EIGENVECTORS AND EIGENVALUES OF LINEAR TRANSFORMATIONS

Definition

Let a linear operator $\hat{A}$, act in a vector space Λ and have values in the same space. In other words, the images and preimages for $\hat{A}$ belong to Λ, i.e. is a linear transformation $\hat{A}$ of the space Λ. Then

An *eigenvector* of a linear transformation $\hat{A}$ corresponding to an *eigenvalue* λ is a *nonzero* element $f \in \Lambda$ such that

$$\hat{A} f = \lambda f . \qquad (1)$$

So, f is a nonzero element of the space Λ, the image of which, when acted on by $\hat{A}$, is the product of a number λ (generally, a complex number) by the same element f.

In general, there are no universal methods for describing or finding eigenvectors and eigenvalues. To find them, one has to use the properties of a specific vector space and a linear transformation.

For example, for the differentiation operator $\hat{A} = \frac{d}{dx}$, acting in the vector space of infinitely differentiable functions $f(x)$, equality (1) takes the form of a differential equation

$$\frac{df}{dx} = \lambda f.$$

In this case, an eigenvector is each solution of this equation that is not identically equal to zero, i.e. a function of the form $f(x) = Ce^{\lambda x} \quad \forall C \neq 0$.

This example illustrates the fact that the word "vector" in the term "eigenvector" is only a tribute to a historically established tradition. An eigenvalue in this example is any complex number.

Let us indicate (these are theorems!) important properties of eigenvectors that follow from definition (1):

1) the set of all eigenvectors corresponding to the same eigenvalue (and supplemented by a zero element) is an invariant subspace of the transformation $\hat{A}$, called an *eigensubspace*;

2) eigenvectors corresponding to different eigenvalues are *linearly independent.*

Looking ahead, let us clarify:

a subspace is called *invariant* for a linear operator if $\hat{A}x \in \Omega \quad \forall x \in \Omega$.

Also note that properties 1) and 2) hold in *any* Λ, not necessarily finite-dimensional.

The case of a finite-dimensional space

Although there is no universal "recipe" for finding eigenvectors, a pleasant exception is the case when the space $\Lambda = \Lambda^n$ is finite-dimensional, i.e. there is a basis in it.

Recall important *theorems* about the properties of a finite-dimensional space:

1) each element in Λ^n (in a specific basis) is completely and uniquely described by its coordinate column, including, it is true: $a = b \quad \Leftrightarrow \quad \| a \| = \| b \|$;
2) operations with elements in Λ^n in the coordinate representation are performed according to the rules of operations with matrices. It is important for us that $\| \lambda f \| = \lambda \| f \|$.
3) each linear transformation $\hat{A}$ has a coordinate representation in the form of $\| \hat{A} \|$. Here $\| \hat{A} \|$ is a n-th order square matrix, the columns of which are the coordinate columns of the images of the basis vectors.
4) for the coordinate representation of the image of any element $x \in \Lambda^n$, i.e. $\| \hat{A} x \|$, the equality is true $\| \hat{A} x \| = \| \hat{A} \| \| x \|$.

This allows (only in Λ^n !) condition (1) to be written as:

$$\| \hat{A} \| \| f \| = \lambda \| f \| \qquad \Rightarrow \qquad \| \hat{A} \| \| f \| - \lambda \| \hat{E} \| \| f \| = \| o \| \ ,$$

or, according to the rules of operations with linear operators, as

$$\| \hat{A} - \lambda \hat{E} \| \| f \| = \| o \| \ , \tag{2}$$

where $\hat{E}$ is the unit (identity) operator, and $\| o \|$ is the zero column, i.e. the coordinate representation of the zero element $o \in \Lambda^n$.

Let us now consider (2) in expanded (coordinate) form.

Let it be $\|\hat{A}\| = \left\| \begin{matrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{matrix} \right\|$ and $\| f \| = \left\| \begin{matrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_n \end{matrix} \right\|$.

