Improper integrals

So far we have considered two types of integrals: indefinite and definite. Let us now consider another class of integrals called *improper*.

Quite often in applications there is a situation where the integration segment $[a,b]$ *is unlimited*, or the integrand function $f(x)$ is *unbounded* in the left semi-neighborhood of point b. And at the same time there is a definite integral $\int\limits_a^\beta f(x)\,dx$ for *any fixed* a and β such that $a<\beta<b$.

Let either $b = +\infty$, or $b < +\infty$, but $\lim\limits_{x \to b} = \infty$. Then you can give

Definition 6.1	*An improper integral* of a function $f(x)$ on the interval $[a, b]$ is a limit of the form $$\lim_{\beta \to b-0} \int\limits_a^\beta f(x)\, dx \ ,$$ for which (to avoid coincidence with the case of a definite integral) we will use the notation $$\int\limits_a^{\to b} f(x)\, dx \ ,$$

Important: limit in Definition 6.1 is *one-sided.*

Definition 6.2	The improper integral can be defined similarly for *lower* limit of integration: $$\int\limits_{a\leftarrow}^{b} f(x)\,dx \;=\; \lim_{\alpha\to+a} \int\limits_{\alpha}^{b} f(x)\,dx \;,$$ If the limit specified in Definition 6.1 *exists* and *is finite*, then this improper integral is usually called *convergent,* otherwise, one speaks of a *non-convergent* (or *divergent*) improper integral.

Definition 6.3	Point «$\to b$» (or, accordingly, «$a \leftarrow$») are called *singular point* of the improper integral. An improper integral can have several singular points. Such an integral is considered to be convergent, if it converges *at all* singular points.

For example, the integral

$$\int\limits_{2}^{+\infty} \frac{\ln x}{x-2}\,dx$$

has two singular points: $x = 2 \leftarrow$ and $x \to +\infty$. Often they simply say that this integral has singular points: «2» and «$+\infty$» .

However, in some cases it is necessary to be more precise.

For example, the integral $\int\limits_{-1}^{1}\frac{dx}{x}$ also has *two* singular points: $x = 0 \leftarrow$ and $x = \rightarrow 0$, and they, of course, should not be designated the same way, like «0» .

Another example is the integral $\int\limits_{1}^{+\infty}\frac{\ln x}{x-1}dx$, whose integrand is not defined at point $x_0 = 1$, but is *bounded* in the right half-neighborhood of this point. Therefore, by Definition 6.3, the point $x_0 = 1$ is not singular.

For such points, you can use the term «point suspected of being singular».

When calculating the values of improper integrals, you can use «improper» analogues of rules *integration by parts* or *replacement of the integration variable.*

Say, when replacing $x = u^2$ and $dx = 2u\,du$ for integral

$$J = \int\limits_0^{\to+\infty} e^{-\sqrt{x}}\,dx = 2\int\limits_0^{\to+\infty} ue^{-u}\,du\,.$$

Further, obviously, in parts

$$J = 2ue^{-u}\Big|_0^{\to+\infty} - \left(-2\int\limits_0^{\to+\infty} e^{-u}\,du\right) = 2ue^{-u}\Big|_0^{\to+\infty} - 2e^{-u}\Big|_0^{\to+\infty} = 2\,.$$

Let's now look at more interesting examples.

`Example` 6.1. Examine the improper integral for convergence

$$\int\limits_{0}^{\to+\infty} \frac{dx}{x^2+1} \,.$$

`Solution`. We have, according to Definition 6.1,

$$\int\limits_{0}^{\to+\infty} \frac{dx}{x^2+1} = \lim_{\beta\to+\infty} \int\limits_{0}^{\beta} \frac{dx}{x^2+1} =$$

$$= \lim_{\beta\to+\infty} \left(\operatorname{arctg} x \Big|_{0}^{\beta} \right) = \lim_{\beta\to+\infty} \operatorname{arctg}\beta = \frac{\pi}{2} \,.$$

This means that this improper integral converges.

