Euclidean space

Axiomatics and basic properties

In an arbitrary vector space, there are no concepts of length, distance, angle size, and other metric characteristics. However, their use becomes possible if we additionally introduce an operation called the *scalar product* in this space, described by the following rules.

Definition 1. Let each ordered pair of elements x and y be assigned a real number in a real vector space, denoted by the symbol (x, y), called the *scalar product*, so that the following axioms are satisfied:

1) $(x, y) = (y, x)$;
2) $(\lambda x, y) = \lambda(x, y)$;
3) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$;
4) $(x, x) \geq 0$, and $(x, x) = 0 \quad \Leftrightarrow \quad x = o$,

then we say that the Euclidean space E is given.

Note: axioms 1)–4) together mean that the scalar product is a form

- *bilinear* (which follows from axioms 2) and 3)) ,
- *symmetric* (which follows from axiom 1) ,
- which generates a *positive definite quadratic* (which follows from axiom 4) *form*.

Any bilinear form with these properties can be used as a scalar product. Different ways of introducing the scalar product will yield different Euclidean spaces

Task 7-01.

1°. If in Λ^n the space of n-dimensional columns $x = \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{Vmatrix}$; $y = \begin{Vmatrix} \eta_1 \\ \eta_2 \\ \dots \\ \eta_n \end{Vmatrix}$, we introduce a scalar product defined by the formula $(x, y) = \sum_{i=1}^{n} \xi_i \eta_i$, then we obtain a Euclidean space E^n.

2°. The space of continuous at $\tau \in [\alpha, \beta]$ functions with scalar product

$$(x, y) = \int_{\alpha}^{\beta} x(\tau) y(\tau) d\tau \tag{1}$$

is Euclidean

Definition 2. In a Euclidean space E we call:
:

1) the *norm* (or *length*) of an element x number $|x| = \sqrt{(x, x)}$;
2) the *distance* between elements x and y number $|x - y|$.

The following inequalities follow from the axiomatics of Euclidean space:

- $\forall\, x, y \in E \quad |\,(x,y)\,| \le |\,x\,||\,y\,|$ the *Cauchy-Bunyakovsky inequality* holds.
- $\forall\, x, y \in E \quad |\,x+y\,| \le |\,x\,| + |\,y\,|$ the *triangle inequality* holds..

Let us check the first of them: we have

$$(x-\tau y, x-\tau y) = (x,x) - 2\tau(x,y) + \tau^2(y,y) \ge 0 \quad \forall\,\tau, \quad \forall x, y \in E.$$

This expression for arbitrary fixed $x, y \in E$ is a square trinomial with respect to τ. This trinomial is non-negative $\forall\,\tau \in R$, which means that its discriminant is not positive. That is,

$$(x,y)^2 - (x,x)(y,y) = (x,y)^2 - |\,x\,|^2 |\,y\,|^2 \le 0.$$

Where the validity of the Cauchy-Bunyakovsky inequality follows.

Note that the Cauchy-Bunyakovsky and triangle inequalities for Euclidean space can have a rather exotic form, such as, say, in examples 1-2°

$$\left| \int_\alpha^\beta x(\tau)y(\tau)\,d\tau \right| \le \sqrt{\int_\alpha^\beta x^2(\tau)d\tau}\sqrt{\int_\alpha^\beta y^2(\tau)d\tau}\ , \quad \sqrt{\int_\alpha^\beta \big(x(\tau)+y(\tau)\,\big)^2 d\tau} \le \sqrt{\int_\alpha^\beta x^2(\tau)\,d\tau} + \sqrt{\int_\alpha^\beta y^2(\tau)\,d\tau}\ .$$

Definition 2. In Euclidean space E, the *value of the angle* between non-zero elements x and y is called the number $\alpha \in [0,\pi]$ satisfying the relation $\cos\alpha = \dfrac{(x,y)}{|x||y|}$.