Then equality (2) can be written as follows:

$$\left\| \begin{matrix} \alpha_{11}-\lambda & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-\lambda & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn}-\lambda \end{matrix} \right\| \left\| \begin{matrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_n \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ \dots \\ 0 \end{matrix} \right\| \quad \text{or} \quad \begin{cases} (\alpha_{11}-\lambda)\,\phi_1 + \alpha_{12}\phi_2 + \dots + \alpha_{1n}\phi_n = 0, \\ \alpha_{21}\,\phi_1 + (\alpha_{22}-\lambda)\,\phi_2 + \dots + \alpha_{2n}\phi_n = 0, \\ \dots \\ \alpha_{n1}\,\phi_1 + \alpha_{n2}\phi_2 + \dots + (\alpha_{nn}-\lambda)\,\phi_n = 0. \end{cases} \qquad (3)$$

System (3) consists of n nonlinear equations with $n+1$ unknowns, which can be solved by an algorithm that uses the specifics of its type.

The idea of this algorithm is that if in (3) the unknown λ is taken as a parameter (we consider λ it as if it were known, but could have different values), then with respect to the components of $\| f \|$ the system (3) turns out to be linear and homogeneous, with the main square matrix $\| \hat{A} - \lambda \hat{E} \|$ of order n.

It follows from Cramer's theorem that for λ, satisfying the condition

$$\det \| \hat{A} - \lambda \hat{E} \| \neq 0$$

the solution of system (3) exists and is unique.

In this case, it is obvious that (3) has a unique solution $\phi_1 = \phi_2 = \ldots = \phi_n = 0$, which cannot be a solution to the original problem, since the eigenvector is *nonzero* by definition.

Let us consider the remaining case $\det\left\| \hat{A} - \lambda\,\hat{E} \right\| = 0$. Here Cramer's theorem cannot help, so we will use another well-known theorem that the maximum number of linearly independent (i.e., nonzero) particular solutions of the linear homogeneous system (3) is $n-r$, where r is the rank of the main matrix of this system.

It follows that for an eigenvector to exist, the inequality $n-r \ge 1$ must be satisfied. That is, we need

$$\mathrm{rg}\left\| \hat{A} - \lambda\,\hat{E} \right\| \le n-1 \ .$$

The rank of a square matrix of order cannot be greater than n, and therefore this condition will be satisfied if $\det\left\| \hat{A} - \lambda\,\hat{E} \right\| = 0$, since the matrix $\left\| \hat{A} - \lambda\,\hat{E} \right\|$ has a unique minor of order n, equal to its determinant.

It follows that, the choice λ from the condition

$$\det\left\| \hat{A} - \lambda \hat{E} \right\| = 0 \ , \tag{4}$$

ensures the existence of a solution to system (3) with components $\| f \|$ that are not equal to zero simultaneously. Therefore,

> in any *finite-dimensional* space, the problem of finding eigenvectors and eigenvalues is reduced to solving equation (4) and sequentially (separately for each found λ) solving system (3).

We also report that:

> equation (4) is usually called characteristic.

Let us recall that the matrix of any linear transformation $\hat{A}$, acting in Λ^n is square, of order n, , and its columns are the images of the basis elements (i.e., the result of the transformation $\hat{A}$ acting on them).

In a finite-dimensional vector space Λ^n, the eigenvectors and eigenvalues of a linear transformation $\hat{A}$ have the following properties.

1) Equation (4) is an algebraic equation of order n, the form and solutions of which are the same in any basis of Λ^n.
2) In a complex vector space Λ^n, each linear transformation $\hat{A}$ has at least one eigenvector.
3) In a real vector space Λ^n, each linear transformation $\hat{A}$ has either at least one eigenvector or a two-dimensional invariant subspace.
4) The dimension of the eigensubspace Ω_λ corresponding to the root λ of the characteristic equation of multiplicity is not less than 1 and not greater than k. That is $1 \le \dim(\Omega_\lambda) \le k$.
5) The matrix of a linear transformation $\hat{A}$ in a basis of eigenvectors of this transformation (in the case where such a basis exists) has a diagonal form, and on its main diagonal are located the eigenvalues of $\hat{A}$.

