Quadratic forms

Bilinear forms

The initial concept in this topic is the bilinear form

Definition 1. Let in a vector space Λ each ordered pair of elements x and y be assigned a number $B(x,y)$ such that

1) $B(\alpha x_1 + \beta x_2, y) = \alpha B(x_1, y) + \beta B(x_2, y)$
$\forall x_1, x_2, y \in \Lambda$; $\forall \alpha, \beta$,
2) $B(x, \alpha y_1 + \beta y_2) = \alpha B(x, y_1) + \beta B(x, y_2)$
$\forall x, y_1, y_2 \in \Lambda$; $\forall \alpha, \beta$,

then we say that in Λ is given a *bilinear form* (or *bilinear function*, *functional*)

For example, the product of two linear forms $F(x)$ and $G(y)$, defined in Λ, $B(x,y) = F(x)G(y)$ is a bilinear form.

Bilinear forms in Λ^n

Let a basis $\{g_1, g_2, ..., g_n\}$ and a bilinear form $B(x,y)$ be given in Λ^n. Let us find a formula for expressing its value through the coordinates of the arguments.

Let us assume that in the basis under consideration $x = \sum_{i=1}^{n} \xi_i g_i$ and $y = \sum_{j=1}^{n} \eta_j g_j$, then, according to Definition 1, the equalities are true

$$\begin{aligned} B(x,y) &= B(\sum_{i=1}^{n} \xi_i g_i, \sum_{j=1}^{n} \eta_j g_j) = \sum_{i=1}^{n} \xi_i B(g_i, \sum_{j=1}^{n} \eta_j g_j) = \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \eta_j B(g_i, g_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \xi_i \eta_j . \end{aligned} \tag{1}$$

Definition 2. The numbers $\beta_{ij} = B(g_i, g_j)$ are called components of the bilinear form $B(x,y)$ in the basis $\{g_1, g_2, ..., g_n\}$, and the matrix $\|B\|_g = \| \beta_{ij} \|$ is called the matrix of the bilinear form in this basis.

It follows from formula (1) that in Λ^n with a basis $\{g_1, g_2, ..., g_n\}$ the bilinear form $B(x,y)$ can be represented in matrix form:

$$B(x,y) = \left\| \xi_1 \quad \xi_2 \quad \dots \quad \xi_n \right\| \left\| \begin{matrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} \end{matrix} \right\| \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \dots \\ \eta_n \end{matrix} \right\| = \| x \|_g^{\mathrm{T}} \| B \|_g \| y \|_g ,$$

where the columns $\|x\|_g$ and $\|y\|_g$ are the coordinate representations of the elements x and y in the given basis.

The matrix of the bilinear form depends on the choice of basis. The rule for changing this matrix when changing the basis has the form

$$\| B \|_{g'} = \| S \|^{\mathrm{T}} \| B \|_g \| S \| ,$$

where $\| S \|$ is the matrix of transition from basis $\{g_1, g_2, ..., g_n\}$ to basis $\{g'_1, g'_2, ..., g'_n\}$.

Definition 3. A bilinear form $B(x,y)$ is called *symmetric* if $\forall x, y \in \Lambda$ we have $B(x,y) = B(y,x)$.

For a bilinear form to be symmetric in Λ^n, it is necessary and sufficient that its matrix be symmetric.

Quadratic forms

Definition 4. Let each element x in a vector space Λ be assigned a number

$$\Phi(x) = B(x,x),$$

where $B(x,y)$ is some bilinear form in Λ, then we say that a *quadratic form* (or *quadratic function*, *quadratic functional*) is given in Λ.

In general, in a real vector space, given a quadratic form $\Phi(x)$, it is impossible to reconstruct the bilinear form $B(x,y)$ that generates it, but this can be done in the case of a symmetric bilinear form.

Indeed, let a quadratic form be generated by a symmetric bilinear form, then the equalities hold

$$\Phi(x+y) = B(x+y, x+y) = B(x,x) + B(x,y) + B(y,x) + B(y,y) =$$
$$= \Phi(x) + 2B(x,y) + \Phi(y),$$

whence

$$B(x,y) = \frac{\Phi(x+y) - \Phi(x) - \Phi(y)}{2}.$$

Using the obtained formula, we can give

Definition 5. In Λ^n a matrix of a bilinear form $\frac{1}{2}\big(\Phi(x+y)-\Phi(x)-\Phi(y)\big)$ is called a matrix of a quadratic form $\Phi(x)$.

By virtue of this definition, a matrix of any quadratic form is symmetric.

