Linear operators in Euclidean space

Earlier we considered the operator of orthogonal projection onto a finite-dimensional subspace in Euclidean space. However, in this space it is possible to distinguish other specific classes of linear operators, for the definition of which the operation of the scalar product is used.

Let us consider three more classes of linear transformations that exist only in Euclidean spaces. These are conjugate, self-conjugate and orthogonal operators.

For greater clarity, we will immediately give the definitions of these operators. And we will consider their main properties separately.

Definition 1. In Euclidean space E :

1°. A linear operator $\hat{A}^{+}$ is called *conjugate* to a linear operator $\hat{A}$ if $\forall x, y \in E$ the equality $(\hat{A}x, y) = (x, \hat{A}^{+} y)$ holds.

2°. A linear operator $\hat{R}$ is called *self-conjugate* if $\forall x, y \in E$ the equality $(\hat{R}x, y) = (x, \hat{R}y)$ holds.

3°. A linear operator $\hat{Q}$ is called *orthogonal* if $\forall x, y \in E$ the equality holds $(\hat{Q}x, \hat{Q}y) = (x, y)$.

Properties of conjugate operators

First, we give examples of conjugate operators.

Example 1. Let us construct an operator conjugate to a linear differentiation operator $\hat{A} = \frac{d}{d\tau}$, which acts in the Euclidean space of infinitely differentiable functions equal to zero outside a certain finite interval, with the scalar product $(x,y) = \int\limits_{-\infty}^{+\infty} x(\tau)y(\tau)d\tau$.

For this, we use the rule of integration "by parts", according to which the equalities hold

$$(\hat{A}x,y) = \int\limits_{-\infty}^{+\infty} \frac{dx(\tau)}{d\tau} y(\tau)d\tau = x(\tau)y(\tau)\Bigg|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} x(\tau)\frac{dy(\tau)}{d\tau}d\tau =$$

$$= \int\limits_{-\infty}^{+\infty} x(\tau)\left(-\frac{dy(\tau)}{d\tau}\right)d\tau = (x, \hat{A}^{+}y).$$

From which it follows that the desired conjugate operator is the operator $\hat{A}^{+} = -\frac{d}{d\tau}$.

Example 2. Let us now consider a finite-dimensional Euclidean space E^n with a basis $\{g_1, g_2, ..., g_n\}$ and find out the relationship between the matrices of linear operators $\hat{A}$ and $\hat{A}^+$ in this basis.

Solution: Let the matrices of operators $\hat{A}$ and $\hat{A}^+$ have the form $\|\hat{A}\|_g$ and $\|\hat{A}^+\|_g$, respectively, and the coordinate columns of the elements x and y in the basis $\{g_1, g_2, ..., g_n\}$ be

$$\|x\|_g = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \ldots \\ \xi_n \end{matrix} \right\| \quad \text{and} \quad \|y\|_g = \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \ldots \\ \eta_n \end{matrix} \right\|,$$

then the equality $(\hat{A}x, y) = (x, \hat{A}^+ y)$ can be written as

$$(\|\hat{A}\|_g \|x\|_g)^{\mathrm{T}} \|\Gamma\|_g \|y\|_g = \|x\|_g^{\mathrm{T}} \|\Gamma\|_g \|\hat{A}^+\|_g \|y\|_g ,$$

where $\|\Gamma\|_g$ is the Gram basis matrix.