A remark on the importance of eigenvectors

Proof of property 5)

Let us assume that for some linear transformation $\hat{A}$, defined in Λ^n, it was possible to find n linearly independent eigenvectors $\{g_1, g_2, \ldots, g_n\}$ for which the equalities

$$\hat{A}g_1 = \lambda_1 g_1 \ ; \quad \hat{A}g_2 = \lambda_2 g_2 \ ; \ldots ; \quad \hat{A}g_n = \lambda_n g_n \ .$$

If we take the set of elements $\{g_1, g_2, \ldots, g_n\}$ as a basis, then these relations can be considered as coordinate decompositions of the *images of the basis elements*:

$$\hat{A}g_k = 0 \cdot g_1 + 0 \cdot g_2 + \ldots + \lambda_k g_k + \ldots + 0 \cdot g_n \ ; \quad \forall k = [1, n] \ .$$

Since these decompositions (as decompositions in the basis) necessarily *exist* and are *unique*, then, based on the definition of the matrix of the linear transformation $\hat{A}$, we can assert that the matrix of a linear transformation in this basis will have a diagonal form:

$$\| \hat{A} \|_f = \begin{Vmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{Vmatrix} ,$$

due to which the study of the properties of the transformation $\hat{A}$ in a basis of eigenvectors turns out to be simpler.

Task 4.01. *In* Λ^2 *find the eigenvalues and eigenvectors of the linear transformation* $\hat{A}$ *, for which we have* $\|\hat{A}\| = \begin{Vmatrix} 1 & 1 \\ 0 & 1 \end{Vmatrix}$.

Solution. 1) The characteristic equation (4) in this case has the form:

$$\det\begin{Vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{Vmatrix} = 0 \quad \text{or} \quad (\lambda-1)^2 = 0.$$

Hence this transformation has one eigenvalue $\lambda_{1,2} = 1$ of multiplicity $k = 2$.

2) For $\lambda = 1$ we create the system (3):

$$\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}\begin{Vmatrix} \phi_1 \\ \phi_2 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \quad \text{or} \quad \begin{cases} 0\cdot\phi_1 + 1\cdot\phi_2 = 0, \\ 0\cdot\phi_1 + 0\cdot\phi_2 = 0 \end{cases}, \text{ whence } \quad \| f_{(1)} \| = \begin{Vmatrix} \phi_1 \\ \phi_2 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}.$$

This means that the desired eigenvector is any element of Λ^2 with a coordinate representation of the form $C\| f_{(1)} \|$, where $C \neq 0$. The one-dimensional eigen-subspace will be the linear span of the element $f_{(1)}$.

Solution is found

Task 4.02. *In* Λ^3 *find the eigenvalues and eigenvectors of the linear transformation* $\hat{A}$ *,*

for which in the standard basis $\left\| \hat{A} \right\| = \left\| \begin{matrix} 1 & -2 & 0 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{matrix} \right\|$.

Solution. **1.** Let us solve the problem by first assuming that Λ^3 is a complex vector space.

1) Let us create the characteristic equation (4)

$$\det \left\| \begin{matrix} 1-\lambda & -2 & 0 \\ 1 & -1-\lambda & -2 \\ 0 & 1 & 1-\lambda \end{matrix} \right\| = 0\text{, according to the "triangle" rule}$$

$$-(\lambda-1)^2(\lambda+1)+4(\lambda-1)=0.$$

or finally $(\lambda-1)(\lambda^2+3)=0$.

Its solutions will be the numbers: $\lambda_1 = 1,\ \lambda_{2,3} = \pm i\sqrt{3}$. These are the eigenvalues $\hat{A}$.

