Indefinite integral

In the case $f(x) = F'(x)$, the function $f(x)$ is called the *is the derivative function of the function* $F(x)$, and the function $F(x)$ — *is the primal function for the function* $f(x)$.

The relation $f(x) = F'(x)$ expresses the derivative through the primal, and the question naturally arises as to how to express the primal through the derivative.

It is convenient to introduce a special notation for all primes of the function $f(x)$.

Definition 2.2. The set of *all* antiderivative functions for some function $f(x)$ is called *indefinite integral* and is denoted as

$$\int f(x)\,dx\ .$$

The question naturally arises: Is it always possible to construct a formula for $F(x)$ using a formula for $f(x)$? And, if possible, how to do it?

Let us remind you that that from the formula $F(x)$ the formula for $f(x)$ can always be constructed. But on the contrary — «alas and ah!» the answer to this question is generally speaking, *is negative.*

Examples of functions $f(x)$ for which antiderivatives are not expressed through elementary functions, they serve e^{x^2}, $\dfrac{\sin x}{x}$, $\dfrac{1}{\ln x}$. Accordingly, integrals of the type

$$\int e^{x^2}\,dx\ , \qquad \int \frac{\sin x}{x}dx\ , \qquad \int \frac{dx}{\ln x}$$

It is customary to call them "non-taken" ones.

At the same time, it should be noted that, in addition to obvious relations like
$$\int F'(x)\,dx = \Big\{\ F(x) + C \quad \forall C\ \Big\}\,,$$
in a number of practically important cases it is still possible to find formula for $F(x)$ according to the known formula for $f(x)$. The total number of these cases varies in the thousands and their description is the contents of reference books that are quite substantial in size and number of pages.

For our purposes it will be quite enough collections of indefinite integrals presented in Tables 2.3a and 2.3b. In what follows we will call these integrals “tabular” and consider them known.

I would like to draw your attention once again on the significance of the assumption of continuity of the function $f(x)$.

Let's take a closer look at the fourth formula (marked with «asterisk») in table 2.3a. You can verify that the given formula for the antiderivative specifies *not all* functions $F(x)$ such that $F'(x) = f(x)$. For example, for the function

$$F(x) = \left[\begin{array}{ll} \ln x + C_1 \, , & \text{if } \; x > 0 \, , \\ \ln(-x) + C_2 \, , & \text{if } \; x < 0 \end{array} \right.$$

derivatives both for the case $x > 0$, and for $x < 0$ are represented the same formula $\frac{1}{x}$, even if $C_1 \neq C_2$.

Really,

$$\frac{d\,(\ln x + C_1)}{dx} = \frac{1}{x}\,, \text{ if } \; x > 0\;,$$
$$\frac{d\,(\ln(-x) + C_2)}{dx} = \frac{1}{(-x)} \cdot (-1) = \frac{1}{x}\,, \text{ if } \; x < 0\;.$$

The point is that the function $f(x) = \dfrac{1}{x}$ is not continuous: for $x_0 = 0$ it has a discontinuity, which leads to violation of equality (2.4.)

However, for practice these problems are not a significant limitation. Let's say the function $f(x) = \dfrac{1}{x}$ can be considered for the cases $x > 0$ and $x < 0$ independently of each other. Then formula (2.4) will be satisfied, since this function is continuous both for all $x > 0$, and for any negative values of its argument.

	$f(x)$	$\int f(x)\,dx$
1°	$x^a, a \neq -1$	$\dfrac{x^{a+1}}{a+1} + C$
2°	e^x	$e^x + C$
3°	$a^x, \quad a > 0, a \neq 1$	$\dfrac{a^x}{\ln a} + C$
4°	$\dfrac{1}{x}$	$\ln\|x\| + C^*$
5°	$\cos x$	$\sin x + C$
6°	$\sin x$	$-\cos x + C$

Table 2.3a

	$f(x)$	$\int f(x)\,dx$
7°	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
8°	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\operatorname{arctg}\dfrac{x}{a} + C$, если $a \neq 0$
9°	$\dfrac{1}{\sqrt{x^2+a^2}}$	$\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert + C$, если $a > 0$
10°	$\sqrt{a^2-x^2}$	$\dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a} + C$, если $a > 0$
11°	$\sqrt{x^2-a^2}$	$\dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert + C$

Table 2.3b

Such a small number of «tabular» integrals turns out to be sufficient to solve a significant number of problems, since we also have formulas at our disposal, allowing *to express indefinite integrals of some functions through integrals of others.* These formulas are summarized in Table 2.4.