`Example` 6.2. Examine the improper integral for convergence

$$\int\limits_{0\leftarrow}^{1} \frac{dx}{\sqrt[3]{x^2}} \; .$$

`Solution`. Using again Definition 6.1, we get

$$\int\limits_{0\leftarrow}^{1} \frac{dx}{\sqrt[3]{x^2}} = \lim_{\alpha\to 0} \int\limits_{\alpha}^{1} \frac{dt}{\sqrt[3]{x^2}} =$$

$$= \lim_{\alpha\to 0} \left(3\sqrt[3]{x} \Big|_{\alpha}^{1} \right) = \lim_{\alpha\to 0} \left(3 - 3\sqrt[3]{\alpha}\right) = 3 \; .$$

And this improper integral also converges.

Note that in the examples given, the improper integrals were of two different types: «improper integrals on an unbounded integration interval» (as in example 6.1) and «improper integrals of unbounded functions», as in example 6.2.

`Example` 6.3. Find out at what values of the parameter p the improper integral converges $\int\limits_{1}^{\to+\infty} \frac{dx}{x^p}$.

`Solution.` For $p \neq 1$ we have

$$\lim_{\beta\to+\infty} \int\limits_{1}^{\beta} \frac{dx}{x^p} = \lim_{\beta\to+\infty} \left(\frac{x^{1-p}}{1-p} \Bigg|_1^\beta \right) =$$

$$= \lim_{\beta\to+\infty} \left(\frac{\beta^{1-p}}{1-p} - \frac{1}{1-p} \right) = \begin{cases} \frac{1}{p-1} , & \text{if } p > 1 , \\ \infty , & \text{if } p < 1 , \end{cases}$$

If $p = 1$, then the integral

$$\lim_{\beta\to+\infty} \int\limits_{1}^{\beta} \frac{dx}{x} = \lim_{\beta\to+\infty} \ln|x| \Bigg|_1^\beta = \lim_{\beta\to+\infty} (\ln\beta - \ln 1) =$$

$$= \lim_{\beta\to+\infty} \ln\beta = +\infty.$$

Consequently, the original improper integral converges for $p > 1$ and diverges at $p \leq 1$.

`Example` 6.4. Find out at what values of the parameter p the improper integral converges $\int\limits_{0\leftarrow}^{1} \frac{dx}{x^p}$

`Solution.` By definition 6.1 for $p \neq 1$ we have

$$\lim_{\alpha\to 0} \int\limits_{\alpha}^{1} \frac{dx}{x^p} = \lim_{\alpha\to 0} \left(\frac{x^{1-p}}{1-p} \Bigg|_{\alpha}^{1} \right) =$$

$$= \lim_{\alpha\to 0} \left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right) = \begin{cases} \frac{1}{1-p} \ , & \text{if } p < 1 \ , \\ \infty \ , & \text{if } p > 1 \ , \end{cases}$$

If $p = 1$, then the integral

$$\lim_{\alpha\to 0} \int\limits_{\alpha}^{1} \frac{dx}{x} = \lim_{\alpha\to 0} \ln|x| \Bigg|_{\alpha}^{1} = \lim_{\alpha\to 0} \Big(\ln 1 - \ln \alpha \Big) =$$

$$= \lim_{\alpha\to 0} (-\ln \alpha) = +\infty.$$

As a result, the original integral converges for $p < 1$ and diverges for $p \geq 1$.

Let us demonstrate the difference between a definite and an improper integral using the example of solving geometric and physical problems.

