# Definite integral

Until now, we have considered the properties of functions using their local quantitative characteristics. These characteristics relate to some small neighborhood of a fixed point, such as: value, limit, derivative, etc.

However, quite often the problem of studying the properties of functions relating to *not small* intervals of the domain of definition arises.

These include, for example, problems of finding numerical characteristics of a function, such as: the length of its graph or area of the figure bounded by the graph functions.

Methods for solving such problems are based on the use of special mathematical concept called *by a definite integral.*

Let both function $f(x)$ and segment $[a, b]$ be finite. The concept under consideration is described the following definitions.

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| Definition 5.1 | Let a segment be given $[\,a, b\,] \in \mathbb{R}$. Break it down to $N$ segments by points $x_k$, satisfying the conditions $$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b\,.$$ We call such a set of intervals *partition of the segment* $[\,a, b\,]$. The number $$\delta_\tau = \max_{k=\overline{1,N}} \left|\, x_k - x_{k-1} \,\right|$$ is called *fineness* of this partition. |

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| Definition 5.2 | For each partition we choose randomly $\forall k = \overline{1, N}$ one number $\xi_k \in [x_{k-1}, x_k]$. Expression of the form $$\sigma_\tau\Big(f, \xi_1, \xi_2, \ldots \xi_N\Big) = \sum_{k=1}^{N} f(\xi_k)\Delta x_k\,, \qquad (5.01)$$ where $\Delta x_k = x_k - x_{k-1}$, we call *Riemann integral sum* for the function $f(x)$. |

| Definition 5.3 | If for any partition $\tau$ and for any choice of numbers $\xi_k$ exists finite $$\lim_{\delta_\tau \to 0} \sigma_\tau = A\,,$$ then this limit is called *Riemannian definite integral* for the function $f(x)$ on the segment $\{\, a, b \,\}$ and is designated $$\int\limits_a^b f(x)\, dx\,.$$ |
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If the limit specified in Definition 5.3 exists, then the function $f(x)$ is called *integrable* (or integrable according to Riemann) on the segment $\{\, a, b \,\}$.

To the description of the concept of a certain integral it is worth adding the following comment.

From Definition 5.3 it follows that the definite integral can be considered as a unique mapping function $f(x)$, defined on $[a, b]$, to number $\int\limits_a^b f(x)\,dx$. Uniqueness follows from the fact that the definite integral is a limit. And the limit, if it exists, is the only one.

The unambiguous mapping of a number to some element from a set of objects in mathematics is usually called *functional.* For example, the functional associates a vector with its length, and a square matrix with its determinants.

Finally, a function of one variable can be considered as a functional on its domain of definition. Or vice versa, functional can be considered as a generalization of the concept of function.

Therefore we can consider the definite integral as *a functional*, defined on the set of functions integrable on the segment $\{\, a, b \,\}$ .

Using Definition 5.03 to justify the integrability of a function (or its non-integrability) can be quite complicated. This is primarily due to the fact that the Riemann integral sum is not unambiguous. It depends on both the method of partitioning and from the choice of points $\xi_k \in [x_{k-1}, x_k]$. Therefore, alternative integrability conditions are of interest.

To describe them, let's give

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| `Definition 5.4` | Let $$M_k = \sup_{x\in[x_{k-1},x_k]} f(x) \quad \text{and} \quad m_k = \inf_{x\in[x_{k-1},x_k]} f(x)\,.$$ Let's denote $$S_\tau = \sum_{k=1}^{N} M_k \Delta x_k \qquad \text{and} \qquad s_\tau = \sum_{k=1}^{N} m_k \Delta x_k\,.$$ $S_\tau$ and $s_\tau$ are called respectively *upper and lower Darboux sums of the function $f(x)$.* |

They will be fair

`Theorem 5.1` **In order for the bounded function $f(x)$ to be integrable on $[\,a,\,b\,]$ is necessary and sufficient so that**

$$\lim_{\delta_\tau\to 0}\bigl(S_\tau - s_\tau\bigr) = 0\,.$$

Additionally, we denote $J^* = \sup\limits_{\delta_\tau} s_\tau$ and $J_* = \inf\limits_{\delta_\tau} S_\tau$ .

`Theorem 5.2` **In order for the bounded function $f(x)$ to be integrable on $[\,a,\,b\,]$ is necessary and sufficient so that $J_* = J^*$ .**

`Corollary 5.1` **In order for the bounded function $f(x)$ to be integrable on $[\,a,\,b\,]$ is necessary and sufficient so that $\forall\,\varepsilon > 0$ such a partition was found $\tau$ , What $S_\tau - s_\tau < \varepsilon$ .**

We illustrate the use of these statements with the following examples.