In Euclidean space E, elements x and y are called *orthogonal* if $(x,y) = 0$.

Orthonormal basis. Orthogonalization of the basis

In a finite-dimensional Euclidean space E^n, a basis will be called orthonormal if

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1, \text{ если } i = j, \\ 0, \text{ если } i \neq j \end{cases} \quad \forall i, j = [1, n].$$

There is a theorem stating that

in every Euclidean space E^n there is an orthonormal basis.

The proof of this theorem is based on the fact that for any set of n linearly independent elements, it is possible to construct a set of pairwise orthogonal elements, each of which is a linear combination of elements of the original set.

It turns out that the computational costs of orthogonalizing the basis are significantly reduced (in comparison, say, with the method of undetermined coefficients) if we apply the so-called *Gram-Schmidt* procedure, the essence of which we will explain by considering

Task 7-02. *Let the scalar product in the vector space of algebraic polynomials of degree no higher than 2, i.e. of the form* $P(x)=\xi_1+\xi_2 x+\xi_3 x^2$*, be given by formula (1) for and* $\alpha=0$ *and* $\beta=1$*. That is, we have a three-dimensional Euclidean space* E^3.

Solution: Let us take three linearly independent elements in this space, forming a standard basis: $\left\{ g_{(1)}(x) = 1, \quad g_{(2)}(x) = x, \quad g_{(3)}(x) = x^2 \right\}$. These elements are not pairwise orthogonal, since, for example,

$$(g_{(1)}(x), g_{(2)}(x)) = \int\limits_0^1 1\cdot x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2} \neq 0\ .$$

The use of the *Gram-Schmidt* procedure for orthogonalization of a given set is as follows.

We will take the first element of the original set as the first element $e_{(1)}$ of the desired orthogonal set. That is, $e_{(1)} = g_{(1)} = 1$.

We will search for the second element of the orthogonal set as a linear combination $e_{(2)} = g_{(2)} + \lambda_{2,1}\, e_{(1)}$, where $\lambda_{2,1}$ is a constant, the value of which we will select so that the *orthogonality* condition is satisfied

$$(e_{(2)}, e_{(1)}) = 0\,. \qquad (2)$$

In our case $e_{(2)}(x) = x + \lambda_{2,1} \cdot 1$, and the orthogonality condition (2) will be

$$(e_{(2)}, e_{(1)}) = (g_{(2)} + \lambda_{2,1}\, e_{(1)}, e_{(1)}) = (g_{(2)}, e_{(1)}) + \lambda_{2,1}\,(e_{(1)}, e_{(1)}) = 0 \qquad \Rightarrow \qquad \lambda_{2,1} = -\frac{(g_{(2)}, e_{(1)})}{(e_{(1)}, e_{(1)})}\,.$$

Which finally gives

$$(e_{(1)}, e_{(1)}) = \int\limits_0^1 1 \cdot 1\; dx = x\Big|_0^1 = 1\,, \quad (g_{(2)}, e_{(1)}) = \int\limits_0^1 x \cdot 1\; dx = \frac{x^2}{2}\Bigg|_0^1 = \frac{1}{2} \;\Rightarrow\; \lambda_{2,1} = -\frac{1}{2}\,.$$

So, $\quad e_{(2)}(x) = x - \dfrac{1}{2}$.

Now, attention! The main idea of the *Gram-Schmidt* method:

We will look for the third element of the orthogonal set in the form:

$$e_{(3)} = g_{(3)} + \lambda_{3,1}\, e_{(1)} + \lambda_{3,2}\, e_{(2)} \ . \tag{3}$$

We will select the values of the constants $\lambda_{3,1}$ and $\lambda_{3,1}$ so that the orthogonality conditions are satisfied simultaneously

$$(e_{(3)}, e_{(1)}) = 0 \quad \text{and} \quad (e_{(3)}, e_{(2)}) = 0 \ . \tag{4}$$