Example 6.5. Find the area of the figure bounded by the graph of the function $y = xe^{-x}$:

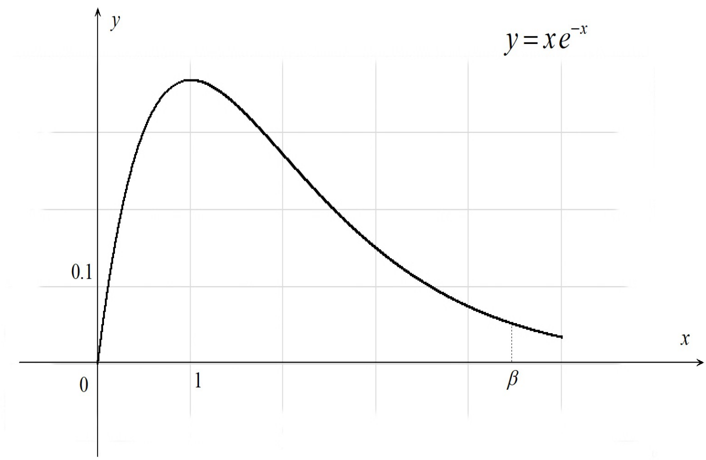

Fig. 1. Area of the figure bounded by the graph $y = xe^{-x}$.

a) at $0 \le x \le 3$,
b) for $0 \le x < +\infty$.

`Solution`. a) The function $y = xe^{-x}$ has non-negative values on the intervals under consideration (see Fig. 1.) Therefore the area under the graph for $0 \le x \le \beta$ is equal to *definite integral*

$$\int\limits_0^\beta xe^{-x}\,dx\,.$$

Note that $e^{-x} = (-e^{-x})'$. Then, using integration by parts, we get

$$\int\limits_0^\beta xe^{-x}\,dx = x\left(-e^{-x}\right)\Big|_0^\beta - \int\limits_0^\beta \left(-e^{-x}\right)\,dx =$$

$$= -xe^{-x}\Big|_0^\beta + \int\limits_0^\beta e^{-x}\,dx = \left(-xe^{-x} - e^{-x}\right)\Big|_0^\beta = 1-(\beta+1)e^{-\beta}\,.$$

This means that for $\beta = 3$ the required area is equal to

$$S = 1 - \frac{4}{e^3}\,.$$

b) for unbounded values of x, the required area is expressed by *improper* integral

$$\int\limits_0^{\to+\infty} xe^{-x}\,dx = \lim_{\beta\to+\infty}\int\limits_0^\beta xe^{-x}\,dx =$$

$$= \lim_{\beta\to+\infty}\left(1 - \frac{\beta+1}{e^\beta}\right) = 1\,.$$

`Example` 6.6. Find with what force the point mass M is attracted

a) a piece of thin thread of length L, located at a distance H from M. The mass per unit length of the thread is μ,

b) the same thread, but of unlimited length.

`Solution.` a) Let us choose a rectangular coordinate system such that the thread is located on the Ox, axis and the massive point M is on the Oy axis at point B (see Fig. 2.)

Let us consider a segment of thread of small length dx, whose middle coincides with the point A, with coordinates $\{x, 0\}$.

Since the mass of this segment is $\mu\ dx$, then, according to the law of universal gravitation, the magnitude of the force of attraction between the segment and the point M is equal to $\left|\overrightarrow{dF}\right| = \gamma \frac{M\mu\ dx}{(AB)^2}$, where γ is the gravitational constant and (by the Pythagorean theorem) $(AB)^2 = x^2 + H^2$.

The force $\overrightarrow{dF}$ is a vector quantity and should be summed using the «parallelogram rule».

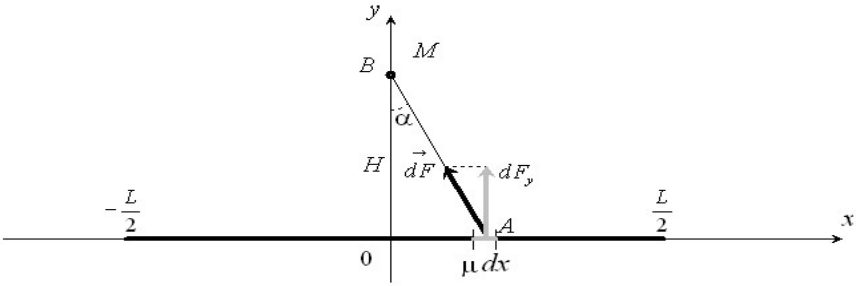

Fig. 2. What is the force of attraction between the point and the thread?