2) We find the eigenvectors corresponding to these eigenvalues by sequentially solving system (3). For $\lambda_1 = 1$ this system will be:

$$\left\| \begin{matrix} 0 & -2 & 0 \\ 1 & -2 & -2 \\ 0 & 1 & 0 \end{matrix} \right\| \left\| \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\| .$$

Considering that the rank of its matrix is 2, we have . $\begin{cases} \phi_1 - 2\phi_2 - 2\phi_3 = 0, \\ \phi_2 = 0. \end{cases}$

Then, as an eigenvector, we can take $\left\| f_{(1)} \right\| = \left\| \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{matrix} \right\| = \left\| \begin{matrix} 2 \\ 0 \\ 1 \end{matrix} \right\|$.

3) Since the problem condition has a real form, the eigenvalues λ_2 and λ_3 are complex conjugates. For the same reason (there is such a theorem), the corresponding eigenvectors will also be complex conjugates. Therefore, to find them, it is sufficient to solve system (3) only with $\lambda_2 = i\sqrt{3}$.

In this case, system (3) has the form:

$$\begin{cases} (1-i\sqrt{3})\,\phi_1 - 2\phi_2 \qquad\qquad = 0, \\ \qquad\qquad \phi_2 + (1-i\sqrt{3})\,\phi_3 = 0. \end{cases}$$

Of course, this system can be solved by the elimination method, but it is more convenient to simply accept that $\phi_2 = 1 - i\sqrt{3}$. Then it is obvious that $\phi_1 = 2$ and $\phi_3 = -1$. As a result, we obtain the eigenvectors

$$\| f_{(2)} \| = \begin{Vmatrix} 2 \\ 1-i\sqrt{3} \\ -1 \end{Vmatrix} \quad \text{or} \quad \| f_{(3)} \| = \begin{Vmatrix} 2 \\ 1+i\sqrt{3} \\ -1 \end{Vmatrix}.$$

And here we obtain $\| f_{(3)} \|$. from $\| f_{(2)} \|$. by complex conjugation.

2. Now we solve this problem, assuming that Λ^3 is a *real* vector space.

In this case, performing the calculations as in 2), we obtain that $\hat{A}$ has only one $\lambda_1 = 1$ and, corresponding to it, a one-dimensional eigensubspace, which is the linear span of the element $\| f_{(1)} \| = \begin{Vmatrix} 2 \\ 0 \\ 1 \end{Vmatrix}$.

By property 3) of a linear operator in a real linear finite-dimensional space

each linear transformation $\hat{A}$ has either at least one eigenvector or a two-dimensional invariant subspace.

There is a theorem that such an invariant space is the linear span of the elements

$$\operatorname{Re}\left\| f_{(2)} \right\| = \operatorname{Re}\left\| \begin{matrix} 2 \\ 1-i\sqrt{3} \\ -1 \end{matrix} \right\| = \left\| \begin{matrix} 2 \\ 1 \\ -1 \end{matrix} \right\| \quad \text{and} \quad \operatorname{Im}\left\| f_{(2)} \right\| = \operatorname{Im}\left\| \begin{matrix} 2 \\ 1-i\sqrt{3} \\ -1 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ -\sqrt{3} \\ 0 \end{matrix} \right\|$$

They are real elements in Λ^3. Moreover, the elements $\operatorname{Re}\left\| f_{(2)} \right\|$ and $\operatorname{Im}\left\| f_{(2)} \right\|$ are linearly independent (*and there is such a theorem*).