By substituting formula (3) into equalities (4), we obtain

$$(e_{(3)}, e_{(1)}) = (g_{(3)} + \lambda_{3,1}\, e_{(1)} + \lambda_{3,2}\, e_{(2)}, e_{(1)}) = (g_{(3)}, e_{(1)}) + \lambda_{3,1}\,(e_{(1)}, e_{(1)}) + \lambda_{3,2}\,(e_{(2)}, e_{(1)}) = 0 \ ,$$

$$(e_{(3)}, e_{(2)}) = (g_{(3)} + \lambda_{3,1}\, e_{(1)} + \lambda_{3,2}\, e_{(2)}, e_{(2)}) = (g_{(3)}, e_{(2)}) + \lambda_{3,1}\,(e_{(1)}, e_{(2)}) + \lambda_{3,2}\,(e_{(2)}, e_{(2)}) = 0 \ .$$

These equalities are simplified to

$$(g_{(3)}, e_{(1)}) + \lambda_{3,1}\,(e_{(1)}, e_{(1)}) = 0 \ , \qquad (g_{(3)}, e_{(2)}) + \lambda_{3,2}\,(e_{(2)}, e_{(2)}) = 0$$

and we get $\lambda_{3,1} = -\dfrac{(g_{(3)}, e_{(1)})}{(e_{(1)}, e_{(1)})}$ and $\lambda_{3,2} = -\dfrac{(g_{(3)}, e_{(2)})}{(e_{(2)}, e_{(2)})}$.

It remains to calculate three scalar products (as $(e_{(1)}, e_{(1)}) = 1$ was found earlier).

We have $\quad (g_{(3)}, e_{(1)}) = \int\limits_0^1 x^2 \cdot 1 \, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} \quad \Rightarrow \quad \lambda_{3,1} = -\frac{1}{3}$.

Similarly

$$(e_{(2)}, e_{(2)}) = \int\limits_0^1 \left(x - \frac{1}{2} \right)^2 dx = \left.\frac{1}{3}\left(x - \frac{1}{2} \right)^3 \right|_0^1 = \frac{1}{3} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{8} = \frac{1}{12} ,$$

$$(g_{(3)}, e_{(2)}) = \int\limits_0^1 x^2 \cdot \left(x - \frac{1}{2} \right) dx = \left.\left(\frac{x^4}{4} - \frac{x^3}{6} \right)\right|_0^1 = \frac{1}{12} \quad \Rightarrow \quad \lambda_{3,2} = -1 \ .$$

Now we find from (3)

$$e_{(3)}(x) = g_{(3)}(x) + \lambda_{3,1}\, e_{(1)}(x) + \lambda_{3,2}\, e_{(2)}(x) = x^2 - \frac{1}{3} \cdot 1 - \left(x - \frac{1}{2} \right) = x^2 - x + \frac{1}{6} .$$

The orthogonal system is constructed.

Finally, we should do the normalization. The corresponding orthonormal system will obviously have the form

$$\left\{ \tilde{e}_{(1)}(x) = \frac{e_{(1)}(x)}{\left| e_{(1)}(x) \right|}, \qquad \tilde{e}_{(2)}(x) = \frac{e_{(2)}(x)}{\left| e_{(2)}(x) \right|}, \qquad \tilde{e}_{(3)}(x) = \frac{e_{(3)}(x)}{\left| e_{(3)}(x) \right|} \right\},$$

since $\left| \dfrac{e_{(j)}(x)}{\left| e_{(j)}(x) \right|} \right| = 1 \quad \forall j = 1,2,3$.