By virtue of the relation $(\|A\|\|B\|)^{\mathrm{T}} = \|B\|^{\mathrm{T}} \|A\|^{\mathrm{T}}$, the last equality can be transformed to the form $\|x\|_g^{\mathrm{T}} (\|\hat{A}\|_g^{\mathrm{T}} \|\Gamma\|_g - \|\Gamma\|_g \|\hat{A}^+\|_g) \|y\|_g = 0.$

Since this equality is valid for any x and y, we conclude that the matrix in parentheses is *zero*, and from the relation

$$\|\hat{A}\|_g^{\mathrm{T}} \|\Gamma\|_g - \|\Gamma\|_g \|\hat{A}^+\|_g = \|O\| \quad \text{it follows} \quad \|\hat{A}^+\|_g = \|\Gamma\|_g^{-1} \|\hat{A}\|_g^{\mathrm{T}} \|\Gamma\|_g ,$$

which, in particular, for an *orthonormal* basis $\{e_1, e_2, ..., e_n\}$ has the form

$$\|\hat{A}^+\|_e = \|\hat{A}\|_e^{\mathrm{T}} .$$

Now we formulate (without proofs that can be found in lecture notes or other resources) the main properties of conjugate operators in the form of the following theorems.

Theorem 1. **Every linear operator in a Euclidean space E^n has a unique conjugate operator.**

Theorem 2. **For any linear operators $\hat{A}$ and $\hat{B}$ acting in E, the equality holds $(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$.**

Theorem 3. **The equality holds $(\hat{A}^+)^+ = \hat{A}$.**

Theorem 4. **The *orthogonal complement of the range* of the operator $\hat{A}$ in E^n is the *kernel* of the operator $\hat{A}^+$.**

Theorem 4 admits an interesting interpretation. If its condition and assertion are written in coordinate form, then we obtain a theorem equivalent to *Fredholm's* theorem on a necessary and sufficient condition for the compatibility of an inhomogeneous system of linear equations. If you are interested, check it yourself.

Self-conjugate operators

A linear operator $\hat{R}$ is called self-conjugate if $\forall x, y \in E \quad (\hat{R}x, y) = (x, \hat{R}y)$.

Example 3. In Euclidean space, operators of the form $\hat{A} + \hat{A}^{+}$, $\hat{A}\hat{A}^{+}$ and $\hat{A}^{+}\hat{A}$, will be self-conjugate for any linear operator $\hat{A}$.

Solution: Indeed, for the operator $\hat{A}^{+}\hat{A}$, for example, we will have that $\forall x, y \in E$ and $(\hat{A}^{+}\hat{A}x, y) = (\hat{A}x, \hat{A}y) = (x, \hat{A}^{+}\hat{A}y)$, from which it follows that it is self-conjugate.

And, if we take the operator $\hat{A}$ from example 1 as $\hat{A} = \dfrac{d}{d\tau}$, then from the previous reasoning it follows that the operator

$$\hat{A}\hat{A}^{+} = \frac{d}{d\tau}\left(-\frac{d}{d\tau}\right) = -\frac{d^2}{d\tau^2}$$

will be self-conjugate. Let's remember this!

Let us formulate the main properties of self-conjugate operators in the form of the following theorems.

Theorem 1. **A linear operator $\hat{R}$ in E^n is self-conjugate if and only if its matrix in each orthonormal basis is *symmetric*.**

Theorem 2. **All eigenvalues of a self-conjugate operator $\hat{R}$ in E^n are real numbers.**

Theorem 3. **The eigenvectors of a self-conjugate operator corresponding to *different* eigenvalues are pairwise *orthogonal*.**

Theorem 4. **Let E' be an *invariant* subspace of a self-conjugate operator $\hat{R}$ acting in E, and let E'' be the orthogonal complement of E' in E. Then E'' is also an *invariant* subspace of $\hat{R}$.**

Theorem 5. **For any self-conjugate operator $\hat{R}$ in E^n, there exists an orthonormal basis consisting of eigenvectors $\hat{R}$.**

Theorem 6. **Two self-conjugate operators $\hat{A}$ and $\hat{B}$ have a *common system* of eigenvectors in if and only if $\hat{A}$ and $\hat{B}$.**

We did not plan to prove these theorems here, but it is difficult to resist and not to give a proof of Theorem 3, rare in its compactness and elegance.

Proof of then Theorem 3.