In the case under consideration, the interaction between the material point and the thread symmetrically relative to the Oy axis. The forces acting along the Ox axis cancel each other out. Therefore, only the forces dF_y parallel to the Oy. axis should be summed.

Taking this symmetry into account, the force of attraction between M and a thread of length L will be equal to the *definite* integral

$$\int\limits_{-\frac{L}{2}}^{\frac{L}{2}} \gamma \frac{M\mu\cos\alpha\, dx}{x^2+H^2} = 2\gamma M\mu H \int\limits_0^{\frac{L}{2}} \frac{dx}{(x^2+H^2)\sqrt{x^2+H^2}} \,,$$

because $dF_y = \left|\overrightarrow{dF}\right| \cos\alpha$, and $\cos\alpha = \frac{H}{\sqrt{x^2+H^2}}$.

Using tables of integrals (or changing the variable $x = \operatorname{th} u$), we obtain

$$F = 2\gamma M\mu H \int\limits_0^{\frac{L}{2}} \frac{dx}{(x^2+H^2)\sqrt{x^2+H^2}} =$$

$$= 2\gamma M\mu \left(\frac{x}{H\sqrt{x^2+H^2}} \right)\Bigg|_0^{\frac{L}{2}} = \frac{2\gamma M\mu}{H} \cdot \frac{L}{\sqrt{L^2+4H^2}} \,.$$

b) For the case of an endless thread, the total force of attraction is expressed by *improper* integral

$$F = 2\gamma M\mu H \int\limits_0^{\to +\infty} \frac{dx}{(x^2+H^2)\sqrt{x^2+H^2}} =$$

$$= \frac{2\gamma M\mu}{H} \cdot \lim_{L\to+\infty} \frac{L}{\sqrt{L^2+4H^2}} =$$

$$= \frac{2\gamma M\mu}{H} \cdot \lim_{L\to+\infty} \frac{1}{\sqrt{1+\left(\frac{2H}{L}\right)^2}} = \frac{2\gamma M\mu}{H} \ .$$

Improper integrals of functions of constant sign

In a significant number of practically important problems, it turns out that it is only necessary to establish the fact of convergence or divergence of the improper integral, *without calculating its value.*

In these cases, you can try to apply *convergence conditions for improper integrals.*

Consider continuous functions $f(x)$ and $g(x)$ defined for any x belonging to the interval from a to $\forall\beta < b$ and such that the following inequality holds: $0 \le f(x) \le g(x)$. Then

$$\int\limits_a^{\to b} f(x)\,dx \;\le\; \int\limits_a^{\to b} g(x)\,dx$$

due to the preservation of non-strict inequalities when passing to the limit (see the theorem about «two policemen»).

It can be argued that from the convergence of the improper integral $\int\limits_a^{\to b} g(x)\,dx$ it follows that the improper integral $\int\limits_a^{\to b} f(x)\,dx$ also converges. Likewise from the divergence of the integral $\int\limits_a^{\to b} f(x)\,dx$ follows the divergence of the integral $\int\limits_a^{\to b} g(x)\,dx$.

Note that the reverse is not true. If $\int\limits_a^{\to b} g(x)\,dx$ diverges, then nothing can be said about the convergence of $\int\limits_a^{\to b} f(x)\,dx$. Similarly, If $\int\limits_a^{\to b} f(x)\,dx$ converges, then nothing can be said about convergence of $\int\limits_a^{\to b} g(x)\,dx$ either.

Let us illustrate the use of these rules with the following examples.

`Example` 6.7. Not finding the value of the improper integral

$$\int\limits_1^{\to+\infty} \frac{dx}{x\sqrt{x}+\frac{1}{x^4}}\,,$$

find out whether it converges or not.