We will calculate the norms of the elements using the formula of point 1) of definition 2. We will obtain

$$\left| e_{(1)}(x) \right| = \sqrt{\left(e_{(1)}(x), e_{(1)}(x) \right)} = \sqrt{1} = 1, \qquad \left| e_{(2)}(x) \right| = \sqrt{\left(e_{(2)}(x), e_{(2)}(x) \right)} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}},$$

$$\left| e_{(3)}(x) \right| = \sqrt{\left(e_{(3)}(x), e_{(3)}(x) \right)} = \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6} \right)^2 dx} = \frac{\sqrt{5}}{30}.$$

As a result, after normalization we arrive at the orthonormal basis

$$\left\{ \tilde{e}_{(1)}(x) = 1, \qquad \tilde{e}_{(2)}(x) = 2\sqrt{3}\left(x - \frac{1}{2} \right), \qquad \tilde{e}_{(3)}(x) = 6\sqrt{5}\left(x^2 - x + \frac{1}{6} \right) \right\}.$$

As a result, after normalization we arrive at the orthonormal basis

Solution is found

Let us make the following remarks on the found solution.

1) The Gram-Schmidt orthogonalization method can also be used in the case where the space E has no basis, and the original set $\{g_{(n)}\}$ is a *sequence* of elements in this space. The algorithm remains the same.

Indeed, let us orthogonalize the first k elements of the sequence and obtain pairwise orthogonal elements $\{e_{(1)},\ e_{(2)},\ \dots,\ e_{(k)}\}$.

We look for the next orthogonalized element using the formula

$$e_{(k+1)} = g_{(k+1)} + \sum_{j=1}^{k} \lambda_{k+1,j} e_{(j)} \ , \tag{5}$$

in which each of the coefficients $\lambda_{k+1,m}$ is found by scalar multiplication of both parts of equality (5) by the element $e_{(m)}$. In this case, due to $(e_{(j)}, e_{(m)}) = 0$, if $m \neq j \ \forall m \in [1, k]$, and the orthogonality condition, we obtain $\lambda_{k+1,m} = -\dfrac{(g_{(k+1)}, e_{(m)})}{(e_{(m)}, e_{(m)})}$.

Here the condition $(e_{(m)}, e_{(m)}) \neq 0$ follows (check this yourself) from the assumption of linear independence of the elements of the sequence $\{g_{(n)}\}$.

2) The Gram-Schmidt orthogonalization process can also be applied to a *linearly dependent* system of elements of Euclidean space. In this case, as a result of some steps, zero elements may be obtained, the discarding of which allows the orthogonalization process to continue.

Coordinate representation of the scalar product

A useful tool for studying the properties of a set of elements $\{f_1, f_2, ..., f_k\}$ in Euclidean space is the *Gram matrix*.

Definition 3. B In Euclidean space E, the *Gram matrix* of a system of elements $\{f_1, f_2, ..., f_k\}$ is a symmetric matrix of the form

$$\|\Gamma\|_f = \begin{Vmatrix} (f_1, f_1) & (f_1, f_2) & \cdots & (f_1, f_k) \\ (f_2, f_1) & (f_2, f_2) & \cdots & (f_2, f_k) \\ \cdots & \cdots & \cdots & \cdots \\ (f_k, f_1) & (f_k, f_2) & \cdots & (f_k, f_k) \end{Vmatrix}.$$

Let a basis $\{g_1, g_2, ..., g_n\}$ be given in E^n. The scalar product of elements $x = \sum_{i=1}^{n} \xi_i g_i$ and $y = \sum_{j=1}^{n} \eta_j g_j$, by definition 1 can be represented as

$$(x, y) = (\sum_{i=1}^{n} \xi_i g_i, \sum_{j=1}^{n} \eta_j g_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \eta_j (g_i, g_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \xi_i \eta_j,$$

where $\gamma_{ij} = (g_i, g_j) \ \forall i, j = [1, n]$ are the components of the matrix $\|\Gamma\|_g$, called the *basis Gram* matrix.