Therefore, if in Λ^n is given a basis $\{g_1, g_2, \dots, g_n\}$, then the quadratic form can be written as

$$\Phi(x)=\sum_{k=1}^{n}\sum_{i=1}^{n}\varphi_{ki}\xi_k\xi_i=\left\| \xi_1 \quad \xi_2 \quad \dots \quad \xi_n \right\| \left\| \begin{matrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{matrix} \right\| \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{matrix} \right\| = \| x \|_g^{\mathrm{T}} \| \Phi \|_g \| x \|_g ,$$

where $\|x\|_g$ is the coordinate column of the element $x=\sum_{i=1}^{n}\xi_i g_i$ in the given basis.

Replacing the basis, in turn, leads to a change in the matrix of the quadratic functional according to the formula

$$\| \Phi \|_{g'} = \| S \|^{\mathrm{T}} \| \Phi \|_g \| S \| .$$

Note that sometimes it is advisable to construct a quadratic form $\Phi(x)$ from a generating bilinear form, having previously symmetrized the latter.

Indeed, for any $B(x,y)$ one can specify a symmetric bilinear form $B^*(x,y)$ that will generate the same quadratic form as $B(x,y)$. For example, let us take $B^*(x,y)=\frac{1}{2}(B(x,y)+B(y,x))$. For such a bilinear form it is obvious that

$$\varphi_{ij}=\frac{\beta_{ij}+\beta_{ji}}{2}=\frac{\beta_{ji}+\beta_{ij}}{2}=\varphi_{ji} \quad \forall i,j=[1,n], \qquad (2)$$

i.e. the numbers φ_{ij} are elements of a symmetric matrix, which is by definition 4 a matrix of a quadratic form $\Phi(x)=B^*(x,x)$

Example 1. Let a bilinear (non-symmetric!) form be given шт Λ^3

$$B\ (x,y)=\xi_1\eta_1+3\xi_2\eta_2-\xi_2\eta_1-3\xi_1\eta_2+2\xi_3\eta_1-\xi_2\eta_3-\xi_3\eta_2=$$
$$=\left\|\begin{matrix}\xi_1 & \xi_2 & \xi_3\end{matrix}\right\|\left\|\begin{matrix}1 & -3 & 0\\ -1 & 3 & -1\\ 2 & -1 & 0\end{matrix}\right\|\left\|\begin{matrix}\eta_1\\ \eta_2\\ \eta_3\end{matrix}\right\|.$$

The quadratic form generated by it in Λ^3 due to formula (2) will have the form

$$\Phi(x)==\left\|\begin{matrix}\xi_1 & \xi_2 & \xi_3\end{matrix}\right\|\left\|\begin{matrix}1 & -2 & 1\\ -2 & 3 & -1\\ 1 & -1 & 0\end{matrix}\right\|\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\|=\xi_1^2+3\xi_2^2-4\xi_1\xi_2+2\xi_1\xi_3-2\xi_2\xi_3.$$

At the same time, the symmetric bilinear form

$$B^*(x,y)=\xi_1\eta_1+3\xi_2\eta_2-2\xi_1\eta_2-2\xi_2\eta_1+\xi_1\eta_3+\xi_3\eta_1-\xi_2\eta_3-\xi_3\eta_2=$$
$$=\left\|\begin{matrix}\xi_1 & \xi_2 & \xi_3\end{matrix}\right\|\left\|\begin{matrix}1 & -2 & 1\\ -2 & 3 & -1\\ 1 & -1 & 0\end{matrix}\right\|\left\|\begin{matrix}\eta_1\\ \eta_2\\ \eta_3\end{matrix}\right\|,$$

having a matrix $\left\|\begin{matrix}1 & -2 & 1\\ -2 & 3 & -1\\ 1 & -1 & 0\end{matrix}\right\|$, will generate in the same quadratic form $\Phi(x)=\xi_1^2+3\xi_2^2-4\xi_1\xi_2+2\xi_1\xi_3-2\xi_2\xi_3$

Lagrange Method

As already noted, the quadratic form in Λ^n is completely and uniquely described by its matrix $\| \Phi \|_g$ in the chosen basis. In this case, in different bases, the matrix of the quadratic form, as we have seen, is different.

Recall that the rule for changing this matrix when changing the basis has the form

$$\| \Phi \|_{g'} = \| S \|^{\mathrm{T}} \| \Phi \|_g \| S \| , \qquad (3)$$

where $\| S \|$ is the matrix of transition from basis $\{g_1, g_2, ..., g_n\}$ to basis $\{g'_1, g'_2, ..., g'_n\}$.