1°	$\int (\ f(x)+g(x)\ )\ dx = \int f(x)\,dx + \int g(x)\,dx$
2°	$\int (\ k\cdot f(x)\ )\ dx = k\cdot\int f(x)\,dx$, где k – const
3°	$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx$
4°	$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du$, где $u = g(x)$

Table 2.4

Such a small number of «tabular» integrals turns out to be sufficient to solve a significant number of problems, since we also have formulas at our disposal, allowing *to express indefinite integrals of some functions through integrals of others.* These formulas are summarized in Table 2.4.

These formulas need proof, and for example let us verify the validity of the formula 3°, often called a rule *integration by parts.* According to paragraph 3° of Table 2.2 and the definition of the integral

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x) \;\Rightarrow$$

$$\Rightarrow \int (f(x)\cdot g(x))'\,dx = \int (f'(x)\cdot g(x) + f(x)\cdot g'(x))\,dx \;\Rightarrow$$

$$\Rightarrow\; f(x)\cdot g(x) = \int f'(x)\cdot g(x)\,dx + \int f(x)\cdot g'(x)\,dx\;,$$

since the equality of the functions implies the equality of their indefinite integrals (but not primitives!) The formula for integration by parts follows from the last relation.

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx\;.$$

Rule 4°, often called *rule of change of variable of integration,* is also quite obvious, because by the definition of the first differential from the equality $u = g(x)$ follows $du = g'(x)\,dx$. Note that quite often (especially if the function $g(x)$ is not too complex) the 4° rule is written as

$$\int f(g(x))\cdot g'(x)\,dx = \int f(g(x))\,d(g(x))\;.$$

Examples of finding integrals

Let's look at examples of finding integrals, based on the following scheme:

A) we reduce the integrand of this definite integral to the form convenient for using tables 2.3 and 2.4;

B) using tables 2.3 and 2.4 we find the indefinite integral. (or some antiderivative);

First, let's demonstrate how, using Table 2.3, knowing the indefinite integral $\int f(x)\,dx$, we can obtain a formula for $\int f(ax+b)\,dx$.

Example 2.2. Let it be required to find

$$\int \cos(2x-7)\,dx\ .$$

Solution. Note that in table 2.3a (formula 5°) there is a table integral

$$\int \cos x\,dx = \sin x + C\ .$$

Let's introduce a new variable $u = 2x - 7$, for which we have

$$du = d(2x-7) = d(2x) + d(-7) = 2\,dx + 0 = 2\,dx \qquad \Rightarrow \qquad dx = \frac{du}{2}\ .$$

Then the required integral using Table 2.4 (formula 2°) is found as follows

$$\int \cos(2x-7)\,dx = \int \cos u\,\frac{du}{2} = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C =$$

$$= \frac{1}{2}\sin(2x-7) + C.$$

Notice that in the answer, instead of $\dfrac{C}{2}$ written C is not a typo. An arbitrary constant divided in half, still remains an arbitrary constant.

Check for yourself that you can use a similar method, for example, get formulas

$$\int e^{3x+1}\,dx = \frac{1}{3}e^{3x+1} + C \qquad \text{and} \qquad \int \frac{dx}{5x+2} = \frac{1}{5}\ln|5x+2| + C\,.$$

Next, we note that, despite the impossibility in the general case of writing the integral in the form of some elementary function, there are cases when it is fundamentally *is always doable.*

To such integrals, first of all, include integrals of *fractional-rational* functions, that is, functions representable in the form of a fraction, the numerator and denominator of which are algebraic polynomials, For example,

$$\int \frac{x^2+4x+3}{x^3-3x^2+8x-6}\,d\,x\,.$$

Let us consider two main methods for integrating fractional-rational functions, in the first of which it is possible to decompose the denominator into linear factors.

Example 2.3. Let it be required to find

$$\int \frac{(x-4)\,dx}{x^2-5x+6} \,.$$

Solution. Since $x^2-5x+6=(x-2)(x-3),$ then let's try to represent the integrand function in the form

$$\frac{x-4}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}\,,$$

where A and B are some numbers whose values we will find from the following chain of equalities.

$$\frac{x-4}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}=$$

$$=\frac{A(x-3)-B(x-2)}{(x-2)(x-3)}=\frac{(A+B)x-(3A+2B)}{x^2-5x+6}\,.$$

Comparing the initial and final links of this chain, it is easy to see that the values of the numbers A and B (for which these equalities are true for all x) are solving a system of linear equations

$$\left\{\begin{array}{rcrcr} A & + & B & = & 1 \\ 3A & + & 2B & = & 4 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{rcr} A & = & 2 \\ B & = & -1 \end{array}\right. .$$

Now, using tables 2.3 and 2.4, we get

$$\int \frac{(x-4)\,dx}{x^2-5x+6} = \int \left(\frac{2}{x-2} - \frac{1}{x-3} \right) dx =$$

$$= 2\int \frac{dx}{x-2} - \int \frac{dx}{x-3} = 2\ln|x-2| - \ln|x-3| + C = \ln\frac{(x-2)^2}{|x-3|} + C\ .$$

Note that this algorithm is sometimes called «by the method of decomposition into simple factors».

