Analysis of functions of several variables for differentiability

It is convenient to demonstrate the method for studying functions of several variables for differentiability for the next problem.

Problem 4.1 *Examine for differentiability at the origin of functions* $f(x, y) = \sqrt[3]{x^2 y^2}$ *and* $g(x, y) = \sqrt[3]{x^3 + y^3}$.

Solution. 1. First, let's remember the definition of differentiability (see theme 03). From there we get the following.
For the function $f(x, y)$ at the point $\|x_0\, y_0\|$ you need to answer the question: *is the equality true*

$$\lim_{\|x\, y\| \to \|x_0\, y_0\|} \frac{f(x, y) - f(x_0, y_0) - A\big(x - x_0\big) - B\big(y - y_0\big)}{\sqrt{\big(x - x_0\big)^2 + \big(y - y_0\big)^2}} = 0, \tag{4.01}$$

where $A = \dfrac{\partial f}{\partial x}(x_0, y_0)$ *and* $B = \dfrac{\partial f}{\partial y}(x_0, y_0)$?

2. Consider the function $f(x, y)$ at the origin. Here there is $f(0, 0) = 0$. According to the rules of differentiation

$$\frac{\partial f}{\partial x} = \frac{2}{3}\sqrt[3]{\frac{y^2}{x}} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2}{3}\sqrt[3]{\frac{x^2}{y}}\,.$$

Calculate partial derivatives using these formulas at the origin obviously not possible. Therefore we use (3.01). We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f\big(0+t, 0\big) - f(0,0)}{t} = \lim_{t\to 0}\frac{\sqrt[3]{t^2\cdot 0^2} - 0}{t} = 0\,.$$

Similarly we get that $\dfrac{\partial f}{\partial y}(0,0) = 0\,.$

Note also that at the origin

$$x - x_0 = x \qquad \text{and} \qquad y - y_0 = y\,.$$

3. The condition (4.01) can be written in form

$$\lim_{\|x\ y\| \to \|0\ 0\|} \frac{\sqrt[3]{x^2y^2} - 0 - 0 \cdot x - 0 \cdot y}{\sqrt{x^2 + y^2}} = 0,$$

or

$$\lim_{\|x\ y\| \to \|0\ 0\|} \frac{\sqrt[3]{x^2y^2}}{\sqrt{x^2 + y^2}} = 0, \tag{4.02}$$

4. Let us move in formula (4.02) to the polar coordinate system, making a change of variables

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi. \end{cases}$$

Then we get

$$0 \leq \frac{\sqrt[3]{x^2y^2}}{\sqrt{x^2 + y^2}} = \frac{\sqrt[3]{r^2\cos^2\varphi \cdot r^2\sin^2\varphi}}{r} \leq \sqrt[3]{r}\,,$$

from which, by the theorem about «two policemen», due to $r \to 0$, justice follows (4.01). So $\sqrt[3]{x^2y^2}$ differentiable at the origin.

5. Consider the function $g(x, y) = \sqrt[3]{x^3 + y^3}$ at the origin. Here there is $f(0, 0) = 0$.
Partial derivatives of this function at the origin will also have to calculate using formula (3.01).

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{\sqrt[3]{t^3 + 0^3} - 0}{t} = 1\ .$$

Similarly we find $\quad \dfrac{\partial f}{\partial y}(0,0) = 1\ .$

At the origin we have

$$x - x_0 = x \qquad \text{and} \qquad y - y_0 = y\,.$$

6. Condition (4.01) has the form

$$\lim_{\|x\,y\|\to\|0\,0\|} \frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}} = 0, \qquad (4.03)$$

Let us show that equality (4.03) is not true, using the negation of Heine's definition of the limit.
Indeed, if in formula (4.03) we pass to the limit with $t \to 0$ along the trajectory

$$\begin{cases} x(t) = 0, \\ y(t) = t, \end{cases}$$

then we get

$$\lim_{t\to 0} \frac{\sqrt[3]{0^3+t^3}-0-t}{\sqrt{0^2+t^2}} = 0,$$

At the same time, on the trajectory

$$\begin{cases} x(t) = t, \\ y(t) = t, \end{cases}$$

for $t > 0$ we will have

$$\lim_{t\to +0} \frac{\sqrt[3]{t^3+t^3}-t-t}{\sqrt{t^2+t^2}} = \frac{\sqrt[3]{2}-2}{\sqrt{2}} \neq 0,$$

Solution found.