**Problem 5.1** *For $x \in [0,1]$, examine the integrability of the function*

$$f(x) = \begin{cases} \operatorname{sgn} \sin \dfrac{\pi}{x}, & \text{if} \quad x \in (0,1], \\ 0\,, & \text{if} \quad x = 0. \end{cases}$$

**Solution.** 1) On the specified set the function $f(x)$ is bounded and has discontinuities of the first kind at the points $x_j = \dfrac{1}{j}$.

2) Note that on the interval $[0,1]$ the difference

$$\sup f - \inf f = 1 - (-1) = 2.$$

Let us divide the segment $[0,1]$ by a point $x^*$ so that inside the right side there is exactly $N$ break points. Then there will be an infinite number of them on the left side.

3) It is obvious that $x^* < \dfrac{1}{N}$ and contribution of the 'left' discontinuity points to the difference of the Darboux sums will not exceed $\dfrac{2}{N}$.

4) For each of the 'right' break points we choose, a segment containing it of the following type

$$x_j \in \left[ x_j - \frac{1}{2N^2}, x_j + \frac{1}{2N^2} \right] .$$

This guarantees that the contribution to the Darboux sum difference due to each 'right' point will not exceed $\dfrac{2}{N^2}$.

5) Since there are $N'right'points$, then the estimate for the difference between the upper and lower Darboux sums for the entire integration interval will be

$$S_\tau - s_\tau < \frac{2}{N} + N \cdot \frac{2}{N^2} = \frac{4}{N}.$$

The constructed partition is uniquely determined by the value $N$.

Let $N > N_\varepsilon = \left[\frac{4}{\varepsilon}\right] + 1$, where $[A]$ denotes the integer part of $A$, then

$$\frac{4}{N} < \frac{4}{N_\varepsilon} = \frac{4}{\left[\frac{4}{\varepsilon}\right] + 1} < \frac{4}{\left[\frac{4}{\varepsilon}\right]} = [\varepsilon] \le \varepsilon.$$

Thus, $\forall \varepsilon > 0 \;\; \exists N_\varepsilon = \left[\frac{4}{\varepsilon}\right] + 1$ such that $\forall N > N_\varepsilon$ : $S_\tau - s_\tau < \varepsilon$. That is, by Corollary 5.1, the function $f(x)$ integrable on the interval $[0, 1]$.

Solution received.

`Problem 5.2` *For $x \in [0,1]$, examine the integrability of the function*

$$f(x) = \begin{cases} 1\,, & \textit{if} \quad x \in \mathbb{Q}, \\ 0\,, & \textit{if} \quad x \notin \mathbb{Q}. \end{cases}$$

`Solution.` Let's construct two partitions, in the first of which $x_k \in \mathbb{Q}$, and in the second — $x_k \notin \mathbb{Q}$.

`Solution found.` Then the condition of Theorem 5.1 is not satisfied and given function (called Dirichlet function) non-integrable.

Note that if a function $f(x)$ is integrable, then function $|\,f(x)\,|$ is also integrable. The opposite is not always true. As an example, consider on segment $[0,1]$ such a function

$$f(x) = \begin{cases} 1\,, & \text{if} \quad x \in \mathbb{Q}, \\ -1\,, & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

When studying integrability may also turn out to be useful the following (sufficient) conditions

`Theorem 5.3` **If the bounded function $f(x)$ is continuous on $[\,a,\,b\,]$, then it is integrable on $\{\,a,\,b\,\}$.**

`Theorem 5.4` **If the bounded function $f(x)$ is monotonic on $[\,a,\,b\,]$, then it is integrable on $\{\,a,\,b\,\}$.**

Let us pay attention to the following here.

First, the value of the definite integral does not depend on how the variable of integration is denoted. For example, instead of $x$ you can use $t$. In other words,

$$\int\limits_a^b f(x)\,dx \qquad \text{and} \qquad \int\limits_a^b f(t)\,dt \qquad \text{are the same thing!}$$

Secondly, if we can find for $f(x)$ its antiderivative $F(x)$, then to calculate definite integral there is a more convenient method than (5.1). You can use the Newton-Leibniz formula:

$$\int\limits_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)\,.$$

In conclusion, we note that $dx$ in the integral notation should be considered on the one hand as a multiplier of the integrand, and on the other hand, as a differential independent integration variable $x$.