Let the equalities $\hat{R}f_1 = \lambda_1 f_1$ and $\hat{R}f_2 = \lambda_2 f_2$ hold for a self-conjugate operator $\hat{R}$ where the nonzero elements f_1 and f_2 are the eigenvectors of the operator $\hat{R}$ and $\lambda_1 \neq \lambda_2$ are the corresponding eigenvalues.
Multiplying these equalities respectively: the first is a scalar from the right by f_2, the second is a scalar from the left by f_1, we obtain

$$\begin{cases}(\hat{R}f_1, f_2) = (\lambda_1 f_1, f_2), \\ (f_1, \hat{R}f_2) = (f_1, \lambda_2 f_2)\end{cases} \quad \text{or} \quad \begin{cases}(\hat{R}f_1, f_2) = \lambda_1 (f_1, f_2), \\ (f_1, \hat{R}f_2) = \lambda_2 (f_1, f_2).\end{cases}$$

Subtracting these equalities term by term and taking into account that $\hat{R}$ is a self-conjugate operator (i.e. the left parts are equal), we arrive at the equality $(\lambda_1 - \lambda_2)(f_1, f_2) = 0$.

Whence, by virtue of $\lambda_1 \neq \lambda_2$, $(f_1, f_2) = 0$.

As an exercise, check the following corollaries:

Corollary 1. **(*Self-conjugateness* criterion): if a linear operator in E^n has a symmetric matrix in some orthonormal basis, then it is self-conjugate.**

Corollary 2. **The dimension of the proper invariant subspace corresponding to some eigenvalue of a self-conjugate operator is equal to the multiplicity of this eigenvalue.**

Corollary 3. **If $\|R\|$ is a symmetric matrix, then there exists an orthogonal matrix $\|Q\|$ such that $\|D\| = \|Q\|^{-1}\|R\|\|Q\| = \|Q\|^{\mathrm{T}}\|R\|\|Q\|$ is diagonal.**

Orthogonal operators

A linear operator $\hat{Q}$ acting in a Euclidean space E is called orthogonal (or isometric) if $\forall x, y \in E$ the equality $(\hat{Q}x, \hat{Q}y) = (x, y)$ holds.

From this definition it follows that an orthogonal operator preserves the norms of elements and the magnitudes of angles between them. Indeed,

$$\left|\hat{Q}x\right| = \sqrt{(\hat{Q}x, \hat{Q}x)} = \sqrt{(x, x)} = |\,x\,| \ ;$$

$$\cos\psi = \frac{(\hat{Q}x, \hat{Q}y)}{\left|\hat{Q}x\right|\left|\hat{Q}y\right|} = \frac{(x, y)}{|\,x\,||\,y\,|} = \cos\varphi \ ; \quad x, y \in E,$$

where φ is the magnitude of the angle between nonzero elements x and y , and ψ is the magnitude of the angle between elements and

Let us formulate (without proof) the main properties of orthogonal operators in the form of the following theorems.

Theorem 7. **. If an orthogonal operator $\hat{Q}$ has an *conjugate* operator, then it also has an *inverse* operator, and $\hat{Q}^{-1} = \hat{Q}^{+}$.**

Theorem 8. **The matrix of an orthogonal operator in E^n in *each orthonormal* basis is *orthogonal.***

Theorem 9. **Any linear operator $\hat{A}$ in E^n with $\det\|\hat{A}\| \neq 0$ can be *uniquely* represented in the form $\hat{A} = \hat{Q}\hat{R}$, where the operator $\hat{Q}$ is orthogonal, and the operator $\hat{R}$ is self-conjugate and has positive eigenvalues.**

Theorem 9 is often called the polar decomposition theorem in mathematical literature.

As an exercise, check the following corollaries:

Corollary 3. **The operators $\hat{Q}^{+}$ and $\hat{Q}^{-1}$ are also orthogonal.**

Corollary 4. **(*Orthogonality* criterion) For a linear operator in E^n to be orthogonal, it is necessary and sufficient that its matrix be *orthogonal* in *some orthonormal* basis.**