with its *derivative* function.

3°. In the space of continuous functions, a linear operator is the operation of *multiplying* a continuous function by an independent variable τ .

Task 3.01 *Is the operator $\hat{A}$ that assigns to each element $x \in \Lambda$ a fixed element $a \in \Lambda$ a linear operator?*

Solution If the element is $a = o$, then $\hat{A}$ is a linear operator. Indeed, if $\hat{A}$ is linear, then, on the one hand,

$$\hat{A}(\lambda a + \mu a) = \lambda \hat{A} a + \mu \hat{A} a = \lambda a + \mu a = (\lambda + \mu) a,$$

but, on the other hand,

$$\forall \lambda, \mu : \hat{A}(\lambda a + \mu a) = a \quad \Rightarrow \quad a = (\lambda + \mu) a \quad \Rightarrow \quad a = o.$$

Solution is found

Operations with linear operators

Definition

Linear operators $\hat{A}$ and $\hat{B}$ are called *equal* (which is denoted by $\hat{A} = \hat{B}$) if

$$\forall x \in \Lambda: \quad \hat{A}x = \hat{B}x .$$

The *sum* of linear operators $\hat{A}$ and $\hat{B}$ is the operator , denoted by $\hat{A} + \hat{B}$, which assigns to each element x of a vector space Λ an element $\hat{A}x + \hat{B}x$.

Lemma

The sum of two linear operators is a linear operator.

Definition

The *zero operator* $\hat{O}$ is the operator that assigns to each element x of a vector space Λ the zero element of this vector space.

Definition

The *opposite operator* to the operator $\hat{A}$ is the operator, denoted by $-\hat{A}$, for which $\hat{A} + \left(-\hat{A}\right) = \hat{O}$ is true.

From the solution of Task 3.01 it follows that the zero operator is linear. It can also be shown that the operator opposite to any linear operator is also linear.

Lemma **For any linear operators $\hat{A}$, $\hat{B}$ and $\hat{C}$, and the relations**

$$\hat{A}+\hat{B}=\hat{B}+\hat{A};$$

$$(\hat{A}+\hat{B})+\hat{C}=\hat{A}+(\hat{B}+\hat{C});$$

$$\hat{A}+\hat{O}=\hat{A};\ \hat{A}+(-\hat{A})=\hat{O}$$

are valid.

Definition The *product* of a number λ by a linear operator $\hat{A}$ is an operator, denoted by $\lambda\hat{A}$, which assigns to each element x of the vector space Λ the element $\lambda(\hat{A}x)$.

Lemma **The product of a number by a linear operator is a linear operator for which the relations**

$$\alpha(\beta\hat{A})=(\alpha\beta)\hat{A};\quad 1\hat{A}=\hat{A};$$
$$(\alpha+\beta)\hat{A}=\alpha\hat{A}+\beta\hat{A};$$
$$\alpha(\hat{A}+\hat{B})=\alpha\hat{A}+\alpha\hat{B}.$$

Theorem **The set of *all linear operators* acting in a vector space Λ is a vector space.**

Definition The *product* (*composition*, or *superposition*) of linear operators $\hat{A}$ and $\hat{B}$ is the operator, denoted by $\hat{A}\hat{B}$, which assigns to each element x of the vector space Λ the element $\hat{A}(\hat{B}x)$.

Theorem **The product of linear operators is a linear operator for which the relations**

$$\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}\ ; \quad \hat{A}(\hat{B}+\hat{C}) = \hat{A}\hat{B}+\hat{A}\hat{C};$$

$$(\hat{A}+\hat{B})\hat{C} = \hat{A}\hat{C}+\hat{B}\hat{C}.$$

are valid.

In the general case, the product of linear operators does not have the *commutation* property (or, in other words, the operators *do not commute*), that is $\hat{A}\hat{B} \neq \hat{B}\hat{A}$.
.

Definition The operator $\hat{A}\hat{B} - \hat{B}\hat{A}$ is called the *commutator* of the operators $\hat{A}$ and $\hat{B}$.

The commutator of commuting operators is the *zero* operator.

Task 3.02 *In the* vector *space of algebraic polynomials* $P_n(\tau)=\sum\limits_{k=0}^{n}\alpha_k\tau^k$ *, find the commutator for the operators:* $\hat{A}$ *, which associates a polynomial with its derivative function, and* $\hat{B}$ *is the operator of multiplication of a polynomial by an independent variable.*

Solution Let us construct the operator $\hat{A}\hat{B}-\hat{B}\hat{A}$. For any $P_n(\tau)$ we have

$$\hat{A}P_n(\tau)=\frac{d}{d\tau}P_n(\tau)=\frac{d}{d\tau}\left(\sum_{k=0}^{n}\alpha_k\tau^k\right)=\sum_{k=1}^{n}k\,\alpha_k\tau^{k-1},$$

$$\hat{B}P_n(\tau)=\tau\left(\sum_{k=0}^{n}\alpha_k\tau^k\right)=\sum_{k=0}^{n}\alpha_k\tau^{k+1}.$$