This means that equality (4.03) is not true and function $\sqrt[3]{x^3+y^3}$ non-differentiable at the origin.

Let us now show that the scheme considered above can be used for a fairly wide class of tasks.

Problem 4.2 *Examine the function for differentiability at the origin*

$$f(x,y) = \begin{cases} \dfrac{\sqrt{1+xy} - e^{\frac{xy}{2}}}{(x^2+y^2)^{\frac{3}{2}}}, & if \quad x^2+y^2 \neq 0\,, \\ 0, & if \quad x^2+y^2 = 0\ . \end{cases}$$

Solution. 1. For the function $f(x,y)$ there is $f(0,0) = 0$. In this case, at the origin of coordinates

$$x - x_0 = x \qquad \text{and} \qquad y - y_0 = y\,.$$

Let's calculate the partial derivatives of this function at the origin by definition 2.01. We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{\sqrt{1+t\cdot 0} - e^{\frac{t\cdot 0}{2}}}{t(t^2+0^2)^{\frac{3}{2}}} = 0\ .$$

Likewise $\dfrac{\partial f}{\partial y}(0,0) = 0\ .$

2. According to the definition of differentiability and the results obtained at the previous solution step, equality must be fair

$$\lim_{\|x\,y\|\to\|0\,0\|} \frac{\sqrt{1+xy}-e^{\frac{xy}{2}}}{(x^2+y^2)^2} = 0\,. \tag{4.04}$$

If we come to the origin along a trajectory, then the limit value must be zero.
Let's check the fulfillment of this condition on the trajectory

$$\begin{cases} x(t)=t, \\ y(t)=t. \end{cases}$$

We will have according to the Taylor formulas

$$\lim_{t\to+0} \frac{\sqrt{1+t^2}-e^{\frac{t^2}{2}}}{4t^4} =$$

$$= \lim_{t\to+0} \frac{1+\frac{1}{2}t^2-\frac{1}{8}t^4-1-\frac{1}{2}t^2-\frac{1}{8}t^4+o(t^4)}{4\,t^4} = -\frac{1}{16} \neq 0\,.$$

This means that equality (4.04) does not hold due to the denial of the definition of the limit according to Heine.

`Solution found.`

The function under consideration is non-differentiable at the origin.

`Problem 4.3` *Examine the function for differentiability at the origin*

$$f(x,y)=\sin\left(3+x^{\frac{2}{9}}y^{\frac{6}{7}}\right).$$

`Solution.` 1. For this function it is obvious that $f(0,0)=\sin 3$,

$$x-x_0=x \qquad \text{and} \qquad y-y_0=y\,.$$

It is easy to verify that the values of the partial derivatives of this function at the origin are zero. Indeed, by definition 2.01 we have

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{\sin\left(3+t^{\frac{2}{9}}\cdot 0^{\frac{6}{7}}\right)-\sin 3}{t}=0\,.$$

It is clear that $\dfrac{\partial f}{\partial y}(0,0)=0$. Note that the use of differentiation rules here was not possible.

2. From the definition of differentiability (4.02) and the obtained estimates it follows that we need to check fulfilling equality

$$\lim_{\|x\,y\|\to\|0\,0\|}\frac{\sin\left(3+x^{\frac{2}{9}}y^{\frac{6}{7}}\right)-\sin 3}{\sqrt{x^2+y^2}}=0\,. \qquad (4.05)$$

Let us transform condition (4.05) as follows

$$\lim_{\|x\,y\|\to\|0\,0\|}\frac{2\sin\dfrac{x^{\frac{2}{9}}y^{\frac{6}{7}}}{2}\cdot\cos\dfrac{6+x^{\frac{2}{9}}y^{\frac{6}{7}}}{2}}{\sqrt{x^2+y^2}}=0$$

and note that the estimate is valid

$$\left|\frac{2\sin\dfrac{x^{\frac{2}{9}}y^{\frac{6}{7}}}{2}\cdot\cos\dfrac{6+x^{\frac{2}{9}}y^{\frac{6}{7}}}{2}}{\sqrt{x^2+y^2}}\right|\le\frac{x^{\frac{2}{9}}y^{\frac{6}{7}}}{\sqrt{x^2+y^2}}\,. \qquad (4.06)$$