Of interest is the problem of finding in Λ^n bases in which the quadratic form has the *simplest* and most convenient form for study.

First, let us make sure that the value of the quadratic form does not depend on the choice of basis. Let

$$\Phi(x) = \| x \|_g^{\mathrm{T}} \| \Phi \|_g \| x \|_g \quad \text{and} \quad \Phi'(x) = \| x \|_{g'}^{\mathrm{T}} \| \Phi \|_{g'} \| x \|_{g'} . \qquad (4)$$

We must show that $\Phi'(x) = \Phi(x)$.

Indeed, on the one hand, according to the transition formulas we have $\| x \|_g = \| S \| \| x \|_{g'}$ and, by virtue of the transposition rule of the product of matrices, we obtain $\| x \|_g^{\mathrm{T}} = \| x \|_{g'}^{\mathrm{T}} \| S \|^{\mathrm{T}}$.

On the other hand, the transition matrix $\| S \|$ is non-singular, therefore it has an inverse matrix $\| S \|^{-1}$. Similarly, for the matrix $\| S \|^{\mathrm{T}}$ there exists $\left(\| S \|^{\mathrm{T}} \right)^{-1}$ and the equalities $\| x \|_{g'} = \| S \|^{-1} \| x \|_g$ and $\| x \|_{g'}^{\mathrm{T}} = \| x \|_g^{\mathrm{T}} \left(\| S \|^{\mathrm{T}} \right)^{-1}$ will be true.

Substituting the last two equalities and formula (3) into formula (4), we obtain

$$\Phi'(x) = \| x \|_{g'}^{\mathrm{T}} \| \Phi \|_{g'} \| x \|_{g'} = \| x \|_g^{\mathrm{T}} \left(\| S \|^{\mathrm{T}} \right)^{-1} \| S \|^{\mathrm{T}} \| \Phi \|_g \| S \| \| S \|^{-1} \| x \|_g .$$

Since the product of matrices has the property of associativity, the calculation of the obtained expression can be started with any pair of adjacent matrices. This gives the required equality

$$\Phi'(x) = \| x \|_g^{\mathrm{T}} \| E \| \| \Phi \|_g \| E \| \| x \|_g = \| x \|_g^{\mathrm{T}} \| \Phi \|_g \| x \|_g = \Phi(x) ,$$

by virtue of the definitions of the inverse and identity matrices.

Now let us clarify the concept of the *simplest type* of matrix. We will consider a *diagonal* matrix to be convenient for use. Or, in relation to a quadratic form, we will give

Definition 6. We will say that a quadratic form $\Phi(x)$ has a diagonal form in the basis $\{g_1, g_2, ..., g_n\} \subset \Lambda^n$, if it is representable in it as

$$\Phi(x) = \sum_{i=1}^{n} \lambda_i \xi_i^2 \ , \qquad (5)$$

where $\lambda_i \ \forall i = [1, n]$ are some numbers.

If, in addition, the numbers λ_i, $i = [1, n]$ take only the values 0 or $\pm$ 1, then we say that the quadratic form in this basis has a *canonical* form.

Valid

Theorem 1. For any quadratic form in Λ^n there is a basis, in which this form has a canonical form

and

Theorem 2. The number of positive, negative and zero coefficients λ_i in formula (5) does not depend on the choice of the canonical basis.

Theorem 2 is traditionally called the theorem of *inertia* of a quadratic form in mathematical literature.

The problem of finding a diagonal or canonical base (that is, a base in which the quadratic form has a diagonal or canonical form, respectively) can be solved in different ways.

The simplest of these is the Lagrange method (or the method of extracting perfect squares), which is widely used in elementary mathematics.

For small-dimensional problems, the implementation of this method can be reduced to constructing a sequence of linear non-degenerate changes of variables. Which is illustrated by

Task 6.01. *In a vector space* Λ^3*, reduce the quadratic form*

$$\Phi(x) = 2\xi_1\xi_2 + 2\xi_1\xi_3 - 2\xi_2\xi_3 \qquad (6)$$

to canonical form and construct a canonical basis for it.