Wherefrom we obtain

$$\hat{B}(\hat{A}P_n(\tau))=\tau\left(\sum_{k=1}^{n}k\,\alpha_k\tau^{k-1}\right)=\sum_{k=1}^{n}k\,\alpha_k\tau^k=\sum_{k=0}^{n}k\,\alpha_k\tau^k,$$

$$\hat{A}(\hat{B}P_n(\tau))=\frac{d}{d\tau}\left(\sum_{k=0}^{n}\alpha_k\tau^{k+1}\right)=\sum_{k=0}^{n}(k+1)\alpha_k\tau^k,$$

$$(\hat{A}\hat{B}-\hat{B}\hat{A})P_n(\tau)=\left(\sum_{k=0}^{n}(k+1)\alpha_k\tau^k\right)-\left(\sum_{k=0}^{n}k\,\alpha_k\tau^k\right)=$$

$$=\sum_{k=0}^{n}\alpha_k\tau^k=P_n(\tau).$$

Therefore, these linear operators *do not commute*.

Solution is found

In Task 3.02 it turned out that the action of the operator $\hat{A}\hat{B}-\hat{B}\hat{A}$ on any element of the vector space of polynomials does not lead to a change in this element. Let us introduce a special name for such an operator.

Definition An operator $\hat{E}$ is called a *unit* (or *identity*) operator if it assigns the same element to each element of the vector space, that is,

$$\hat{E}x = x \quad \forall x \in \Lambda .$$

The following relations hold: $\hat{A}\hat{E} = \hat{E}\hat{A} = \hat{A} \quad \forall \hat{A}$, as well as linearity and uniqueness $\hat{E}$..

Definition An operator $\hat{B}$ is called the inverse of a linear operator $\hat{A}$ (is denoted by $\hat{A}^{-1}$), if $\hat{A}\hat{B} = \hat{B}\hat{A} = \hat{E}$.

Example In the vector space of functions $f(\tau)$, having on $[\alpha,\beta]$ the derivative of any order and satisfying the conditions $f^{(k)}(\alpha)=0$; $k = 0,1,2,\ldots$, the differentiation operator $\hat{A}f = \dfrac{df}{d\tau}$ and operator $\hat{B}f = \int\limits_{\alpha}^{\tau} f(\sigma)d\sigma$ (the integration operator with a variable upper limit) are mutually inverse.

Indeed,

$$\hat{A}\hat{B}f = \frac{d}{d\tau}\int\limits_{\alpha}^{\tau} f(\sigma)d\sigma = f(\tau) = \hat{E}f \quad \text{and} \quad \hat{B}\hat{A}f = \int\limits_{\alpha}^{\tau}\frac{df}{d\sigma}d\sigma = f(\tau) - f(a) = f(\tau) = \hat{E}f \ .$$

Notes.

1°. Not every linear operator has an inverse operator. For example, the zero operator $\hat{O}$ has no inverse. Indeed, let $\hat{O}x = o$ for all $\forall x \in \Lambda$, then for any $\hat{B}$:

$$(\hat{B}\hat{O})x = \hat{B}(\hat{O}x) = o \quad \forall x \in \Lambda ,$$

and, therefore, the equality $\hat{B}\hat{O} = \hat{E}$ does not hold for any $\hat{B}$.

2°. The inverse operator, if it exists, is unique.

3°. In the case of an infinite-dimensional vector space, the validity of the condition $\hat{A}\hat{B} = \hat{E}$ may not imply the fulfillment of the condition $\hat{B}\hat{A} = \hat{E}$.

This is the case, for example, in the space of polynomials

$$P_n(\tau) = \sum_{k=0}^{n} \alpha_k \tau^k$$

for a pair of operators $\hat{A}$ and $\hat{B}$, where $\hat{B}$ is the operator of multiplication of a polynomial by an independent variable, and the operator $\hat{A}$ assigns to the polynomial $\sum\limits_{k=0}^{n} \alpha_k \tau^k$ the polynomial $\sum\limits_{k=1}^{n} \alpha_k \tau^{k-1}$.