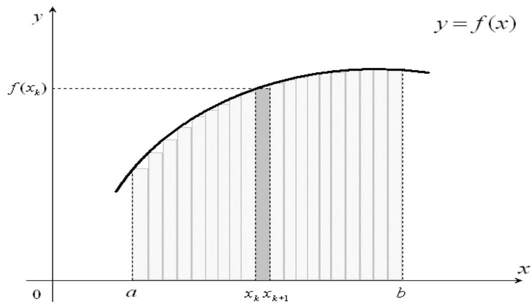


Fig. 6.1. Geometric meaning of the definite integral

## Geometric meaning of the definite integral

Let us now find out the *geometric meaning* of a definite integral of a function that has non-negative values on $[a, b]$.

In Fig. 6.1 shows the graph of the function $y = f(x)$ and partition of the segment $[a, b]$ into $N$ parts by points $x_1, x_2, \ldots x_{N-1}$. It is easy to see that in Fig. 6.1 an example of integral sum (5.01)

$$\sum_{k=0}^{N-1} f(x_k)\Delta x_k,$$

is shown.

This sum is equal to the area of a «step» figure formed from $N$ rectangles, the $k-$th of which has base length $\Delta x_k = x_{k+1} - x_k$ and height $f(x_k)$.

When passing to the limit $N \to \infty$ this area tends to $S$ — the area of the figure bounded below by the $Ox$. On the left and right the figure is bounded by vertical straight lines passing through the points $x = a$ and $x = b$. At last it is bounded above by the graph of the function $y = f(x)$. Therefore, in this case the equality will be true

$$S = \int\limits_a^b f(x)\,dx \quad . \tag{5.03}$$

The area of any geometric figure must be a *non-negative* number. Hence in the general case, one should perform the geometric interpretation of the definite integral *taking into account the sign* of the integrand $y = f(x)$.

## Properties of a definite integral

Let us now present (without proof) the main properties of the definite integral that follow from its definition.

1°. *Integrating the sum of functions*:

$$\int\limits_a^b \Big( f(x) + g(x) \Big) \, dx = \int\limits_a^b f(x) \ ; dx + \int\limits_a^b g(x) \, dx \ .$$

2°. *Integrating the product of a number and a function*:

$$\int\limits_a^b \Big(k \cdot f(x)\Big) \, dx = k \cdot \int\limits_a^b f(x) \, dx \ ; \ .$$

3°. *Additivity of integration, that is, integration of a function by combining integration segments*:

$$\int\limits_a^b f(x)\,dx + \int\limits_b^c f(x)\,dx \;=\; \int\limits_a^c f(x)\,dx \;.$$

4°. *By definition, the following equalities are also considered valid*:

$$\int\limits_a^b f(x)\,dx = -\int\limits_b^a f(x)\,dx \quad \text{and} \quad \int\limits_a^a f(x)\,dx = 0\;.$$

5°. *Integration by parts*:

$$\int\limits_a^b f(x)\,g'(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int\limits_a^b f'(x)\,g(x)\,dx \quad .$$

6°. *Replacement of integration variable*:

$$\int\limits_a^b f\Big(g(x)\Big)g'(x)\,dx = \int\limits_\alpha^\beta f(u)\,du,$$

where $\quad u = g(x) \quad$ and $\quad \alpha = g(a)\,,\ \beta = g(b)$

The properties of the definite integral may be useful for solving problems, related to inequalities.

$7^\circ$. *If* $\forall x \in [a,b]: \quad f(x) \geq g(x)$, *then*

$$\int\limits_a^b f(x)\,dx \geq \int\limits_a^b g(x)\,dx \ .$$

$8^\circ$. *For any integrable function* $f(x)$

$$\left| \int\limits_a^b f(x)\,dx \right| \leq \int\limits_a^b \left|f(x)\right|dx\,.$$

$9^\circ$. *If* $\forall x \in [a,b]: \quad m \leq f(x) \leq M$, *then*

$$m\int\limits_a^b g(x)\,dx \quad \leq \quad \int\limits_a^b f(x)\,g(x)\,dx \quad \leq \quad M\int\limits_a^b g(x)\,dx\,.$$

## Relationship between antiderivative and derivative functions

As already noted, in the case $f(x) = F'(x)$ the function $f(x)$ is called *a derivative function of the function* $F(x)$, and the function $F(x)$ is *an antiderivative function for the function* $f(x)$.