Solution. Since the quadratic form in the condition is given in coordinate form, we will consider the initial basis to be standard, i.e. formed by elements with coordinate columns $\left\{ \|g_1\| = \left\| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\|, \ \|g_2\| = \left\| \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\|, \ \|g_3\| = \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\| \right\}$. In this basis, the matrix of the quadratic form, by virtue of (2), will have the form

$$\|\Phi\| = \left\| \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix} \right\| .$$

There are no perfect squares in formula (6), so we will create them "artificially" using the following change of variables $\begin{cases} \xi_1 = \xi_1' + \xi_2', \\ \xi_2 = \xi_1' - \xi_2', \\ \xi_3 = \xi_3'. \end{cases}$ After substituting into (6) and simplifications, we obtain a formula in "primed variables" of the form $\Phi(x) = \xi_1'^2 - \left(\xi_2' - \xi_3'\right)^2 + \xi_2'^2$.

Now we make the second substitution, following from obvious relations $\begin{cases} \xi_1'' = \xi_1', \\ \xi_2'' = \xi_2' - \xi_3', \\ \xi_3'' = \xi_3'. \end{cases}$ namely $\begin{cases} \xi_1' = \xi_1'', \\ \xi_2' = \xi_2'' + \xi_3'', \\ \xi_3' = \xi_3'', \end{cases}$ allowing us to obtain a canonical form in variables with "two primes" $\Phi(x) = \xi_1''^2 - \xi_2''^2 + \xi_3''^2$.

Finally, to construct the canonical basis, we find formulas for the transition from the original system of variables to the canonical one, expressing the "oldest" variables (without primes) through the "newest" (with two primes). We obtain (check this yourself) $\begin{cases} \xi_1 = \xi_1'' + \xi_2'' + \xi_3'', \\ \xi_2 = \xi_1'' - \xi_2'' - \xi_3'', \\ \xi_3 = \xi_3''. \end{cases}$ Therefore, the matrix of the transition from the original basis to the canonical one has the form $\| S \| = \left\| \begin{matrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{matrix} \right\|$. It is obviously non-degenerate and its columns (as is known!) are the coordinate columns of the basis to which we have passed, in the basis from which we have passed.

In other words, from the transition matrix we obtain a canonical basis

$$\left\{ \| g_1'' \| = \left\| \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right\| , \quad \| g_2'' \| = \left\| \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} \right\| , \quad \| g_3'' \| = \left\| \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\| \right\},$$

in which the matrix of the quadratic form has a diagonal form

$$\| \Phi \| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\| .$$

Solution is found

Note: the basis in which the quadratic functional has a diagonal or canonical form is not unique, just as the canonical or diagonal form of the quadratic functional in Λ^n is not unique, however, by virtue of the inertia theorem, in any diagonal or canonical basis the formula for the quadratic form from example 2 will have two positive coefficients, one negative and no zero ones.

Signed definiteness of quadratic forms

Using elementary transformations for the diagonal basis of a quadratic form

The Lagrange method as a method for directly extracting perfect squares is not always the simplest (in terms of computational effort) procedure. Sometimes, reducing the matrix of a quadratic functional to a diagonal (or canonical) form can be performed more efficiently by using a certain set of elementary transformations.

The connection between the diagonal representation of a quadratic form and the canonical one is almost obvious. This is a change of basis that normalizes nonzero values of λ_k while preserving their sign. For example, by dividing each coordinate ξ_k by $|\lambda_k|$.

Indeed, when moving from the original basis $\{g_1, g_2, \ldots, g_n\}$ to a new one $\{g'_1, g'_2, \ldots, g'_n\}$ with a transition matrix $\| S \|$, the matrix of a quadratic functional changes according to the rule $\| \Phi \|_{g'} = \| S \|^{\mathrm{T}} \| \Phi \|_g \| S \|$.

We also note that according to the transpose rule of the product of matrices, we have

$$\| \Phi \|_{g'}^{\mathrm{T}} = \left(\| S \|^{\mathrm{T}} \| \Phi \|_g \| S \| \right)^{\mathrm{T}} = \| S \|^{\mathrm{T}} \| \Phi \|_g^{\mathrm{T}} \left(\| S \|^{\mathrm{T}} \right)^{\mathrm{T}} = \| S \|^{\mathrm{T}} \| \Phi \|_g \| S \| = \| \Phi \|_{g'} ,$$

that is, the symmetry of the matrix $\| \Phi \|_g$ follows from the symmetry of the matrix $\| \Phi \|_{g'}$.