Coordinate representation of linear operators

Let in Λ^n be given a basis $\{g_1, g_2, ..., g_n\}$ and a linear operator $\hat{A}$ that is a mapping from Λ^n in Λ^m with basis $\{f_1, f_2, ..., f_m\}$. It is known that $\forall x \in \Lambda^n$ there is a unique decomposition

$$x = \sum_{i=1}^{n} \xi_i g_i \ , \quad \text{that is} \quad \| x \|_g = \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{Vmatrix}.$$

Similarly, in Λ^m there is a unique decomposition $y = \hat{A}x$ for which, due to linearity, a representation of the form is valid

$$y = \hat{A}x = \hat{A}\left(\sum_{i=1}^{n} \xi_i g_i \right) = \sum_{i=1}^{n} \xi_i \hat{A} g_i .$$

Taking into account the possibility and uniqueness in space Λ^m of a decomposition of the form $\hat{A} g_i = \sum_{k=1}^{m} \alpha_{ki} f_k \quad \forall i = [1, n]$, we obtain that

$$y = \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \alpha_{ki} \xi_i \right) f_k .$$

On the other hand, if $\| y \|_f = \begin{Vmatrix} \eta_1 \\ \eta_2 \\ \dots \\ \eta_m \end{Vmatrix}$ is a coordinate representation of y in Λ^m, then we have the equality $y = \sum_{k=1}^{m} \eta_k f_k$. Finally, due to the uniqueness of the decomposition of an element of a finite-dimensional space by a basis, we obtain $\eta_k = \sum_{i=1}^{n} \alpha_{ki} \xi_i \ ; k = [1, m]$.

These relations allow us to find a coordinate representation of the images of elements of a vector space by the coordinate representation of the preimage. In this case, we note that each linear operator of the form $\hat{A} : \Lambda^n \to \Lambda^m$ in a pair of specific bases is completely and uniquely described by a matrix of size with elements α_{ki}.

Definition A matrix of size $m \times n$, the columns of which are formed by the components of the elements $\hat{A}g_i$:

$$\| \hat{A} \|_{fg} = \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{Vmatrix},$$

is called the *matrix of a linear operator* $\hat{A}$ in the bases

$\{g_1, g_2, \dots, g_n\} \in \Lambda^n$ and $\{f_1, f_2, \dots, f_m\} \in \Lambda^m$.

In matrix form, the relations $\eta_k = \sum_{i=1}^{n} \alpha_{ki} \xi_i \; ; k = [1, m]$ have the form $\| y \|_f = \| \hat{A} \|_{fg} \| x \|_g$, which can be easily verified using their two-index notation:

$$\eta_{k1} = \sum_{i=1}^{n} \alpha_{ki} \, \xi_{i1} \; ; \quad k = [1, m].$$

The result obtained is formulated as

Theorem **There is a one-to-one correspondence between the set of all linear operators of the form $\hat{A} : \Lambda^n \to \Lambda^m$ and the set of all matrices of size $m \times n$.**

Task 3.03. *In the vector space of algebraic polynomials of degree no higher than* n*, of the form* $P_m(\tau)=\sum\limits_{k=0}^{m}\xi_k\tau^k$ *with a basis* $\{1,\tau,\tau^2,\dots,\tau^n\}$*, find the matrix of the differentiation operator* $\hat{D}=\dfrac{d}{d\tau}$ *with respect to the variable* τ.

Solution: 1) The dimension of the vector space of algebraic polynomials of degree no higher than n is equal to $n+1$. And the differentiation operator acting in this space is a linear transformation.

2) The image of the basis element τ^k under the action of the differentiation operator for $1\le k\le n$ will be the function $k\tau^{k-1}$, and for $k=0$ - a polynomial equal to 0 at $\forall\tau$.

3) По By definition, the columns of the matrix of the operator are the *coordinate columns* of the images of the basis elements. And for the image of the $k+1$-th basis element, the coordinate decomposition has the form

$$k\tau^{k-1}=0\cdot 1+0\cdot\tau+\ldots+k\cdot\tau^{k-1}+0\cdot\tau^k+\ldots+0\cdot\tau^n\,.$$

Therefore, the desired matrix will be

$$\|\hat{D}\|=\begin{Vmatrix}0&1&0&\dots&0&0&0\\0&0&2&\dots&0&0&0\\0&0&0&\dots&0&0&0\\\dots&\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0&n-1&0\\0&0&0&\dots&0&0&n\\0&0&0&\dots&0&0&0\end{Vmatrix}$$

Solution is found

Task 3.04. *In the vector space of square matrices of the second order* $A = \begin{Vmatrix} -2 & 1 \\ -6 & -7 \end{Vmatrix}$, *a transformation* $\hat{M}$ *is defined by the formula* $Y = \hat{M}\,X \equiv A\,X$, *where* $A = \begin{Vmatrix} -2 & 1 \\ -6 & -7 \end{Vmatrix}$, *and the matrix* Y *is the image of the matrix* X *under this transformation.*

It is required to find the matrix of this transformation in the standard basis $\{\, e_1 \quad e_2 \quad e_3 \quad e_4 \,\}$, *for which*

$$e_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix},\ e_2 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix},\ e_3 = \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix},\ e_4 = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}.$$