The following two examples show how to act in case when the denominator of a fractional rational function fails factorize it linearly.

Example 2.4. Find the integral

$$\int \frac{dx}{x^2+4}\,.$$

Solution. Using the formula 2° (from table 2.4), we perform the following transformations of the corresponding indefinite integral

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{1+\left(\dfrac{x}{2}\right)^2} = \frac{1}{2}\int \frac{\dfrac{dx}{2}}{1+\left(\dfrac{x}{2}\right)^2} =$$

Now we use the definition of differential and the 4° rule from Table 2.4, considering $u = \dfrac{x}{2}$.

$$= \frac{1}{2}\int \frac{d\left(\dfrac{x}{2}\right)}{1+\left(\dfrac{x}{2}\right)^2} = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + C\,.$$

The final equality follows from formula 8° of table 2.4.

Example 2.5. Find

$$\int \frac{x^2+x+1}{x^2+1}\,dx\ .$$

Solution. Because the

$$\frac{x^2+x+1}{x^2+1} = 1 + \frac{x}{x^2+1}\ ,$$

then applying sequentially formulas 1° and 4° of Table 2.4, we obtain

$$\int \frac{x^2+x+1}{x^2+1}\,dx = \int \left(1 + \frac{x}{x^2+1}\right)\,dx = x + \int \frac{x}{x^2+1}\,dx =$$

$$= x + \frac{1}{2}\int \frac{d(x^2)}{x^2+1} = x + \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = x + \frac{1}{2}\ln\left|x^2+1\right| + C\ .$$

Here, when using 4°, we assumed $u = x^2+1$, whence the equalities for differentials follow: $d(x^2+1) = d(x^2) = 2x\,dx$.

Example 2.5. Найти

$$\int \frac{x^2+x+1}{x^2+1}\,dx\ .$$

Решение. Поскольку

$$\frac{x^2+x+1}{x^2+1} = 1 + \frac{x}{x^2+1}\ ,$$

то применяя последовательно формулы 1° и 4° таблицы 2.4, получим

$$\int \frac{x^2+x+1}{x^2+1}\,dx = \int \left(1+\frac{x}{x^2+1}\right)\,dx = x + \int \frac{x}{x^2+1}\,dx =$$

$$= x + \frac{1}{2}\int \frac{d(x^2)}{x^2+1} = x + \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = x + \frac{1}{2}\ln\left|x^2+1\right| + C\ .$$

Здесь, при использовании 4° мы полагали $u = x^2+1$, откуда следуют равенства для дифференциалов: $d(x^2+1) = d(x^2) = 2x\,dx$.

Example 2.7. Find

$$\int x \cdot \cos x \, dx \; .$$

Solution. To find the indefinite integral we apply rule of integration by parts, that is, the 3° formula from Table 2.4. We will assume that $f(x) = \sin x$, and $g(x) = x$. Then the corresponding the indefinite integral can be transformed as follows.

$$\int x \cdot \cos x \, dx \; = \int x \cdot (\sin x)' \, dx \; = x \cdot \sin x - \int (x)' \cdot \sin x \, dx \; =$$

Since $(x)' = 1$,, using the sixth formula of Table 2.3 we obtain

$$= x \cdot \sin x - \int \sin x \, dx \; = x \cdot \sin x - (-\cos x) + C = x \cdot \sin x + \cos x + C \; .$$

Example 2.8. Find

$$\int \operatorname{arctg} x \, dx \; .$$

Solution. Let us apply the formula 3° *integration by parts,* using the equality $(x)' = 1$,

$$\int \operatorname{arctg} x \, dx \;=\int 1 \cdot \operatorname{arctg} x \, dx \;= x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx \;=$$

then - formula 4° of table 2.4, which, taking into account $u = x^2$ gives

$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{d(x^2)}{1+x^2} = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C \; .$$

To conclude the review, we give an example in which it turns out It is advisable to *double* use the rule of integration by parts.

Example 2.9. Find

$$\int e^x \sin x \, dx \; .$$

Solution. Applying *integration by parts* we obtain

$$\int e^x \sin x \, dx \; = e^x (-\cos x) - \int e^x (-\cos x) \, dx \; ,$$

that is, an expression containing an integral that is no simpler than the original one. However, repeated integration by parts gives the relation

$$\int e^x \sin x \, dx \; = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \; ,$$

from which it follows

$$2 \cdot \int e^x \sin x \, dx \; = -e^x \cos x + e^x \sin x + C \quad \Rightarrow$$

$$\Rightarrow \quad \int e^x \sin x \, dx \; = \frac{1}{2} e^x (\sin x - \cos x) + C \; .$$