This material is optional	We will now consider the matrix $\\| S \\|$ as the matrix of some elementary transformation of the matrix $\\| \Phi \\|_g$ such that multiplication it by $\\| \Phi \\|_g$ from the left reduces the latter to an upper triangular form. That is, the matrix $\\| \Phi \\|_g \\| S \\|$ is upper triangular. According to the properties of elementary transformations, it is known that in this case, multiplication of the matrix $\\| \Phi \\|_g$ by $\\| S \\|^{\mathrm{T}}$ from the left will reduce it to a lower triangular form. Then, if the matrix $\\| \Phi \\|_g \\| S \\|$ is multiplied from the left by $\\| S \\|^{\mathrm{T}}$, the resulting matrix $\\| S \\|^{\mathrm{T}} \\| \Phi \\|_g \\| S \\|$ will be both upper and lower triangular, i.e. diagonal. Let the matrix $\\| S \\|$ be such that the matrix $\\| \Phi \\|_{g'} = \\| S \\|^{\mathrm{T}} \\| \Phi \\|_g \\| S \\|$ becomes diagonal. In this case, the matrix $\\| S \\|$ (as the matrix of the transition from the original standard basis to the "diagonal" basis) by definition consists of columns obtained by applying the "diagonalizing" transformation to the columns of the identity matrix. Therefore, having performed diagonalization $\\| \Phi \\|_g$ by a certain set of elementary transformations (*performed at each step of the procedure both with its rows and with its columns*) and, having applied the same set of elementary transformations only to the columns of the identity matrix, we obtain simultaneously - both the diagonal form of the matrix of the quadratic form $\\| \Phi \\|_{g'}$, - and $\\| x \\|_g = \\| S \\| \\| x \\|_{g'}$ the formulas for the transition from the original (standard) basis $\{g_1, g_2, ..., g_n\}$ to the basis $\{g'_1, g'_2, ..., g'_n\}$ in which the matrix of the quadratic form turns out to be diagonal.

Here and below, for brevity, we will call the basis in which the quadratic form has a diagonal (canonical form) diagonal (canonical).

The application of the above algorithm (which should be remembered and which should be used) is illustrated by the following examples..

Task 6-02. *Diagonalize in* Λ^2 *a quadratic form*

$$\Phi(x) = -3\xi_1^2 + 10\xi_1\xi_2 + 8\xi_2^2. \qquad (1)$$

Solution In the original basis, the form $\Phi(x)$ has a matrix $\begin{Vmatrix} -3 & 5 \\ 5 & 8 \end{Vmatrix}$, then the expanded original matrix will be $\begin{Vmatrix} -3 & 5 & | & 1 & 0 \\ 5 & 8 & | & 0 & 1 \end{Vmatrix}$.

1°. We perform the following elementary operations:

- first, we replace the second row of the left side of the original matrix with the sum of the first row multiplied by a factor of 5, and the second row multiplied by a factor of 3, that is,

$$5 \cdot Row_1 + 3 \cdot Row_2 \to Row_2 \quad \text{we obtain} \quad \begin{Vmatrix} -3 & 5 & | & 1 & 0 \\ 0 & 49 & | & 0 & 1 \end{Vmatrix},$$

- then, in both parts (both the left and the right) of the resulting matrix, we replace the second column with the sum of the first column multiplied by 5, and the second column multiplied by 3, that is,

$$5 \cdot Col_1 + 3 \cdot Col_2 \to Col_2 \quad \text{we obtain.} \quad \begin{Vmatrix} -3 & 0 & | & 1 & 5 \\ 0 & 147 & | & 0 & 3 \end{Vmatrix}.$$

2°. According to the theory presented earlier, the left side of the augmented matrix is a matrix of quadratic form in the diagonal basis, and the right matrix is the transition matrix from the original basis to the diagonal one

$$\| \Phi \|_{g'} = \begin{Vmatrix} -3 & 0 \\ 0 & 147 \end{Vmatrix} \qquad \| S \| = \begin{Vmatrix} 1 & 5 \\ 0 & 3 \end{Vmatrix}$$

or, in coordinates

$$\Phi(x') = -3\xi'^2_1 + 147\xi'^2_2 \quad (2) \qquad \begin{cases} \xi_1 = \xi'_1 + 5\xi'_2 \\ \xi_2 = \quad\; 3\xi'_2 \end{cases} \quad (3)$$

3°. Let's check by substituting formulas (3) into (1). We get

$$\Phi(x') = -3(\xi'_1 + 5\xi'_2)^2 + 10(\xi'_1 + 5\xi'_2)_1(3\xi'_2) + 8(3\xi'_2)^2 =$$

$$= -3\xi'^2_1 - 30\xi'_1\xi'_2 - 75\xi'^2_2 + 30\xi'_1\xi'_2 + 150\xi'^2_2 + 72\xi'^2_2 = -3\xi'^2_1 + 147\xi'^2_2 .$$

Solution is found

Task 6-03. *Diagonalize in* Λ^3 *a quadratic form*

$$\Phi(x) = -2\xi_1^2 - \xi_2^2 - 4\xi_3^2 - 8\xi_1\xi_2 + 2\xi_1\xi_3 - 8\xi_2\xi_3 .$$

Solution. In the original basis, the form $\Phi(x)$ has a matrix $\left\|\begin{matrix} -2 & -4 & 1 \\ -4 & -1 & -4 \\ 1 & -4 & -4 \end{matrix}\right\|$.