Then, in the task under consideration, the linear transformation $\hat{A}$ has an invariant subspace

$$\left\| \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{matrix} \right\| = C_1 \left\| \begin{matrix} 2 \\ 1 \\ -1 \end{matrix} \right\| + C_2 \left\| \begin{matrix} 0 \\ \sqrt{3} \\ 0 \end{matrix} \right\| \quad \forall C_1, C_2 \in \mathbf{R}.$$

Task 4.03. Find in Λ^3 real eigenvalues and eigenvectors of a linear transformation $\hat{A}$, for which in the standard basis

$$\|\hat{A}\| = \left\| \begin{matrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{matrix} \right\|.$$

Solution. 1) We create and solve the characteristic equation (4)

$$\det \left\| \begin{matrix} -1-\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 1 & -1-\lambda \end{matrix} \right\| = 0,$$

expanding along the first row, we obtain $(-1-\lambda)\cdot\det\left\| \begin{matrix} -\lambda & 0 \\ 1 & -1-\lambda \end{matrix} \right\| = 0$ or finally $\lambda(\lambda+1)^2 = 0$. From where the eigenvalues are numbers: $\lambda_1 = 0$, $\lambda_{2,3} = -1$.

2) We find the eigenvectors corresponding to these eigenvalues by sequentially solving system (3). For $\lambda_1 = 0$ system (3):

$$\left\| \begin{array}{rrr} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{array} \right\| \left\| \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\| , \quad \text{or} \quad \begin{cases} \phi_1 & = 0, \\ \phi_1 + \phi_2 - \phi_3 & = 0. \end{cases}$$

Here we can take as an eigenvector $\left\| \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\|$.

3) Since the eigenvalue has a multiplicity of 2, the eigensubspace corresponding to it can be either one-dimensional or two-dimensional.

We create system (3) $\left\| \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \right\| \left\| \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\| \quad \Rightarrow \quad \{ \phi_1 + \phi_2 = 0$.

For this system $n = 3$, and the rank of its main matrix is equal to 1. Consequently, this system will have two linearly independent (and, therefore, nonzero) particular solutions, which are eigenvectors.

Taking ϕ_2 as the main variable, ϕ_1 and ϕ_3 as free, we find these solutions

$$\| f_{(2)} \| = \left\| \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} \right\| \quad \text{and} \quad \| f_{(3)} \| = \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\| .$$

Note that these eigenvectors can serve as a basis in the two-dimensional eigen-subspace of the transformation corresponding to the eigenvalue $\lambda = -1$.

Solution is found

Test 4.04. *In the vector space of square matrices, find the eigenvalues and eigenvectors of the operator , which associates each such matrix with the result of its multiplication from the right by the matrix* $\begin{Vmatrix} -7 & -2 \\ 8 & 1 \end{Vmatrix}$.

Solution. 1) Let us first recall that this space is finite-dimensional, of dimension 4. Further, the result of multiplying any square matrix of the 2nd order by any fixed square matrix of the same order is again a square matrix of the 2nd order. Therefore, the operator $\hat{A}$ is a transformation. (As a small exercise, show that this transformation is linear.)

Thus, to find the eigenvectors and eigenvalues, it turns out to be possible to use an algorithm based on solving equation (4) and the system of equations (3).

To use this algorithm, we need to know the transformation matrix, which in turn depends on the basis. Since the basis is not specified in the problem statement, we can choose it based on our own preferences.

Recall that in 4-dimensional, any ordered set of four linearly independent elements can serve as a basis. Since we are considering a vector space of 2x2 matrices, this set can, for example, have the form:

$$\left\{ \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\|;\ \left\| \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right\|;\ \left\| \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \right\|;\ \left\| \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\| \right\}.$$

From the obvious equality

$$\left\| \begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{matrix} \right\| = \alpha_1 \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\| + \alpha_2 \left\| \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right\| + \alpha_3 \left\| \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \right\| + \alpha_4 \left\| \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\| \qquad (5)$$

it follows that the coordinate representation of each 2x2 matrix in our basis will be a 4-component column $\| \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \|^{\mathrm{T}}$.

Let us now find the matrix of the transformation specified in the problem statement. Let us recall that the matrix of the linear transformation of a 4-dimensional linear transformation is a square matrix of size 4x4, the columns of which are the coordinate representations of the images of the basic elements.