We have previously noted that this matrix is symmetric, due to the commutativity of the scalar product, and is the matrix of a symmetric bilinear form defining the scalar product. Then, using the coordinate form of the bilinear form, the coordinate representation of the scalar product can be written as follows:

$$(x,y)=\| x \|_g^{\mathrm{T}} \| \Gamma \|_g \| y \|_g = \| \xi_1 \quad \xi_2 \quad \dots \quad \xi_n \| \begin{Vmatrix} (g_1,g_1) & (g_1,g_2) & \dots & (g_1,g_n) \\ (g_2,g_1) & (g_2,g_2) & \dots & (g_2,g_n) \\ \dots & \dots & \dots & \dots \\ (g_n,g_1) & (g_n,g_2) & \dots & (g_n,g_n) \end{Vmatrix} \begin{Vmatrix} \eta_1 \\ \eta_2 \\ \dots \\ \eta_n \end{Vmatrix},$$

where $\|x\|_g$ and $\|y\|_g$ are the coordinate representations (columns) of the elements x and y in the basis $\{g_1, g_2, \dots, g_n\}$.

Finally, note that in an orthonormal basis $\| \Gamma \|_e = \| E \|$, and, therefore, the formula for the scalar product takes the form $(x,y)=\| x \|_e^{\mathrm{T}} \| y \|_e = \sum\limits_{i=1}^{n} \xi_i \eta_i$.

The Gram matrix has the following important properties.

Theorem 1. **The system of elements $\{f_1, f_2, \dots, f_k\}$ in E is linearly independent if and only if the determinant of the Gram matrix of this system is positive.**

Corollary 1. **For a basis Gram matrix $\|\Gamma\|_g$ in any basis $\det \|\Gamma\|_g > 0$.**

Theorem 2. **The system of elements $\{f_1, f_2, \dots, f_k\}$ in E is linearly dependent if and only if the determinant of the Gram matrix of this system is zero.**

The properties of the Gram matrix can be used to solve various problems. Let us demonstrate this by solving

Task 7-03. *Prove that Vandermonde determinant*

$$\det\begin{Vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{Vmatrix} > 0. \tag{6}$$

Solution: in E, the vector space of functions continuous on the interval $[0,1]$, with the scalar product given by formula (1) for $\alpha = 0$ and $\beta = 1$, we consider linearly independent elements $\{1, x, x^2, \dots, x^n\}$.

For these elements, we find all possible pairwise scalar products

$$(x^p, x^q) = \int_0^1 x^{p+q} dx = \left.\frac{x^{p+q+1}}{p+q+1}\right|_0^1 = \frac{1}{p+q+1}.$$

We form from them the Gram matrix, which has the form of the matrix in formula (6) for $p = 0,1,\dots,n-1$ and $q = 0,1,\dots,n-1$.

Since the elements we have chosen are linearly independent (prove this yourself, using, for example, Cramer's theorem), then by virtue of Theorem 1 the determinant of this matrix is positive.

Solution is found

Orthogonal Matrices. Orthogonal Projections and Orthogonal Complements in Euclidean Space

The scalar product operation in Euclidean space allows us to significantly *expand* the scope of application of linear operators and quadratic forms, but this requires the introduction of several additional concepts.

Definition 1. A square matrix $\|Q\|$ satisfying the equality $\|Q\|^{-1} = \|Q\|^{\mathrm{T}}$ is called *orthogonal*.

It is clear that solving systems of linear equations whose underlying matrix is orthogonal is a joy compared to, say, solving using Cramer's method. It is also useful to remember such properties of orthogonal matrices as:

$$\|Q\|^{\mathrm{T}}\|Q\| = \|Q\|\|Q\|^{\mathrm{T}} = \|E\| \quad \text{and} \quad \det\|Q\| = \pm 1 .$$

In addition, the following theorems will be valid in Euclidean space.

Theorem 1. **Orthogonal matrices (and only they!) in E^n can serve as transition matrices from one orthonormal basis to another.**

Indeed, let there be two different orthonormal bases $\{e_1, e_2, ..., e_n\}$ and $\{e'_1, e'_2, ..., e'_n\}$ in E^n with a transition matrix $\|S\|$ from the first basis to the second.