1°. At the first step of the procedure, we perform the following elementary operations:

- replace the second row of the original matrix with the difference of the second and third rows;
- replace the second column in the resulting matrix with the difference of the second and third columns,

as a result of which we obtain a matrix of the form $\left\|\begin{matrix} -2 & -5 & 1 \\ -5 & 3 & 0 \\ 1 & 0 & -4 \end{matrix}\right\|$.

After this, replacing the second column in the identity matrix $\left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|$ with the difference of the second and third, we obtain $\left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}\right\|$.

2º. At the second step:

- we first replace the first row with the tripled first, added to the second, taken with a coefficient of 5.
- Then the same transformation is performed with the columns.

We obtain the following two matrices:

$$\begin{Vmatrix} -93 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & -4 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -5 & -1 & 1 \end{Vmatrix}.$$

3º. At the third step, we replace the third row with the first, added to the third, taken with a coefficient of 31.

Having performed the same transformations with the columns, we obtain the matrices

$$\begin{Vmatrix} -93 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3751 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 3 & 0 & 3 \\ 5 & 1 & 5 \\ -5 & -1 & 26 \end{Vmatrix}.$$

Thus, we conclude that the transition to the basis $\begin{Vmatrix} 3 \\ 5 \\ -5 \end{Vmatrix}; \begin{Vmatrix} 0 \\ 1 \\ -1 \end{Vmatrix}; \begin{Vmatrix} 3 \\ 5 \\ 26 \end{Vmatrix}$ according to the transition formulas $\begin{cases} \xi_1 = 3\xi_1' + 3\xi_3', \\ \xi_2 = 5\xi_1' + \xi_2' + 5\xi_3', \\ \xi_3 = -5\xi_1' - \xi_2' + 26\xi_3', \end{cases}$ will give us the following (one of the possible) diagonal form of the original quadratic form:

$$\Phi(x) = -93\xi_1'^2 + 3\xi_2'^2 - 3751\xi_3'^2 .$$

Solution is found

Study of the sign of a quadratic functional

Despite the non-uniqueness of the coordinate description, quadratic forms have a number of important properties that are invariant with respect to (i.e., independent of) the choice of a basis in Λ^n. We have already discussed three such numerical characteristics. These are the numbers of positive, negative, and zero λ_k $k=[1,n]$ for coordinate representations of a quadratic form in diagonal (or canonical) bases.

In practice, these (and closely related) characteristics are also used under other names.

Definition 1.

1°. The maximum number of positive coefficients of the diagonal (canonical) form of a quadratic form $\Phi(x)$ in Λ^n is called its *positive index of inertia* and is denoted by $\operatorname{rg}_+\Phi$.
2°. The maximum number of negative coefficients of the diagonal (canonical) form of a quadratic form $\Phi(x)$ in Λ^n is called its *negative index of inertia* and is denoted by $\operatorname{rg}_-\Phi$.
3°. The difference between the positive and negative indices of inertia is called the *signature* of the quadratic form $\Phi(x)$ in Λ^n and is denoted by
$$\operatorname{sgn}\Phi = \operatorname{rg}_+\Phi - \operatorname{rg}_-\Phi.$$
4°. The maximum number of non-zero coefficients of the canonical form of the quadratic form $\Phi(x)$ in Λ^n is called its *rank* and is denoted by $\operatorname{rg}\Phi$.

The independence of the choice of the diagonal basis in Λ^n the numerical characteristics of the quadratic forms specified in Definition 1 follows from the inertia theorem.

Numerical characteristics of quadratic forms that are invariant for any (not necessarily, say, diagonal) bases also prove useful when solving applied problems.

Such properties exist for quadratic forms. For example, as we have already seen, the value of any quadratic form is the same in all bases.

There are also more interesting cases. For example, in Λ^2 the form Λ^2 has positive values for any nonzero $x \in \Lambda^2$. And the value of the quadratic form (as we have seen) is the same in all bases. This means that this quadratic form will have this property in each basis.