The relation $f(x) = F'(x)$ expresses the derivative through the antiderivative, and the question naturally arises of how to express the antiderivative through the derivative.

The answer is given by the following formula

$$F(x) = \int\limits_a^x f(t)\,dt\ , \tag{5.04}$$

where $a$ is any fixed number from the domain of definition of the function $f(x)$. The definite integral on the right side of equality (5.04) is called *a definite integral with a variable upper limit.*

Let us verify that equality (5.04) implies $F'(x) = f(x)$ for the case when the function $f(x)$ is continuous. Really, for any fixed point $x_0$ from the domain of definition $f(x)$ the following relations hold true:

$$F'(x_0) = \lim_{\Delta\to 0} \frac{F(x_0+\Delta) - F(x_0)}{\Delta} = \lim_{\Delta\to 0} \frac{1}{\Delta} \cdot \int\limits_{x_0}^{x_0+\Delta} f(t)\, dt =$$

$$= \lim_{\Delta\to 0} \frac{f(x_0)}{\Delta} \cdot \int\limits_{x_0}^{x_0+\Delta} 1 \cdot dt = \lim_{\Delta\to 0} f(x_0) \cdot \frac{1}{\Delta} \cdot \Delta = f(x_0) \ ,$$

by virtue of the Newton-Leibniz formula and by the properties of $2^\circ$ and $3^\circ$ for a definite integral. (In fact, we assumed that on a small segment $[x_0, x_0+\Delta]$ the function $f(x)$ has a constant value $f(x_0)$.) Thus, we come to the conclusion that each function of the form

$$F(x) = \int\limits_{a}^{x} f(t)\, dt$$

is an antiderivative for the function $f(x)$ and each antiderivative for $f(x)$ is representable in this form.

Let us now try to find a way to describe the set *of all* antiderivatives for the continuous function $f(x)$.

As we have seen, integral (5.04) is (for a fixed $a$) a function of the variable $x$, which will also be one of the antiderivatives for $f(x)$ functions. If we change the value of the lower limit of integration by putting some $b$, instead of $a$, then we get *another* antiderivative function.

Let us denote this new antiderivative as $F_1(x)$ and find the connection between $F_1(x)$ and $F(x)$. By properties of a definite integral

$$F_1(x) = \int\limits_b^x f(t)\,dt = \int\limits_b^a f(t)\,dt \ + \ \int\limits_a^x f(t)\,dt = F(x) + C \ ,$$

where $C = \int\limits_b^a f(t)\,dt$ is some constant. Consequently, all antiderivatives of the continuous function $f(x)$ can differ from each other only by an arbitrary constant.

# Curvilinear integrals

Let a smooth line $\Gamma$ be parametrically defined in $R^3$ .

$$\| \vec{r}(t) \| = \left\| \begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array} \right\| \quad t \in [t_0, t_1]\,, \qquad (5.06)$$

where $x(t), y(t), z(t)$ are continuously differentiable functions.

And let also each point in $R^3$, having a coordinate representation $\| \vec{r} \| = \left\| \begin{array}{c} x \\ y \\ z \end{array} \right\|$, the single number $f(x, y, z)$ is assigned.

Then we'll give

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| `Definition 5.5` | The number denoted as $$\int\limits_{\Gamma} f(x,y,z)\,dl$$ and equal $$\int\limits_{t_0}^{t_1} f\big(x(t),y(t),z(t)\big)\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\,dt\,,$$ is called *a curvilinear* ( or, simply, *line* ) *integral of the 1st kind* functions $f(x,y,z)$ along the line $\Gamma$. |

Let in $R^3$ (same as the previous case) a smooth line $\Gamma$ is given parametrically.

And let also each point in $R^3$, having a coordinate representation $\|\vec{r}\| = \left\| \begin{array}{c} x \\ y \\ z \end{array} \right\|$, a single vector $\vec{F}(x,y,z)$, is assigned having a coordinate representation

$$\|\vec{F}\| = \left\| \begin{array}{c} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{array} \right\| ,$$

where the functions $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ have continuous partial derivatives with respect to all their arguments.

In this case let's give

**Definition 5.6** The number denoted as

$$\int\limits_{\Gamma} P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

and equal

$$\int\limits_{t_0}^{t_1} \Big(P\big(x(t),y(t),z(t)\big)\,x'(t) +$$

$$+\,Q\big(x(t),y(t),z(t)\big)\,y'(t) + R\big(x(t),y(t),z(t)\big)\,z'(t)\Big)\,dt,$$

is called *curvilinear* ( or, simply, *line* ) *integral of the 2nd kind* of vector functions $\vec{F}(x,y,z)$ along the line $\Gamma$.