3. Let us go to the right side of inequality (2.11) to the polar coordinate system, making a change of variables

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi. \end{cases}$$

Then the assessment will be fair

$$\frac{x^{\frac{2}{9}}y^{\frac{6}{7}}}{\sqrt{x^2+y^2}} = \frac{r^{\frac{2}{9}}\cos^{\frac{2}{9}}\varphi \cdot r^{\frac{6}{7}}\sin^{\frac{6}{7}}\varphi}{r} \le r^{\frac{5}{63}}\,,$$

from which (since $r \to 0$) justice follows (4.05).

Solution found.

Thus, we come to the conclusion that considered function $\sin\left(3 + x^{\frac{2}{9}}y^{\frac{6}{7}}\right)$ is differentiable at the origin.

`Problem 4.4` *Examine the function for differentiability at the origin*

$$f(x,y)=\begin{cases} 0 & at \quad x^2+y^2=0, \\ e^{\dfrac{-2}{x^2+y^2-\frac{5}{4}xy}} & at \quad x^2+y^2\neq 0. \end{cases} .$$

`Solution.` 1. For a given function $f(0,0)=0$. We have

$$x-x_0=x \qquad \text{and} \qquad y-y_0=y\,.$$

The values of the partial derivatives of this function at the origin are zero. This follows from definition 3.01. For example, according to L'Hopital's rule

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{e^{\dfrac{-2}{t^2+0^2-\frac{5}{4}t\cdot 0}}}{t}=\lim_{u\to\infty}\frac{u}{e^{2u^2}}=0\,.$$

Here we put $u=\dfrac{1}{t}$. Same as $\dfrac{\partial f}{\partial y}(0,0)=0$. Note that the use of differentiation rules in this problem impossible.

2. We need to check the equality

$$\lim_{\|x\,y\|\to\|0\,0\|} \frac{e^{\dfrac{-\,2}{x^2+y^2-\frac{5}{4}xy}}}{\sqrt{x^2+y^2}} = 0\,.$$

Let's make the following change of variables (where $r>0$)

$$\begin{cases} x = \dfrac{\cos\varphi}{r}, \\ y = \dfrac{\sin\varphi}{r}. \end{cases}$$

Then the equality being verified takes the form

$$\lim_{\|r\,\forall\varphi(r)\|\to\|\infty\,\forall\varphi\|} r\,e^{\dfrac{-\,2r^2}{1-\dfrac{5}{8}\sin 2\varphi}} = 0\,.$$

It is fair due to the assessment

$$\frac{3}{8} \le 1-\frac{5}{8}\sin 2\varphi \le \frac{13}{8}$$

Solution found.

and L'Hopital's rules. This means that the function $f(x,y)$ under consideration is differentiable at the origin.

Differentials of a function of several variables

From Definition 4.02 and Theorem 3.1 it follows that difference of values of the differentiable function $f(x,y)$ for close points $\|x\,y\|$ and $\|x_0\,y_0\|$ is primarily determined by the amount

$$\frac{\partial f}{\partial x}(x_0,y_0)\big(x-x_0\big)+\frac{\partial f}{\partial y}(x_0,y_0)\big(y-y_0\big)\,.$$

This sum depends arbitrarily on $\|x_0\,y_0\|$, but it depends linearly in $\Delta x = x - x_0$ and $\Delta y = y - y_0$.

Since the point $\|x\,y\|$ is chosen independently from $\|x_0\,y_0\|$, then Δx and Δy also *are independent* of $\|x_0\,y_0\|$. This allows us to enter new *function* of *four* variables $x_0, y_0, \Delta x$ and Δy the following type

$$df\big(x_0,y_0,\Delta x,\Delta y\big)=\frac{\partial f}{\partial x}(x_0,y_0)\,\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\,\Delta y\,.$$

Note that here *df* is not a product, but a symbol indicating this new function.