`Solution.` On the interval $[1,+\infty)$ the following inequality holds

$$0 \leq \frac{1}{x\sqrt{x}+\frac{1}{x^4}} \leq \frac{1}{x\sqrt{x}}\ .$$

Therefore, from the convergence of the integral

$$\int\limits_{1}^{\to+\infty} \frac{dx}{x\sqrt{x}}$$

(see example 6.3, for $p = \frac{3}{2} > 1$) the convergence of the integral will follow

$$\int\limits_{1}^{\to+\infty} \frac{dx}{x\sqrt{x}+\frac{1}{x^4}}\ .$$

Example 6.8. Not finding the value of the improper integral

$$\int\limits_{0\leftarrow}^{\frac{1}{2}} \frac{dx}{x\sqrt{x}-x^4}\ ,$$

find out whether it converges or not.

Solution. On the interval $\left(0, \frac{1}{2}\right]$ inequality is true

$$\frac{1}{x\sqrt{x}-x^4} \geq \frac{1}{x\sqrt{x}} \geq 0\ .$$

Therefore, from the divergence of the integral

$$\int\limits_{0\leftarrow}^{\frac{1}{2}} \frac{dx}{x\sqrt{x}}$$

(see example 6.4, for $p = \frac{3}{2} > 1$) the integral will diverge

$$\int\limits_{0\leftarrow}^{\frac{1}{2}} \frac{dx}{x\sqrt{x}-x^4}\ .$$

When studying the convergence of improper integrals it may be useful

Theorem 6.1. **If in some neighborhood of the singular point $\to b$ for nonnegative functions $f(x)$ and $g(x)$ there is a finite**

$$\lim_{x\to b}\frac{f(x)}{g(x)} = k \neq 0\,,$$

then the integrals $\int\limits_a^{\to b} f(x)\,dx$ and $\int\limits_a^{\to b} g(x)\,dx$ converge or diverge simultaneously.

In particular, the statement of this theorem will be true for equivalent functions.

`Example` 6.9. Examine the improper integral for convergence

$$\int\limits_{1}^{\to+\infty} \left(\ln \frac{x^2+3x}{x^2} \right) \sin \frac{1}{\sqrt{x}} \, dx \ .$$

`Solution.` Note that the integrand is positive in any sufficiently small neighborhood of the singular point $+\infty$.
In each such neighborhood the following equivalence relations hold:

$$\ln \frac{x^2+3x}{x^2} = \ln \left(1 + \frac{3}{x} \right) \sim \frac{3}{x} \qquad \text{and} \qquad \sin \frac{1}{\sqrt{x}} \sim \frac{1}{\sqrt{x}} .$$

Then the integrand will be equivalent to the function $\dfrac{1}{x\sqrt{x}}$,
And the integral under study will converge or diverge simultaneously with the integral

$$\int\limits_{1}^{\to+\infty} \frac{dx}{x\sqrt{x}} ,$$

which, as it was shown in Example 6.4, converges.

`Example` 6.10. Examine the improper integral for convergence

$$J = \int\limits_{2}^{\to+\infty} \frac{dx}{x^{\lambda} \ln^{\mu} x}\,.$$

`Solution`. *1) Singular point* $\to +\infty$.
If $\lambda > 1$ then we can take $\lambda = 1 + 2\varepsilon \quad \forall \varepsilon > 0$. Then $\forall \mu$ we have the estimate

$$\frac{1}{x^{1+2\varepsilon} \ln^{\mu} x} = \frac{1}{x^{1+\varepsilon}} \cdot \frac{1}{x^{\varepsilon} \ln^{\mu} x}\,,$$

where the second factor tends to zero. From the first factor the integral converges at the singular point $\to +\infty$. This means J converges.

2) Let $\lambda < 1$. Then we can put $\lambda = 1 - 2\varepsilon \ \forall \varepsilon > 0$. In this case $\forall \mu$ we have the estimate

$$\frac{1}{x^{1-2\varepsilon} \ln^{\mu} x} = \frac{1}{x^{1-\varepsilon}} \cdot \frac{x^{\varepsilon}}{\ln^{\mu} x},$$

where the second factor tends to infinity. The integral diverges from the first factor. So J diverges.