This property becomes obvious if we apply the Lagrange method (the method of isolating perfect squares):

$$\Phi(x) = \xi_1^2 + \xi_1\xi_2 + \xi_2^2 = \xi_1^2 + \xi_1\xi_2 + \frac{\xi_2^2}{4} + \frac{3\xi_2^2}{4} = \left(\xi_1 + \frac{\xi_2}{2}\right)^2 + \left(\frac{\xi_2\sqrt{3}}{2}\right)^2 .$$

An interesting question from the point of view of applications arises: is it possible to draw conclusions about the presence (or absence) of such properties for quadratic forms only, for example, based on the type of their matrix (without the Lagrange transformation, etc.)?

In order to answer this question correctly, the concept of *sign definiteness* of a quadratic form is introduced.

Definitipon 2.

1°. A quadratic form $\Phi(x)$ is called *positive* (*negative*) *definite* on the subspace $\Omega^+ \subset \Lambda$ if $\Phi(x) > 0$ ($\Phi(x) < 0$) for any nonzero $x \in \Omega^+$.

2°. If Ω^+ (or Ω^-) coincides with Λ, then the quadratic form is said to be *positive* (*negative*) *definite*.

3°. If $\Phi(x) \geq 0$ ($\Phi(x) \leq 0$) for all nonzero $x \in \Lambda$, then the quadratic form is said to be *positive* (*negative*) *semidefinite*. Sometimes such quadratic forms are also called nonnegative (nonpositive) definite.

4°. If on the set $x \in \Lambda$ $\Phi(x)$ has both positive and negative values, then the quadratic form is said to be *nondefinite*.

Note that Definition 2 does not make any assumptions about the existence of a basis in Λ.

On the other hand, in Λ^n from the inertia theorem (as is not very difficult to notice) it follows

Theorem 1. **The maximum dimension of a subspace in Λ^n on which a quadratic form is positively (negatively) defined is equal to the positive (negative) index of inertia of this form.**

In the case when for some reason the application of Definition 2 requires significant computational resources, one can try to use the following conditions, called the "Sylvester criterion".

Let us first recall that for any matrix $\left\| \alpha_{ij} \right\|$, a minor of order k is the determinant of its square submatrix formed by elements with some sets of row and column indices: $\{ i_1, i_2, \dots i_k \}$ and $\{ j_1, j_2, \dots j_k \}$. In the case when these sets are identical, the minors are called *principal*.

Theorem 2. (Sylvester's criterion)	**For a quadratic form to be positive definite in Λ^n, it is necessary and sufficient that all principal minors of its matrix of the form** $$\Delta_k = \det\begin{Vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1k} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{k1} & \varphi_{k2} & \cdots & \varphi_{kk} \end{Vmatrix}; \quad k = [1,n], \qquad (1)$$ **be positive.**

Theorem 3. **For a quadratic form to be negative definite in Λ^n, it is necessary and sufficient that the principal minors of *even* order of the form (1) of the matrix of this form be *positive*, and those of *odd* order be *negative*.**

Let us explain the proof of Theorem 3.

Let the quadratic form $\Phi(x)$ be negative definite, then the form $-\Phi(x)$ will obviously be positive definite. Applying Theorem 2 (Sylvester's criterion for positive definiteness) to it, we obtain for the principal minor of the k-th order (using the linear property of the determinant, i.e., taking -1 out of each row) the condition

$$\Delta_k = \det\begin{Vmatrix} -\varphi_{11} & -\varphi_{12} & \cdots & -\varphi_{1k} \\ -\varphi_{21} & -\varphi_{22} & \cdots & -\varphi_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ -\varphi_{k1} & -\varphi_{k2} & \cdots & -\varphi_{kk} \end{Vmatrix} = (-1)^k \det\begin{Vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1k} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{k1} & \varphi_{k2} & \cdots & \varphi_{kk} \end{Vmatrix} > 0 \quad \forall k = [1,n] \ ,$$

from which follows the assertion of Theorem 3.

Theorem 4. **For the absence of the sign-definiteness property of a quadratic form in Λ^n, it is necessary and sufficient that** $\begin{cases} \text{rg}_+ \Phi \ge 1, \\ \text{rg}_- \Phi \ge 1. \end{cases}$

Optionally (outside the course program), for the curious, we will inform you that, for example:

> A quadratic form is positive semi-definite if and only if *all* (and not just angular, of the form (1)) of its principal minors are non-negative.