Solution: 1) We have: the dimension of the vector space of square matrices of the second order is 4, and the linearity of this transformation follows from the property of distributivity of matrix multiplication:

$$\| A \| \left(\lambda_1 \| X \|_1 + \lambda_2 \| X \|_2 \right) = \lambda_1 \| A \| \| X \|_1 + \lambda_2 \| A \| \| X \|_2 = \lambda_1 \| Y \|_1 + \lambda_2 \| Y \|_2$$

2) The coordinate representation (coordinate column) of an element in the vector space of square matrices of the second order of the form $\left\| \begin{matrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{matrix} \right\|$ is a 4-component column $\left\| \begin{matrix} \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix} \right\|$. The coordinate representation of a linear transformation in this space will be a square matrix of the 4th order, the columns of which are the coordinate representations of the images of the basis elements.

3) Let's find these representations. We have

$$\hat{M}e_1 = \begin{Vmatrix} -2 & 1 \\ -6 & -7 \end{Vmatrix} \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix} = \begin{Vmatrix} -2 & 0 \\ -6 & 0 \end{Vmatrix} \quad \Rightarrow \quad \| \hat{M}e_1 \| = \begin{Vmatrix} -2 \\ 0 \\ -6 \\ 0 \end{Vmatrix}$$

$$\hat{M}e_2 = \begin{Vmatrix} -2 & 1 \\ -6 & -7 \end{Vmatrix} \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 0 & -2 \\ 0 & -6 \end{Vmatrix} \quad \Rightarrow \quad \| \hat{M}e_2 \| = \begin{Vmatrix} 0 \\ -2 \\ 0 \\ -6 \end{Vmatrix}$$

$$\hat{M}e_3 = \begin{Vmatrix} -2 & 1 \\ -6 & -7 \end{Vmatrix} \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ -7 & 0 \end{Vmatrix} \quad \Rightarrow \quad \| \hat{M}e_3 \| = \begin{Vmatrix} 1 \\ 0 \\ -7 \\ 0 \end{Vmatrix}$$

$$\hat{M}e_4 = \begin{Vmatrix} -2 & 1 \\ -6 & -7 \end{Vmatrix} \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix} = \begin{Vmatrix} -2 & 0 \\ -6 & 0 \end{Vmatrix} \quad \Rightarrow \quad \| \hat{M}e_4 \| = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ -7 \end{Vmatrix}$$

4) From the obtained coordinate representations of the images of the basic elements, we compose the desired transformation matrix $\hat{M}$

$$\left\| \hat{M} \right\|_e = \left\| \begin{matrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ -6 & 0 & -7 & 0 \\ 0 & -6 & 0 & -7 \end{matrix} \right\|$$

Solution is found

Operations with linear operators in coordinate (matrix) form

We will further consider operators of the form $\hat{A}:\Lambda^n \to \Lambda^n$, i.e. linear transformations acting in Λ^n with a basis $\{g_1, g_2,..., g_n\}$, the matrix of which is square, of order n. The previously introduced operations with matrices allow us to describe operations with linear operators in a specific basis in the following form

1°. Comparison of operators: $\hat{A} = \hat{B} \quad \Leftrightarrow \quad \| \hat{A} \|_g = \| \hat{B} \|_g$.

2°. Addition of operators: $\|\hat{A} + \hat{B}\|_g = \|\hat{A}\|_g + \|\hat{B}\|_g$.

3°. Multiplication of an operator by a number: $\| \lambda \hat{A} \|_g = \lambda \| \hat{A} \|_g$.

4°. Product of operators: $\|\hat{A}\hat{B}\|_g = \|\hat{A}\|_g \|\hat{B}\|_g$.

5°. Inversion of operators: $\|\hat{A}^{-1}\|_g = \|\hat{A}\|_g^{-1}$.

Corollary. **The dimension of the** vecto**r space of linear mappings of the form** $\Lambda^n \to \Lambda^m$ **is equal to** $m \times n$**.**

Change of the matrix of a linear operator when changing the basis

Let us find out how the matrix $\| \hat{A} \|_{fg}$ of a linear mapping $\hat{A} : \Lambda^n \to \Lambda^m$ changes when changing the bases. Let in Λ^n be given two bases $\{g_1, g_2, ..., g_n\}$ and $\{g'_1, g'_2, ..., g'_n\}$, connected by the transition matrix $\| G \|$, and in - two bases $\{f_1, f_2, ..., f_m\}$ and $\{f'_1, f'_2, ..., f'_m\}$ with the transition matrix $\| F \|$. Find the relation connecting $\| \hat{A} \|_{fg}$ and $\| \hat{A} \|_{f'g'}$.