In these bases, the Gram matrix is identity, so the equality $\|\Gamma\|_{e'} = \|S\|^{\mathrm{T}} \|\Gamma\|_e \|S\|$, or $\|E\| = \|S\|^{\mathrm{T}} \|S\|$ follows from the relation. And, since the transition matrix is non-singular, we have $\|S\|^{-1} = \|S\|^{\mathrm{T}}$.

Theorem 2. **The eigenvalues of a linear transformation that has an orthogonal matrix in an orthonormal basis E^n are equal in absolute value to one.**

Try to prove this theorem (or find its proof in some resource) as an exercise.

Next, let a subspace E_1 be given in E . Consider a set of elements x in $E_2 \subset E$ orthogonal to all elements of E_1. Then we can give

Definition 2. In a Euclidean space E , a set E_2 is a collection of elements x such that $(x, y) = 0 \quad \forall y \in E_1 \subset E$ is called the *orthogonal complement* of the set E_1.

The following are true:

Theorem 3. **If E_2 is the orthogonal complement of a subspace $E_1 \subset E$, then E_1 is the orthogonal complement of E_2.**

and

Theorem 4. **The orthogonal complement of an k-dimensional subspace $E_1 \subset E^n$ is a subspace of dimension $n-k$.**

In a regular three-dimensional vector space, an example of an orthogonal complement to the coordinate plane Oxy is the coordinate axis Oz. The converse is also true.

Finally, we give

Definition 3. In a Euclidean space, an element E is called an orthogonal projection of an element x onto a subspace E^* if

1°. $y \in E^*$;

2°. $(x-y,u)=0 \quad \forall u \in E^*$.

Very useful for many applications is

Theorem 5. **If $E^* \subset E$ is an k-dimensional subspace, then an element y, an orthogonal projection $x \in E$ onto E^*, exists and is unique.**

Let's analyze its proof.

If there is a basis $\{g_1, g_2, ..., g_k\}$ in E^*, then the element $y \in E^*$ can be represented as $y = \sum_{i=1}^{k} \xi_i g_i$.

The condition $(x-y,u)=0 \ \forall u \in E^*$ is equivalent to the orthogonality of the vector to *each* of the basis elements of the subspace E^*, that is,

$$(x-y, g_j) = 0 \ \forall j = [1,k],$$

and, therefore, the numbers ξ_i , $i=[1,k]$ can be found from a system of linear equations

$$(x - \sum_{i=1}^{k} \xi_i g_i, g_j) = 0 \quad \forall j = [1,k] \quad \text{or} \quad \sum_{i=1}^{k} (g_i, g_j)\xi_i = (x, g_j) \quad \forall j = [1,k].$$

Since the basic matrix of this system (as the Gram basis matrix of a set of linearly independent elements $g_1, g_2, ..., g_k$) is *non-singular*, then by Cramer's theorem a solution to this system exists and is unique.

Note also that if the basis $\{e_1, e_2, ..., e_k\}$ in E^* the subspace is orthonormal, then the orthogonal projection of an element onto is an element of the form $y = \sum_{i=1}^{k} (x, e_i) e_i$.

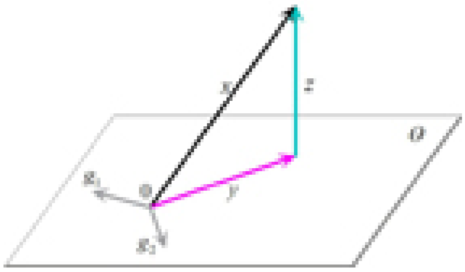
Task 7-04. In Euclidean space E^4 with initial standard orthonormal basis and scalar product, find the orthogonal projection of the element $x = \left\| \begin{matrix} -4 \\ -2 \\ 0 \\ 11 \end{matrix} \right\|$

1) onto Θ − the linear span of the elements $g_1 = \left\| \begin{matrix} -1 \\ 1 \\ 0 \\ 0 \end{matrix} \right\|$ and $g_2 = \left\| \begin{matrix} 2 \\ 2 \\ 2 \\ 3 \end{matrix} \right\|$;

2) onto the orthogonal complement of Θ .