Can be useful when calculating curvilinear integrals their some specific properties.

**Property 5.1** The value of the integral of the 1st kind does not depend on which point $\vec{r}(t_0)$ and $\vec{r}(t_1)$ will be the beginning and end lines $\Gamma$.

**Property 5.2** The value of the integral of the 2nd kind changes its sign to the opposite one when changing the direction of the line bypass.

`Property 5.3` If for an integral of the 2nd kind

$$J = \int\limits_{\Gamma} P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

there is a continuously differentiable function $u(x,y,z)$ such that

$$\frac{\partial u}{\partial x} = P(x,y,z), \qquad \frac{\partial u}{\partial y} = Q(x,y,z), \qquad \frac{\partial u}{\partial z} = R(x,y,z),$$

then the value of the integral will be equal

$$J = u(x_1, y_1, z_1) - u(x_0, y_0, z_0)\,.$$

In this case, the integrand is *total differential* functions $u(x,y,z)$, a necessary condition for which will be the fulfillment of the system of equalities

$$\left\{ \begin{array}{l} \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}, \\ \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}, \\ \dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}. \end{array} \right. \tag{5.07}$$

It is easy to check that, for example, the first equality of this system in the notation introduced above has the form

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\,.$$

`Problem 5.3` *Calculate a curvilinear integral of the 1st kind*

$$J = \oint\limits_{\Gamma} (x+y)\, dl\,,$$

*where $\Gamma$ is a closed loop (as indicated by the circle on the integral symbol) — line lying on the plane $Oxy$, which is the right triangle $OAB$ with vertices at points $O(0,\,0)$, $A(1,\,0)$, $B(0,\,1)$. Contour $\Gamma$ is oriented counterclockwise, the beginning of the contour traversal — the origin of coordinates.*

`Solution.` Let's perform the parameterization in the following simple way:

$$x(t) = \begin{cases} t & \text{at} \quad t \in [0,\,1), \\ 2-t & \text{at} \quad t \in [1,\,2), \\ 0 & \text{at} \quad t \in [2,\,3) \end{cases}$$

$$y(t) = \begin{cases} 0 & \text{at} \quad t \in [0,\,1), \\ t-1 & \text{at} \quad t \in [1,\,2), \\ 3-t & \text{at} \quad t \in [2,\,3) \end{cases}$$

Function $z(t) = 0 \quad \forall t \in [0,3)$.

Due to the additivity property of the definite integral and definitions 5.5

$$J = J_{OA} + J_{AB} + J_{BO},$$

and for $OA: \quad \sqrt{x'^2(t)+y'^2(t)+z'^2(t)} = 1,$

for $AB: \quad \sqrt{x'^2(t)+y'^2(t)+z'^2(t)} = \sqrt{2},$

for $BO: \quad \sqrt{x'^2(t)+y'^2(t)+z'^2(t)} = 1\,.$

That's why

`Solution found.`

$$J = \int\limits_0^1 t \cdot 1\, dt + \int\limits_1^2 1 \cdot \sqrt{2}\, dt + \int\limits_2^3 (3-t) \cdot 1\, dt = 1 + \sqrt{2}\,.$$

**Problem 5.4** *Calculate line integral of the 2nd kind*

$$J = \oint\limits_{\Gamma} e^{-y}\,dx - \Big(xe^{-y} + 2y\Big)\,dy\,,$$

*where $\Gamma$ is the same contour and with the same orientation, as in problem 5.3.*

**Solution.** Note that, since in this problem

$$P(x,y,z) = e^{-y}\,, \quad Q(x,y,z) = -xe^{-y} - 2y\,, \quad R(x,y,z) = 0\,,$$

then all equalities in system (5.07) are satisfied.
Indeed, here it is enough to check What

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -e^{-y}\,.$$

Then $u(x,y,z) = xe^{-y} - y^2$ .

By property 5.3 we have $J = u(x_1,y_1,z_1) - u(x_0,y_0,z_0)$ , but in our case the circuit is closed and, therefore, $\begin{cases} x_0 = x_1, \\ y_0 = y_1, \\ z_0 = z_1, \end{cases}$

this means $J = 0$ .