In this case, the following holds

Theorem **The matrix of a linear operator $\| \hat{A} \|_{f'g'}$ in the bases $\{g'_1, g'_2, ..., g'_n\}$ and $\{f'_1, f'_2, ..., f'_m\}$ is connected with the matrix of the same operator $\| \hat{A} \|_{fg}$ in the bases $\{g_1, g_2, ..., g_n\}$ and $\{f_1, f_2, ..., f_m\}$ by the relation**

$$\| \hat{A} \|_{f'g'} = \| F \|^{-1} \| \hat{A} \|_{fg} \| G \| .$$

Corollary **The matrix of a linear transformation changes according to the rule when moving from basis $\{g_1, g_2, \ldots, g_n\}$ to basis $\{g'_1, g'_2, \ldots, g'_n\}$ in Λ^n**

$$\| \hat{A} \|_{g'} = \| S \|^{-1} \| \hat{A} \|_g \| S \| .$$

The determinant of the matrix of a linear transformation does not depend on the choice of basis in Λ^n.

Indeed, from $\det \| \hat{A} \|_{g'} = \det (\| S \|^{-1} \| \hat{A} \|_g \| S \|)$, due to

$$\det (\| S \|^{-1} \| \hat{A} \|_g \| S \|) = (\det \| S \|^{-1})(\det \| \hat{A} \|_g)(\det \| S \|)$$

and

$$\det \| S \|^{-1} = \frac{1}{\det \| S \|}, \quad \text{where} \quad \det \| S \| \neq 0 ,$$

we obtain that, $\det \| \hat{A} \|_{g'} = \det \| \hat{A} \|_g$.

Domain and kernel of a linear operator

Considering a linear operator acting in a vector space as a generalization of the concept of a function, it is natural to raise the question of the domain and range of linear operators.

By the domain of a linear operator $\hat{A}$ we will understand the set of images of all elements $x \in \Lambda$, that is, elements of the form $\hat{A}x$. In this case, it is obvious that for any linear operator its domain coincides with Λ.

The answer to the question: “What is the domain of a linear operator?” is given by

Theorem **Тогда Let $\hat{A}$ be a linear operator acting in a vector space Λ. Then**

1°. The set of elements $\hat{A}x \ \forall x \in \Lambda$ is a subspace in Λ.

2°. If, in addition $\Lambda = \Lambda^n$, with a basis $\{g_1, g_2, \dots, g_n\}$, then the dimension of this subspace is equal to $\operatorname{rg} \| \hat{A} \|_g$.

Corollary **The dimension of the range of a linear operator $\hat{A}$ acting on some subspace of a vector space $\Lambda^* \subseteq \Lambda$ does not exceed $\dim(\Lambda^*)$.**

Another important characteristic of a linear operator $\hat{A}$ is the set of elements of the vector space Λ called the *kernel* of the linear operator $\hat{A}$ and denoted by $\ker\hat{A}$.

Definition The *kernel* of a linear operator $\hat{A}$ consists of elements $x \in \Lambda$, such that $\hat{A}x = o$.

Theorem **If $\Lambda = \Lambda^n$ and $\operatorname{rg}\hat{A} = r$, then $\ker\hat{A}$ is a subspace of Λ^n and $\dim(\ker\hat{A}) = n - r$.**

Task 3.05. *Let in the bases*

$$\left\{ e_1 = \left\| \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \quad e_2 = \left\| \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right\| \quad e_3 = \left\| \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} \right\| \quad e_4 = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\| \right\} \subset \Lambda^4$$

$$\text{and} \left\{ g_1 = \left\| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\| \quad g_2 = \left\| \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\| \quad g_3 = \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\| \right\} \subset \Lambda^3$$

the matrix of a linear mapping $\Lambda^4 \xrightarrow{\hat{A}} \Lambda^3$ *have the form*

$$\left\| \hat{A} \right\|_{ge} = \left\| \begin{matrix} 1 & -3 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -2 & -1 & 2 \end{matrix} \right\| .$$

It is required to find the bases both in the kernel and in the range of values of a given linear mapping $\hat{A}$.

Solution: 1) By definition, the kernel of a linear operator $\hat{A}$ is the set of all elements $x \in \Lambda$ such that $\hat{A}x = o$. Moreover, in a finite-dimensional space, this equation can be written in matrix form $\|\hat{A}\|\|x\| = \|o\|$, which for the problem being solved will be

$$\begin{Vmatrix} 1 & -3 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -2 & -1 & 2 \end{Vmatrix} \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \quad \Rightarrow \quad \begin{cases} \xi_1 - 3\xi_2 \qquad + \xi_4 = 0, \\ -\xi_1 + \xi_2 - \xi_3 + \xi_4 = 0, \\ \quad -2\xi_2 - \xi_3 + 2\xi_4 = 0. \end{cases}$$

The general solution of the last homogeneous system of linear equations has the form

$$\begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Vmatrix} = \lambda_1 \begin{Vmatrix} 3 \\ 1 \\ -2 \\ 0 \end{Vmatrix} + \lambda_2 \begin{Vmatrix} -1 \\ 0 \\ 2 \\ 1 \end{Vmatrix} \quad \forall \lambda_1, \lambda_2$$

and is a description of the kernel, the mapping specified in the problem statement.