Solution: 1°. Note (justify this) that the elements g_1 and g_2 not only generate the linear span of Θ, but also form an orthogonal basis in it. The dimension of Θ in this problem is 2.

2°. According to definition 3 and the triangle rule, we have the following relationships: let y be the orthogonal projection x onto Θ, then



$$z = x - y\,,$$

$$y = \lambda_1 g_1 + \lambda_2 g_2\,,$$

$$z = x - \lambda_1 g_1 - \lambda_2 g_2\,.$$

The orthogonality condition for each element z of Θ will be:

$$\begin{cases} (g_1, z) = 0, \\ (g_2, z) = 0 \end{cases} \forall z \qquad \text{or} \qquad \begin{cases} (g_1, g_1)\lambda_1 + (g_1, g_2)\lambda_2 = (g_1, x), \\ (g_2, g_1)\lambda_1 + (g_2, g_2)\lambda_2 = (g_2, x). \end{cases} \qquad (1)$$

Having found λ_1 and λ_2 from system (1), we obtain y which is the desired orthogonal projection onto the linear shell Θ .

In order to obtain system (1), we calculate the following five scalar products:

$$\begin{aligned}
(g_1, g_1) &= (-1)(-1) + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 &&= 2, \\
(g_1, g_2) &= (-1)\cdot 2 + 1\cdot 2 + 0\cdot 2 + 0\cdot 3 &&= 0, \\
(g_2, g_2) &= 2\cdot 2 + 2\cdot 2 + 2\cdot 2 + 3\cdot 3 &&= 21, \\
(g_1, x) &= (-1)(-4) + 1\cdot(-2) + 0\cdot 0 + 0\cdot 11 &&= 2, \\
(g_2, x) &= 2\cdot(-4) + 2\cdot(-2) + 2\cdot 0 + 3\cdot 11 &&= 21,
\end{aligned}$$

Then system (1) will have the form and, accordingly, an obvious solution

$$\begin{cases} 2\cdot\lambda_1 + 0\cdot\lambda_2 = 2, \\ 0\cdot\lambda_1 + 21\cdot\lambda_2 = 21 \end{cases} \quad\Rightarrow\quad \lambda_1 = 1, \quad \lambda_2 = 1.$$

Note that the main matrix of this system is diagonal, due to the elements g_1 and g_2 are orthogonal.

Now we find the answer to the first question of the problem: the orthogonal projection of the element onto the linear hull of the elements g_1 and g_2 will be equal to $y = \lambda_1 g_1 + \lambda_2 g_2$. That is, $y = 1\cdot\begin{Vmatrix}-1\\1\\0\\0\end{Vmatrix} + 1\cdot\begin{Vmatrix}2\\2\\2\\3\end{Vmatrix} = \begin{Vmatrix}1\\3\\2\\3\end{Vmatrix}$.

3°. The answer to the second question is already practically obtained if we note that the orthogonal complement to Θ (by definition 2) is the linear span of the element $z = x - \lambda_1 g_1 - \lambda_2 g_2$.
Show that it follows from Theorem 3 that the orthogonal projection of the element x onto the orthogonal complement to Θ will be precisely the element of the form $z = x - \lambda_1 g_1 - \lambda_2 g_2$.

When solving problems that require finding the orthogonal projection onto some subspace, it should be remembered that the subspace can be defined not only as the *linear span* of some elements, but also using a *homogeneous system* of linear equations.