Both points $\|x\,y\|$ and $\|x_0\,y_0\|$ arbitrary, therefore in cases when they are not included in the same formula at the same time, they can be used with the same notation $\|x\,y\|$.

If we designate $\Delta f = f(x,y) - f(x_0,y_0)$ and omit the zero indices, then formula (3.06) for a differentiable function $f(x,y)$ will be written like this

$$\Delta f = df\big(x,y,\Delta x,\Delta y\big) + o\left(\sqrt{\Delta x^2+\Delta y^2}\right) .$$

Moreover, for independent variables x and y we will assume by definition that $\Delta x = dx$ and $\Delta y = dy$.

Then you can give

`Definition 4.1`	Function of four variables, arbitrarily depends on x and y, and linear in dx and dy $$df = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy\,. \qquad (4.07)$$ It is called *first differential* or, simply, *differential* of function $f(x,y)$ at point $\|x\,y\|$.

If we fix the values of dx and dy, then df can be considered as an ordinary function of two variables and we can create for it the first differential.

In other words,

$$d(df) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy\right)\delta x +$$

$$+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy\right)\delta y\ ,$$

where δx and δy are differentials of independent variables at the second differentiation.

Since we can (by definition) take $\delta x = dx$ and $\delta y = dy$, then the formula for $d(df)$ and, according to the rules of differentiation, takes the form

$$d^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\,. \qquad (4.08)$$

Here we assumed that the second mixed derivatives for the function $f(x,y)$ are continuous at the point $\|x\,y\|$. We also introduced the notation $d(df) = d^2 f$.

Let's give

Definition 4.2	Function of four variables $d^2 f$, arbitrarily dependent from x and y and being a *quadratic form* of dx and dy, called *second differential* function $f(x, y)$ at point $\|x\, y\|$.

Arguing similarly, in the conditions of existence continuous partial derivatives of the appropriate order, it is possible to define differentials of higher orders than the second. So for the function $f(x, y)$ the differential of order m has the form

$$d^m f = \sum_{k=0}^{m} C_m^k \frac{\partial^m f}{\partial x^{m-k} \partial y^k} dx^{m-k} dy^k .$$

Problem 4.5 — *Find the first and second differentials of the function* $f(x, y) = \sin\left(x^2 + \ln y\right)$ *at point* $\|\, 0\, 1\|$.

`Solution.` 1. Find the partial derivatives and their values at the point $\|\,0\;1\|$. We have

$$\frac{\partial f}{\partial x} = \cos\bigl(x^2+\ln y\bigr)\cdot 2x = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \cos\bigl(x^2+\ln y\bigr)\cdot\frac{1}{y} = 1\,.$$

For the second derivatives we find

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -\sin\bigl(x^2+\ln y\bigr)\cdot 4x^2 + 2\cos\bigl(x^2+\ln y\bigr)\cdot 2x = 2\,,\\ \frac{\partial^2 f}{\partial x\partial y} &= -2x\left(\sin\bigl(x^2+\ln y\bigr)\cdot\frac{1}{y}\right) = 0\,,\\ \frac{\partial^2 f}{\partial y^2} &= -\frac{1}{y}\bigl(\sin\bigl(x^2+\ln y\bigr)\bigr) - \cos\bigl(x^2+\ln y\bigr)\frac{1}{y^2} = -1\,. \end{aligned}$$

2. From where, according to (4.07) and (4.08), we get

`Solution found.`

$$df = dy \qquad \text{and} \qquad d^2 f = 2dx^2 - dy^2\,.$$

In some cases, for example when the function $f(x, y)$ is specified implicitly, you can search for differentials using the rules of differentiation without finding partial derivatives. This demonstrates

Problem 4.6 *Find the first and second differentials of the function $f(x, y)$ given by the equation*

$$\frac{\pi}{4} + f(x, y) - \frac{x^2}{2} - \frac{y^2}{2} - \operatorname{arctg} f(x, y) = 0 \qquad (4.09)$$

at point $\| 1\ 1 \|$, *if it is known that* $f(1, 1) = 1$.