Note that Theorem 4 is not very convenient for solving problems, since here it is actually required to know some diagonal basis.

However, as an alternative, one can try to find out under what conditions Theorems 2 and 3 are violated simultaneously.

As an illustration, consider

Task 6-04. Investigate in Λ^2 for $\lambda \in \mathbf{R}$ sign-definiteness for any quadratic form $\Phi(x) = \lambda\xi_1^2 + 4\xi_1\xi_2 + (\lambda - 4)\xi_2^2$.

Solution. 1°. The matrix of the quadratic form here will be $\|\Phi\| = \begin{Vmatrix} \lambda & 2 \\ 2 & \lambda - 4 \end{Vmatrix}$. It has only two principal minors of the form specified in Theorem 2:

$$\Delta_1 = \det\|\lambda\| = \lambda \quad \text{and}$$
$$\Delta_2 = \det\begin{Vmatrix} \lambda & 2 \\ 2 & \lambda - 4 \end{Vmatrix} = \lambda^2 - 4\lambda - 4 = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

where $\lambda_1 = 2 - 2\sqrt{2} \approx -0.8$ and $\lambda_2 = 2 + 2\sqrt{2} \approx 4.8$.

2°. A quadratic form is positive definite by the Sylvester criterion (Theorem 2) if and only if

$$\begin{cases} \Delta_1 > 0, \\ \Delta_2 > 0 \end{cases} \Leftrightarrow \begin{cases} \lambda > 0, \\ (\lambda-\lambda_1)(\lambda-\lambda_2) > 0 \end{cases} \Rightarrow \lambda > \lambda_2 .$$

3°. A quadratic form is negative definite by the Sylvester criterion (Theorem 3) if and only if

$$\begin{cases} \Delta_1 < 0, \\ \Delta_2 > 0 \end{cases} \Leftrightarrow \begin{cases} \lambda < 0, \\ (\lambda-\lambda_1)(\lambda-\lambda_2) > 0 \end{cases} \Rightarrow \lambda < \lambda_1 .$$

4°. Using points 2° and 3°, we come to the conclusion that Theorems 2 and 3 are simultaneously violated in the case $\lambda \in (\lambda_1, \lambda_2)$. Moreover, the form of the violation is a strict inequality of the form $(\lambda - \lambda_1)(\lambda - \lambda_2) < 0$. Therefore, $\Phi(x)$ is not sign-definite for $\lambda \in (\lambda_1, \lambda_2)$.

5°. Among the real λ ones, only two values remained unexplored: $\lambda = \lambda_1 = 2 - 2\sqrt{2}$ and $\lambda = \lambda_2 = 2 + 2\sqrt{2}$. We will not be clever here. We will simply find specific formulas for in these cases.

1) for $\lambda = 2 - 2\sqrt{2}$

$$\begin{aligned}\Phi(x) &= \left(2 - 2\sqrt{2}\right)\xi_1^2 + 4\xi_1\xi_2 - \left(2 + 2\sqrt{2}\right)\xi_2^2 = \\ &= (-2)\left(\left(\sqrt{2} - 1\right)\xi_1^2 - 2\xi_1\xi_2 + \left(\sqrt{2} + 1\right)\xi_2^2\right) = \\ &= (-2)\left(\left(\sqrt{\sqrt{2} - 1}\,\xi_1\right)^2 - 2\sqrt{\sqrt{2} - 1}\,\xi_1\sqrt{\sqrt{2} + 1}\,\xi_2 + \left(\sqrt{\sqrt{2} + 1}\,\xi_2\right)^2\right),\end{aligned}$$

since, as you can see, $\sqrt{\sqrt{2} - 1}\sqrt{\sqrt{2} + 1} = \sqrt{\left(\sqrt{2}\right)^2 - 1} = 1$.

Then we finally get that

$$\Phi(x) = (-2)\left(\sqrt{\sqrt{2} - 1}\,\xi_1 - \sqrt{\sqrt{2} + 1}\,\xi_2\right)^2 \le 0 \ .$$

Here $\Phi(x) = 0$ some nonzero points, for example, of the form

$$x = \{ \sqrt{\sqrt{2} + 1}\, t \ ; \ \sqrt{\sqrt{2} - 1}\, t \} \quad \forall t \ne 0 \ ,$$

i.e., the quadratic form $\Phi(x)$ for $\lambda = 2 - 2\sqrt{2}$ is negative semidefinite (non-positively defined).

2) Show independently that for $\lambda = 2 + 2\sqrt{2}$ the quadratic form is positively semidefinite

Solution is found