**Solution found.**

**Problem 5.5** *Calculate a curvilinear integral of the 1st kind*

$$J = \int\limits_{\Gamma} z^2 \, dl \, ,$$

*where $\Gamma$ line of intersection of the plane $x + y + z = 0$ and spheres of radius $a$ and centered at the origin.*

**Solution.** From the course of analytical geometry it is known that in an orthonormal coordinate system, the line $\Gamma$ can be described by following system of equations

$$\begin{cases} x^2 + y^2 + z^2 & = & a^2, \\ x + y + z & = & 0. \end{cases}$$

«Naive» method of parameterization (let's say if we put $z(t) = t$ and find $x(t)$ with $y(t)$ from the system) leads to calculations that are, to put it mildly, very complex. See for yourself.

It makes sense to do things differently. Let $\left\{O\,,\ \vec{e}_1\,,\ \vec{e}_2\,,\ \vec{e}_3\right\}$ is original orthonormal basis. Let's find a new basis, in which the line $\Gamma$ (being obviously a circle) would have *canonical*, that is, the simplest form.

To do this, we take as basis vector (with number 3), for example, vector

$$\|\,\vec{e}\,'_3\| = \left\|\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right\| .$$

Let the other two unit basis vectors $\vec{e}\,'_1$ and $\vec{e}\,'_2$ lie in the plane $x+y+z=0$, leave the origin and perpendicular to each other.

To find them, consider the equation $x+y+z=0$ as a *system of linear equations*, consisting of one equation with three unknowns. Then (this is shown in the linear algebra course) the set of *all* solutions to the system can be written as

$$\left\|\begin{array}{c} x \\ y \\ z \end{array}\right\| = \lambda \left\|\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right\| + \mu \left\|\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right\| \qquad \forall \lambda,\, \mu \in \mathbb{R}\,.$$

Check for yourself that for $\lambda = 1$ and $\mu = 0$ we have a particular solution

$$\| \vec{e}\,'_1 \| = \left\| \begin{array}{r} -1 \\ 1 \\ 0 \end{array} \right\| ,$$

and for $\lambda = 1$ and $\mu = -2$ the particular solution is

$$\| \vec{e}\,'_2 \| = \left\| \begin{array}{r} 1 \\ 1 \\ -2 \end{array} \right\| .$$

These two vectors are orthogonal and therefore linearly independent. Note also that each of these vectors orthogonal to $\vec{e}\,'_3$ by construction.

After normalizing the vectors $\left\{\vec{e}\,'_1\,,\ \vec{e}\,'_2\,,\ \vec{e}\,'_3\right\}$ we obtain a new orthonormal basis $\left\{\vec{e}\,''_1\,,\ \vec{e}\,''_2\,,\ \vec{e}\,''_3\right\}$, *transition formulas* to which look like

$$\left\{\begin{array}{ccccccc} x & = & -\dfrac{1}{\sqrt{2}}x" & + & \dfrac{1}{\sqrt{6}}y" & + & \dfrac{1}{\sqrt{3}}z"\,, \\ y & = & \dfrac{1}{\sqrt{2}}x" & + & \dfrac{1}{\sqrt{6}}y" & + & \dfrac{1}{\sqrt{3}}z"\,, \\ z & = & 0\cdot x" & - & \dfrac{2}{\sqrt{6}}y" & + & \dfrac{1}{\sqrt{3}}z"\,. \end{array}\right.$$

In this basis equation of circle $\Gamma$ it will be written like this

$$\left\{\begin{array}{ccc} x"^2 + y"^2 & = & a^2, \\ z" & = & 0 \end{array}\right.$$

and the parameterization of $\Gamma$ can be done in the form

$$\left\{\begin{array}{ccc} x"(t) & = & a\cos t\,, \\ y"(t) & = & a\sin t\,, \qquad t\in[\,0,2\pi)\,. \\ z"(t) & = & 0\,. \end{array}\right.$$

With this parameterization

$$\sqrt{x''^2(t) + y''^2(t) + z''^2(t)} = a\,.$$

The integrand function will take the form

$$z^2 = \left(-\frac{2}{\sqrt{6}}y'' + \frac{1}{\sqrt{3}}z''\right)^2 = \left(\frac{2}{\sqrt{6}}\right)^2 a^2 \sin^2 t\ .$$

It leads to

Solution found.

$$J = \frac{2}{3}a^3\int\limits_0^{2\pi} \sin^2 t\,dt = \frac{2a^3}{3}\frac{1}{2}\int\limits_0^{2\pi}\Bigl(1-\cos 2t\Bigr)\,dt = \frac{2\pi a^3}{3}\,.$$