In our case, the images of the basic elements are defined as follows:

$$\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} -7 & -2 \\ 8 & 1 \end{Vmatrix} = \begin{Vmatrix} -7 & -2 \\ 0 & 0 \end{Vmatrix},$$

with a coordinate column $\begin{Vmatrix} -7 & -2 & 0 & 0 \end{Vmatrix}^{T}$

$$\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} -7 & -2 \\ 8 & 1 \end{Vmatrix} = \begin{Vmatrix} 8 & 1 \\ 0 & 0 \end{Vmatrix},$$

with a coordinate column $\begin{Vmatrix} 8 & 1 & 0 & 0 \end{Vmatrix}^{T}$

$$\begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} -7 & -2 \\ 8 & 1 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ -7 & -2 \end{Vmatrix},$$

with a coordinate column $\begin{Vmatrix} 0 & 0 & -7 & -2 \end{Vmatrix}^{T}$

$$\begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} -7 & -2 \\ 8 & 1 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ 8 & 1 \end{Vmatrix},$$

with a coordinate column $\begin{Vmatrix} 0 & 0 & 8 & 1 \end{Vmatrix}^{T}$.

From the obtained coordinate representations, we form the transformation matrix $\hat{A}$

$$\| \hat{A} \| = \begin{Vmatrix} -7 & 8 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -7 & 8 \\ 0 & 0 & -2 & 1 \end{Vmatrix},$$

2) Now we create the characteristic equation (4). It has the form

$$\det\begin{Vmatrix} -7-\lambda & 8 & 0 & 0 \\ -2 & 1-\lambda & 0 & 0 \\ 0 & 0 & -7-\lambda & 8 \\ 0 & 0 & -2 & 1-\lambda \end{Vmatrix} = 0,$$

Applying the rule of expansion of the determinant by row and using the formulas of abbreviated multiplication, we obtain the equatione

$$(\lambda+3)^4 = 0 .$$

This means that the transformation specified in the condition has a single eigenvalue $\lambda_{1,2,3,4} = -3$ of multiplicity 4.

It remains to find the eigenvectors. System (3) for $\lambda = -3$ will be as follows:

$$\left\|\begin{matrix} -4 & 8 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & -2 & 4 \end{matrix}\right\| \left\|\begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\| \quad \text{or, in component-wise form,}$$

$$\begin{cases} -\phi_1 + 2\phi_2 = 0, \\ -\phi_3 + 2\phi_4 = 0. \end{cases}$$

Taking ϕ_2 and ϕ_3 as the main unknowns, and ϕ_1 and ϕ_4 as free, we obtain two linearly independent eigenvectors

$$\| f_{(1)} \| = \left\|\begin{matrix} 2 \\ 1 \\ 0 \\ 0 \end{matrix}\right\| \quad \text{and} \quad \| f_{(2)} \| = \left\|\begin{matrix} 0 \\ 0 \\ 2 \\ 1 \end{matrix}\right\| ,$$

which form the basis of the two-dimensional eigensubspace of the operator $\hat{A}$ corresponding to an eigenvalue $\lambda = -3$ of multiplicity 4.

In conclusion, we note that we have found the eigenvectors in the form of their coordinate representations. The eigenvectors themselves (if we take into account equality (5)) are matrices of the 2nd order $\begin{Vmatrix} 2 & 1 \\ 0 & 0 \end{Vmatrix}$ and $\begin{Vmatrix} 0 & 0 \\ 2 & 1 \end{Vmatrix}$.

For the sake of curiosity, we will check that these matrices satisfy the definition of an eigenvector.

Indeed, the equalities are true

$$\hat{A} f = \begin{Vmatrix} 2 & 1 \\ 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} -7 & -2 \\ 8 & 1 \end{Vmatrix} = \begin{Vmatrix} -6 & -3 \\ 0 & 0 \end{Vmatrix} = -3 \cdot \begin{Vmatrix} 2 & 1 \\ 0 & 0 \end{Vmatrix} = \lambda f .$$

The left end of this chain of equalities is the result of the action of the operator on the eigenvector, and the right end is the multiplication of the eigenvalue by this vector.

Solution is found