3) Finally, let $\lambda = 1$. Then, for $\mu = 1$ the divergence is obvious, and for $\mu \neq 1$ according to the Newton-Leibniz formula

$$J = \int\limits_{2}^{\to +\infty} \frac{dx}{x \ln^{\mu} x} = \int\limits_{2}^{\to +\infty} \frac{d \ln x}{\ln^{\mu} x} = \frac{1}{1-\mu} \ln^{1-\mu} x \Bigg|_{2}^{\to +\infty}$$

we find that J converges for $\mu > 1$ and diverges for $\mu < 1$.

`Example` 6.11. Examine the improper integral for convergence

$$J = \int\limits_{0\leftarrow}^{1} \frac{|\ln x \,|\, dx}{x^{\lambda}}\,.$$

`Solution`. Here the singular point is $0 \leftarrow$. Using similar reasoning to that used in solving Example 6.10, show yourself that in this case J converges for $\lambda < 1$ and diverges for $\lambda \geq 1$.

Examples 6.10 and 6.11 provide some support a well-known joke among theorist physicists about that «the logarithm inside the integral behaves like a constant».

`Example` 6.12. Examine the improper integral for convergence

$$J = \int\limits_{0\leftarrow}^{\rightarrow+\infty} \frac{\left(\sqrt{x}+x^3\right)^{\lambda} dx}{\left(x^8+2\right)\ln\left(e^x-x\right)}\,.$$

`Solution.` 1) This integral has two singular points $0 \leftarrow$ and $\rightarrow +\infty$. In the neighborhood of each singular point the integrand is of constant sign. Let's apply the comparison rule, isolating the main parts of this function.

In the neighborhood of the singular point «0» we have:

$$\ln\left(e^x-x\right) = \ln\left(1+\frac{x^2}{2}+o(x^2)\right) \sim \frac{x^2}{2},$$

$$x^8+2 \sim 2\,,$$

$$\left(\sqrt{x}+x^3\right)^{\lambda} \sim x^{\frac{\lambda}{2}}$$

That's why $J \sim \int\limits_{0\leftarrow}^{1} \frac{x^{\frac{\lambda}{2}}}{x^2}\,dx = \int\limits_{0\leftarrow}^{1} \frac{dx}{x^{2-\frac{\lambda}{2}}}\,dx\,,$ which converges for $2-\dfrac{\lambda}{2}<1 \implies \lambda>2\,.$

2) In the neighborhood of the singular point «$+\infty$», selecting the main parts gives:

$$\ln\Big(e^x - x\Big) \sim x\,,$$

$$x^8 + 2 \sim x^8\,,$$

$$\Big(\sqrt{x} + x^3\Big)^{\lambda} \sim x^{3\lambda}$$

Due to what $J \sim \int\limits_1^{+\infty} \frac{x^{3\lambda}}{x^9}\,dx = \int\limits_1^{+\infty} \frac{dx}{x^{9-3\lambda}}\,dx\,,$ which converges for $9 - 3\lambda > 1 \implies \lambda < \frac{8}{3}$.

Combining the results of 1) and 2), we get, that J converges for $2 < \lambda < \frac{8}{3}$. For other values of the parameter, the integral diverges.

`Example` 6.13. Examine the improper integral for convergence

$$J = \int\limits_{0\leftarrow}^{\rightarrow+\infty} \frac{\sqrt{x - \operatorname{th} x}\; dx}{\left(\sqrt{x + 2x^2} - \sqrt{x}\right)^{\lambda}}\,.$$

`Solution`. 1) This integral also has two singular points $0 \leftarrow$ and $\rightarrow +\infty$, in the vicinity of which the integrand function is of constant sign. Let's consider the main parts in the numerator and denominator separately.