`Solution.` 1. First you need to check that at the given point function $f(x, y)$ has the value 1. Indeed,

$$\frac{\pi}{4} + 1 - \frac{1}{2} - \frac{12}{2} - \operatorname{arctg} 1 = 0\,.$$

2. We find the first differential by formally differentiating (4.09). We have at a given point

$$df - x dx - y dy - \frac{df}{1 + f^2} = 0 \qquad (4.10)$$

or

$$df - dx - dy - \frac{df}{2} = 0 \qquad \Longrightarrow \qquad df = 2dx + 2dy\,.$$

3. We find the second differential by differentiating (4.10) and counting (according to the definition of the second differential), that dx and dy (but not df!) are constants. We get

$$d(df) - dx\cdot dx - x\cdot d(dx) - dy\cdot dy - y\cdot d(dy) -$$

$$-\frac{d(df)\cdot(1+f^2) - df\cdot 2f\,df}{(1+f^2)^2} = 0\,.$$

Since $d(dx) = 0$ and $d(dy) = 0$, then

$$d^2f - dx^2 - dy^2 - \frac{d^2f}{1+f^2} + \frac{2f(df)^2}{(1+f^2)^2} = 0\,.$$

The first differential is $df = 2dx + 2dy$. Then

$$d^2f - dx^2 - dy^2 - \frac{1}{2}d^2f + 2(dx+dy)^2 = 0\,.$$

Solution found.

It gives

$$d^2f = 2dx^2 - 8dxdy - 2dy^2$$

Taylor formula for a function of several variables

Let function $f(x,y)$ have continuous partial derivatives in the neighborhood of the point $\|x_0\,y_0\|$ up to order 3 inclusive. Then it can be shown that the formula

$$f(x,y) = f(x_0,y_0) + \sum_{m=1}^{N} \frac{1}{m!}\, d^m f + o\left(\left(\sqrt{dx^2+dy^2}\right)^N\right) .$$

(commonly called *N-th order Taylor formula*) is valid.

Particular forms of this formula are for $N=1$

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + o\left(\sqrt{dx^2+dy^2}\right) .$$

and for $N=2$

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + $$
$$+\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + o\left(dx^2+dy^2\right) . \tag{4.11}$$

They are often used in applications for local approximation of functions of several variables power functions of the first and second orders.

Typically this approximation is used for studying the properties of the function $f(x,y)$ in a small neighborhood of the point $\|x_0\,y_0\|$.

An example of using the Taylor formula is

Problem 4.7 *Represent with the Taylor formula of order $N = 2$ function*

$$f(x, y) = x^3 + y^3 - 3xy$$

in the vicinity of points $A)$ $\|0\,0\|$ *and* $B)$ $\|1\,1\|$.

Solution. 1. This problem can be solved by two methods. In the first of them we find the partial derivatives up to the second order inclusive. We have

$$\frac{\partial f}{\partial x} = 3x^2 - 3y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3y^2 - 3x\,.$$

Where do we find it from?

$$\frac{\partial^2 f}{\partial x^2} = 6x\,, \qquad \frac{\partial^2 f}{\partial x \partial y} = -3\,, \qquad \frac{\partial^2 f}{\partial y^2} = 6y\,.$$

Then, according to (4.07) and (4.08), we obtain for the case A), putting $\Delta x = x$, $\Delta y = y$ and taking into account that $f(0,0) = 0$ approximation

$$f\big(x,y\big) = -3\Delta x\Delta y + o\left(\Delta x^2 + \Delta y^2\right) \ .$$

Similarly, for case B), putting

$$\Delta x = x - 1, \qquad \Delta y = y - 1$$

and taking into account, that $f(1,1) = -1$, we find

$$f\big(x,y\big) = -1 + 6\Delta x^2 - 3\Delta x\Delta y + 6\Delta y^2 + o\left(\Delta x^2 + \Delta y^2\right) \ .$$

2. The second way to solve the problem is based on the following fact.
If a representation of a function by the Taylor formula exists, then it is the only one.