As is known, this kernel is a vector space. In this case, the basis in the kernel can be a fundamental system of solutions, for example, a pair of elements

$$\left\{ \begin{Vmatrix} 3 \\ 1 \\ -2 \\ 0 \end{Vmatrix}, \begin{Vmatrix} -1 \\ 0 \\ 2 \\ 1 \end{Vmatrix} \right\}.$$

2) The set of values of a linear mapping, according to the definition, has the form $y = \hat{A}x \quad \forall x \in \Lambda$. In the finite-dimensional case, this equality takes the form $\| y \| = \| \hat{A} \| \| x \|$, which for the problem being solved leads to a system of linear equations

$$\begin{Vmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Vmatrix} = \begin{Vmatrix} 1 & -3 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -2 & -1 & 2 \end{Vmatrix} \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Vmatrix} \quad \Rightarrow \quad \begin{cases} \xi_1 - 3\xi_2 \qquad + \xi_4 = \eta_1, \\ -\xi_1 + \xi_2 - \xi_3 + \xi_4 = \eta_2, \\ \quad -2\xi_2 - \xi_3 + 2\xi_4 = \eta_3. \end{cases} \qquad \text{(I)}$$

Here $\begin{Vmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Vmatrix}$ is the coordinate representation of an arbitrary element $y \in \Lambda^3$.

Now note that the set of values of the mapping $\hat{A}$ is the set of all $y \in \Lambda^3$ those for which system (I) is consistent. We will find this set using the Kronecker-Capelli theorem.

First, we will reduce the extended matrix of system (I) to a simplified form by successive elementary transformations. To do this, it is sufficient to subtract the sum of the first two from the third row and write the result in place of the third. We obtain

$$\left\| \begin{array}{cccc|c} 1 & -3 & 0 & 1 & \eta_1 \\ -1 & 1 & -1 & 1 & \eta_2 \\ 0 & -2 & -1 & 2 & \eta_3 \end{array} \right\| \sim \left\| \begin{array}{cccc|r} 1 & -3 & 0 & 1 & \eta_1 \\ -1 & 1 & -1 & 1 & \eta_2 \\ 0 & 0 & 0 & 0 & -\eta_1-\eta_2+\eta_3 \end{array} \right\|$$

Since elementary transformations do not change the ranks of matrices, it is obvious that the rank of the main matrix of system (I) will be equal to the rank of the extended one if $-\eta_1-\eta_2+\eta_3=0$. And this, according to the Kronecker-Capelli theorem, means the compatibility of system (I).

The last equality specifies the desired set of values of the linear operator $\hat{A}$. It can be considered as a homogeneous system consisting of one equation with three unknowns, the general solution of which has the form

$$\left\| \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \end{array} \right\| = \mu_1 \left\| \begin{array}{r} 1 \\ -1 \\ 0 \end{array} \right\| + \mu_2 \left\| \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\| \quad \forall \mu_1, \mu_2 \ .$$

Its fundamental system of solutions (for example, columns $\left\| \begin{array}{r} 1 \\ -1 \\ 0 \end{array} \right\|$ and $\left\| \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\|$) can be taken as a basis in the set of values of the linear mapping $\hat{A}$.

Solution is found

When solving problems, it should be taken into account that in the finite-dimensional case, for the chosen basis, the matrix of the linear operator exists and is unique. Therefore, any method can be used to construct it, including selection.

For example, let in Λ^4 a linear transformation map four linearly independent elements $\{ x_1, x_2, x_3, x_4 \}$ to four elements $\{ y_1, y_2, y_3, y_4 \}$, respectively.

And let the columns of the matrix $\| X \|$ be the coordinate representations of the elements $\{ x_1, x_2, x_3, x_4 \}$ in some (for example, standard) basis, and the columns of the matrix $\| Y \|$ be the coordinate representations of the elements $\{ y_1, y_2, y_3, y_4 \}$.

Then, based on the definition of the product of matrices, it can be shown that in this basis the transformation matrix $\| \hat{A} \|$ must satisfy the equalities

$$\| Y \| = \| \hat{A} \| \| X \| \quad \text{or} \quad \| \hat{A} \| = \| Y \| \| X \|^{-1} .$$

As was shown in Theme 01, the product $\| Y \| \| X \|^{-1}$ is conveniently calculated using the following scheme:

1) We form a combined matrix $\| Y | X \|$.
2) We reduce the right submatrix to the identity matrix by some set of elementary transformations.
3) We change the left submatrix by the same set of elementary transformations.