For a better understanding of this fact, try (as an exercise) to solve

Task 7-05. *In a Euclidean space E^4 with an initial orthonormal basis and a standard scalar product, find y, which is the orthogonal projection of an element* $x = \begin{Vmatrix} 1 \\ -1 \\ -2 \\ 2 \end{Vmatrix}$ *onto Θ. Here Θ is a subspace defined by a system of linear equations*

$$\begin{cases} -\xi_1 \phantom{{}-\xi_2} - 2\xi_3 - \xi_4 = 0, \\ \xi_1 - \xi_2 + 3\xi_3 \phantom{{}-\xi_4} = 0. \end{cases}$$

In this problem, I got the answer $y = \begin{Vmatrix} 1 \\ -2 \\ -1 \\ 1 \end{Vmatrix}$.

Task 7-06. *In a Euclidean space E^4 with a standard scalar product in some orthonormal basis, a homogeneous system of linear equations*

$$\begin{cases} \xi_1 + \xi_2 - \xi_3 - \xi_4 = 0, \\ 2\xi_1 + \xi_2 \qquad\qquad = 0 \end{cases}$$

defines a subspace E^*. Find in this basis the matrix of a linear transformation that is an orthogonal projection of elements E^4 onto E^*.

Solution:

1°. A pair of elements and can be taken as a basis for the subspace E^*, whose coordinate representations in the original basis $\{e_1, e_2, e_3, e_4\}$ are linearly independent solutions of a homogeneous system of linear equations defining E^*, for example,

$$\| g_1 \|_e = \begin{Vmatrix} -1 \\ 2 \\ 1 \\ 0 \end{Vmatrix}; \quad \| g_2 \|_e = \begin{Vmatrix} -1 \\ 2 \\ 0 \\ 1 \end{Vmatrix}.$$

2°. Since $\dim E^* = 2$, the dimension of the orthogonal complement E^* according to Theorem 3 is also equal to 2. It is convenient to take as a basis in this orthogonal complement the elements g_3 and g_4, such that $\| g_3 \|_e = \begin{Vmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Vmatrix}$; $\| g_4 \|_e = \begin{Vmatrix} 2 \\ 1 \\ 0 \\ 0 \end{Vmatrix}$, since they are linearly independent and orthogonal to each element from the subspace E^*, as formed from the coefficients of the system of linear equations specified in the problem statement.

3°. Elements g_1, g_2, g_3 and g_4 are linearly independent by construction and form a basis in E^4, and each element of E^4 can be represented, and uniquely, as a linear combination of elements of this basis $\{g_1, g_2, g_3, g_4\}$.

The desired operator $\hat{A}$ of orthogonal projection of elements E^4 onto E^4 must obviously satisfy the relations

$$\hat{A}g_1 = g_1; \quad \hat{A}g_2 = g_2; \quad \hat{A}g_3 = o; \quad \hat{A}g_4 = o,$$

by virtue of which its matrix in the basis $\{g_1, g_2, g_3, g_4\}$ will have the following form:

$$\| \hat{A} \|_g = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}.$$

4°. On the other hand, the matrix of transition from basis $\{e_1, e_2, e_3, e_4\}$ to basis $\{g_1, g_2, g_3, g_4\}$

$$\| S \| = \begin{Vmatrix} -1 & -1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{Vmatrix},$$

but since $\| \hat{A} \|_g = \| S \|^{-1} \| \hat{A} \|_e \| S \|$ and, therefore $\| \hat{A} \|_e = \| S \| \| \hat{A} \|_g \| S \|^{-1}$, then, using the rules for calculating the product of matrices, we find that

$$\| \hat{A} \|_e = \begin{Vmatrix} -1 & -1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{Vmatrix} \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \begin{Vmatrix} -1 & -1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{Vmatrix}^{-1} =$$

$$= \frac{1}{11} \begin{Vmatrix} 2 & -4 & -1 & -1 \\ -4 & 8 & 2 & 2 \\ -1 & 2 & 6 & -5 \\ -1 & 2 & -5 & 6 \end{Vmatrix}.$$

Solution is found