Then, when replacing the variables $\Delta x = x$ and $\Delta y = y$ and the use of equality

$$\Delta x^3 + \Delta y^3 = o\left(\Delta x^2 + \Delta y^2\right)$$

in case A) we have

$$f(x, y) = -3xy + x^3 + y^3 = -3\Delta x\Delta y + o\left(\Delta x^2 + \Delta y^2\right) \,.$$

In case B) we use a change of variables of a different type $\Delta x = x - 1$ And $\Delta y = y - 1$. As a result, this gives us $x = \Delta x + 1, \quad y = \Delta y + 1$ and correspondingly,

$$f(x, y) = \left(\Delta x + 1\right)^3 + \left(\Delta yx + 1\right)^3 - 3\left(\Delta x + 1\right)\left(\Delta y + 1\right) =$$

after opening the parentheses and bringing similar terms

$$= -1 + 6\Delta x^2 - 3\Delta x\Delta y + 6\Delta y^2 + o\left(\Delta x^2 + \Delta y^2\right) \,.$$

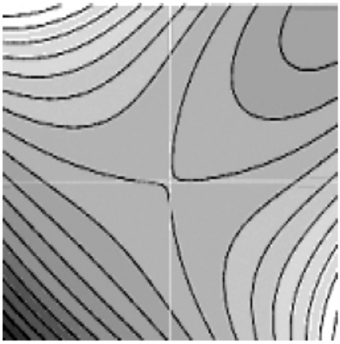

Fig. 2A

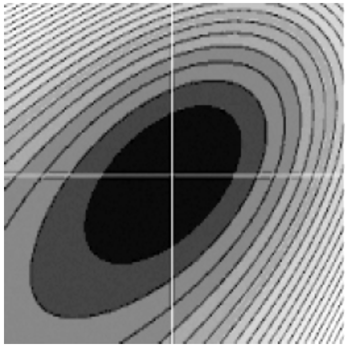

Fig. 2B

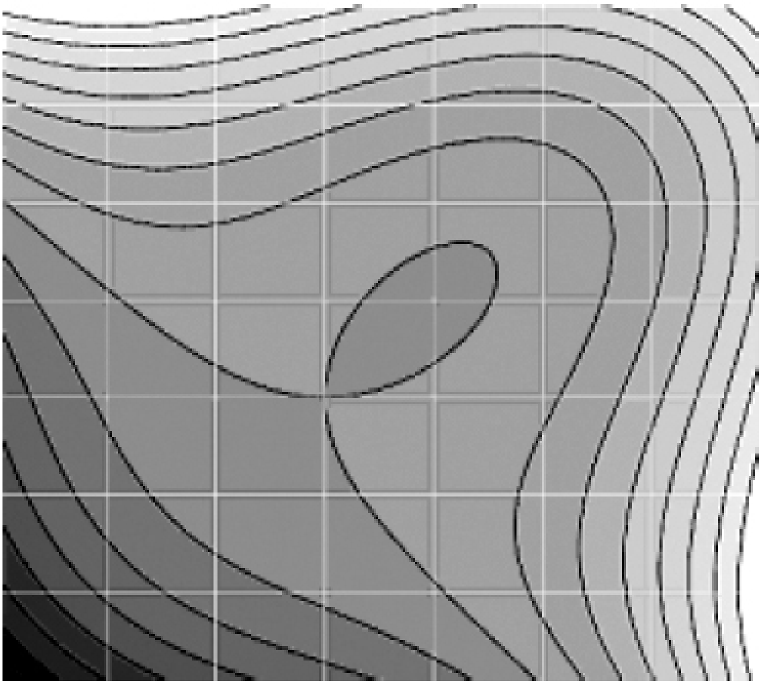

Fig.3

3. Let us now use the obtained approximations to find out nature of behavior in small neighborhoods of points A) and B).

Note that the approximation graphs in coordinates $\|\Delta x\,\Delta y\|$ (up to values of the order of smallness $o\left(\Delta x^2 + \Delta y^2\right)$) are second order surfaces.

Moreover, in case A) this surface there is a hyperbolic paraboloid, dot $\|0\,0\|$ saddle, the gradient vector in it is zero, but there is no extremum here.

In case B) the second-order surface is also a paraboloid, but elliptical. At the point $\|1\,1\|$ gradient vector is null and there is a local minimum here.

The noted properties are illustrated in Fig. 2A and 2B. In Fig. Figure 3 shows the general picture of the isolines of the function under consideration $f(x, y) = x^3 + y^3 - 3xy$.

`Solution found.`