Then the desired product appears in place of the left submatrix $\| \hat{A} \| = \| Y \| \| X \|^{-1}$.

Types of linear mappings

As already noted, in cases where the range of the operator does not belong to the domain of definition, we should talk about a mapping. For mappings, special cases of so-called *bijective*, *injective* and *surjective* mappings are also distinguished. Let us consider these cases in more detail.

Definition

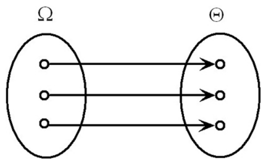
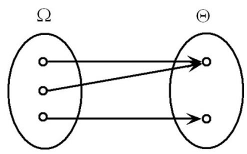
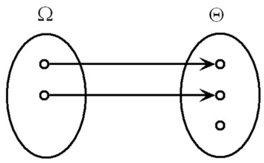
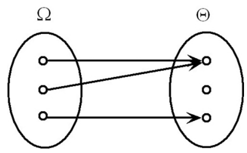
A mapping $y = \hat{A}x, \quad x \in \Omega, \quad y \in \Theta$ of a set Ω to a set Θ is called *injective* (or *injection*) if the condition $\hat{A}x_1 = \hat{A}x_2$ implies

$$x_1 = x_2, \quad x_1, x_2 \in \Omega .$$

A mapping $y = \hat{A}x, \quad x \in \Omega, \; y \in \Theta$ of a set Ω onto a set Θ is called *surjective* (or *surjection*) if each element of Θ has a preimage in Ω.

Finally, a mapping that is both injective and surjective is *one-to-one*, or a bijection.

Examples of mappings of different types

Mapping type	Injective	Non-injective
Surjective		
Non-surjective		

In the case of injection, the set of all values of the operator $y = \hat{A}x$, $x \in \Omega,\ y \in \Theta$ may not coincide with Θ.

In the case of surjection, the preimage of any element of Θ always exists in Ω, but, generally speaking, it is not unique

Note also that in the finite-dimensional case, surjectivity of a mapping $\hat{A}:\Lambda^n \to \Lambda^m$ means that the condition $\Theta = \Lambda^m$ is satisfied, and injectivity means that the condition $\ker\hat{A} = \{o\}$ is satisfied. An alternative form of the conditions of injectivity and surjectivity in the finite-dimensional case is given by

Theorem **The rank of the matrix of a linear operator that is a surjective mapping is equal to the number of *its rows*, and the rank of the matrix of an injective mapping is equal to the number of its *columns*.**

In other words, for $\hat{A}:\Lambda^n \to \Lambda^m$ injectivity is equivalent to the satisfaction of the equalities $\mathrm{rg}\| \hat{A} \|_{fg} = \dim(\Lambda^n) = n$, and surjectivity is equivalent to $\mathrm{rg}\| \hat{A} \|_{fg} = \dim(\Lambda^m) = m$.

Task 3.06. *Linear mapping* $\hat{A}: \ \Lambda^3 \to \Lambda^3$*: in some basis is given by matrix* $\left\| \hat{A} \right\| = \left\| \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{matrix} \right\|$.

Find its kernel and set of values. Find out whether this mapping is injective or surjective.

Solution. 1°. Let $\|x\| = \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix}$ and $\|y\| = \begin{Vmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Vmatrix}$ be coordinate representations of the preimage and image of the operator $y = \hat{A}x$, respectively. Then the kernel is the set of elements such that $\hat{A}x = o$, is defined in the coordinate representation by a system of linear equations

$$\|\hat{A}\|\|x\| = \|o\| \quad \text{or} \quad \begin{cases} \xi_1 + 2\xi_2 + 3\xi_3 = 0, \\ 2\xi_1 + 3\xi_2 + 4\xi_3 = 0, \\ 3\xi_1 + 5\xi_2 + 7\xi_3 = 0, \end{cases}$$

whose general solution is

$$\begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix} = \lambda \begin{Vmatrix} 1 \\ -2 \\ 1 \end{Vmatrix}.$$

From this we conclude that the kernel of a linear mapping $\hat{A}$ is the linear span of an element $\begin{Vmatrix} 1 \\ -2 \\ 1 \end{Vmatrix}$, and since it does not consist only of the zero element, this mapping is non-injective.

The same conclusion can be reached by taking into account that

$$\text{rg}\begin{Vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{Vmatrix} = \text{rg}\begin{Vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{Vmatrix} = 2 < 3,$$

where 3 is the number of columns of the mapping matrix.

2°. The range of a linear mapping $\hat{A}$ consists of elements $y \in \Theta$, such that $y = \hat{A}x \ \forall x \in \Omega$. In coordinate form, the membership of an element y in the range means compatibility of the system of linear equations

$$\begin{Vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{Vmatrix} \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix} = \begin{Vmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Vmatrix}.$$

Therefore, we need to find out for which values η_1, η_2, η_3 this system of linear equations is compatible. This can be done, for example, using the Kronecker–Capelli theorem, by comparing the ranks of the main and extended matrices of this system.

Then, from the condition

$$\operatorname{rg}\left\Vert\begin{array}{ccc|c} 1 & 2 & 3 & \eta_1 \\ 2 & 3 & 4 & \eta_2 \\ 3 & 5 & 7 & \eta_3 \end{array}\right\Vert = \operatorname{rg}\left\Vert\begin{array}{ccc|c} 1 & 2 & 3 & \eta_1 \\ 0 & 1 & 2 & 2\eta_1 - \eta_2 \\ 0 & 0 & 0 & -\eta_1 - \eta_2 + \eta_3 \end{array}\right\Vert = \operatorname{rg}\begin{Vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{Vmatrix} = 2$$

we find that for compatibility it is necessary and sufficient that $\eta_1 + \eta_2 - \eta_3 = 0$, which, in turn, means that the range of the mapping $\hat{A}$ consists of elements of the form

$$\begin{Vmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Vmatrix} = \lambda_1 \begin{Vmatrix} -1 \\ 1 \\ 0 \end{Vmatrix} + \lambda_2 \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} \quad \forall \lambda_1, \lambda_2,$$

which are solutions of the equation $\eta_1 + \eta_2 - \eta_3 = 0$.

Finally, we note that since not every element $y \in \Theta = \Lambda^3$ has a preimage in $\Omega = \Lambda^3$, this mapping is *not surjective*.

Linear forms

In conclusion, let us consider a simple, but important for practice, special case of a linear mapping of the form $\hat{A}:\{\Lambda \xrightarrow{\hat{A}} \Lambda^1\}$. Here, the result of the action $\hat{A}$ on the element $x \in \Lambda$ can naturally be denoted functionally as $f(x)$, i.e. $\hat{A}x = f(x)$.

A dependence of this type is called a *linear function*, a *linear functional*, or a *linear form*, since the set of values is numerical.

It is important to remember that in the finite-dimensional case, i.e. for $\Lambda = \Lambda^n$, the transformation matrix has, by virtue of formula (1), the size $1 \times n$, i.e. is a $n-$component *row*. This follows from our (previously made) choice of the $n-$component *column* as a coordinate representation of the element of space $\Lambda = \Lambda^n$.

The formula for finding the values of a linear form $f(x) \in \mathbf{R} \quad x \in \Lambda^n$ in Λ^n is an ordinary linear function of n numerical variables. Indeed, let the coordinate column be in the basis $\{g_1, g_2, ..., g_n\} \in \Lambda^n$, and the values of the form on the basis elements be $\phi_k = f(g_k)\ k = [1, n]$, which yields $\| \hat{A} \|_g = \| f \|_g = \| \ \phi_1 \ \phi_2 \ \dots \phi_n \ \|$. Then

$$f(x) = \sum_{k=1}^{n} \phi_k \xi_k = \| f \|_g^{\mathrm{T}} \| x \|_g .$$

Task 3.07: *Let Λ^n be a vector space of algebraic polynomials $x = P_4(\tau) = \xi_1 + \xi_2 \tau + \xi_3 \tau^2 + \xi_4 \tau^3$ of degree no greater than 3, with a basis of the form $\{ g_1(\tau) = 1, g_2(\tau) = \tau, g_3(\tau) = \tau^2, g_4(\tau) = \tau^3 \}$, and let the linear form be a definite integral of the polynomial in the range from 0 to 1. Find the coordinate representation of this form in the given basis.*

Solution: In this case $f(x)=\int\limits_0^1 P_4(\tau)\,d\tau$, and the desired coordinate representation of the linear operator $\hat{A}:\{\,\Lambda^4 \to \Lambda^1\,\}$ is a four-component row of the form

$$\| f \|_g = \| \; f(P_4(g_1(\tau))) \quad f(P_4(g_2(\tau))) \quad f(P_4(g_3(\tau))) \quad f(P_4(g_4(\tau))) \; \|.$$

Specifically

$$f(P_4(g_1(\tau))) = \int\limits_0^1 1\cdot d\tau = 1\,,$$

$$f(P_4(g_2(\tau))) = \int\limits_0^1 \tau\, d\tau = \frac{1}{2}\,,$$

$$f(P_4(g_2(\tau))) = \int\limits_0^1 \tau^2 d\tau = \frac{1}{3}\,,$$

$$f(P_4(g_2(\tau))) = \int\limits_0^1 \tau^3 d\tau = \frac{1}{4}\,.$$

Where does the answer to the task follow: $\| f \|_g = \left\| \; 1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \; \right\|$.

Solution is found