For singular point «0»

$$\sqrt{x - \operatorname{th} x} = \sqrt{\frac{x^3}{3} + o(x^4)} \sim x\sqrt{x}\,,$$
$$\left(\sqrt{x + 2x^2} - \sqrt{x}\right)^{\lambda} \sim x^{\frac{3\lambda}{2}}\,,$$

since

$$\sqrt{x+2x^2} - \sqrt{x} =$$

$$= \frac{2x^2}{\sqrt{x+2x^2} + \sqrt{x}} = \frac{2x\sqrt{x}}{\sqrt{1+2x\sqrt{x}} + 1} \sim x\sqrt{x}\,.$$

Then $\quad J \sim \int\limits_{0\leftarrow}^{1} \frac{x^{\frac{3}{2}}}{x^{\frac{3\lambda}{2}}}\,dx = \int\limits_{0\leftarrow}^{1} \frac{dx}{x^{\frac{3\lambda}{2}-\frac{3}{2}}}\,dx\,,$ and this integral converges for $\quad \frac{3\lambda}{2} - \frac{3}{2} < 1 \quad \Longrightarrow \quad \lambda < \frac{5}{3}.$

2) For the singular point «$+\infty$», isolating the main parts, Using the same method as in 1), we find

$$\sqrt{x+2x^2} - \sqrt{x} =$$

$$= \frac{2x^2}{\sqrt{x+2x^2} + \sqrt{x}} = \frac{2x}{\sqrt{\frac{1}{x}+2} + \frac{1}{\sqrt{x}}} \sim x\,.$$

Then we get that

$$J \sim \int\limits_1^{+\infty} \frac{\sqrt{x}}{x^\lambda}\,dx = \int\limits_1^{+\infty} \frac{dx}{x^{\lambda-\frac{1}{2}}}\,dx\,,$$

which converges for $\lambda - \frac{1}{2} > 1 \quad \Longrightarrow \quad \lambda > \frac{3}{2}\,.$

Finally we get, that J converges for $\frac{3}{2} < \lambda < \frac{5}{3}$. For other values of the parameter, the integral diverges.

Improper integrals of alternating functions

In the case when the integrand in the improper integral is not constant sign, establishing the fact of convergence (or divergence) using comparison criteria or, say, Theorem 6.1, turns out to be impossible.

Here you will have to use either definitions 6.1 — 6.3, or special criteria, which we will consider further.

Let us first clarify

Definition 6.4	Improper integral $\int\limits_{\alpha}^{\to b} f(x)\,dx$ is called *absolutely convergent* if integral $\int\limits_{\alpha}^{\to b} \vert f(x) \vert\,dx$ converges. Improper integral $\int\limits_{\alpha}^{\to b} f(x)\,dx$ is called *conditionally convergent* if integral converges $\int\limits_{\alpha}^{\to b} f(x)\,dx$, but at the same time the integral $\int\limits_{\alpha}^{\to b} \vert f(x) \vert\,dx$ is divergent.

Let us now consider the criteria that can be used for improper integrals of alternating functions.

Let the function $f(x)$ be defined by $\forall x \in [\alpha, b)$, where the symbol b denotes either the number $b > \alpha$, or $+\infty$. Let also $\forall \beta < b$ there is a *definite* integral $\int\limits_{\alpha}^{\beta} f(x)\,dx$. Then it is fair

Theorem 6.2 (Cauchy criterion). **Improper integral** $\int\limits_{\alpha}^{\to b} f(x)\,dx$ **converges if and only if** $\forall \varepsilon > 0 \quad \exists \Delta_\varepsilon \in [\alpha, b)$ **such that** $\forall \delta_1, \delta_2 \in (\Delta_\varepsilon, b)$ **the inequality is satisfied**

$$\left| \int\limits_{\delta_1}^{\delta_2} f(x)\,dx \right| < \varepsilon \,.$$

To justify the divergence of improper integrals, it is convenient to use

`Theorem 6.3 (negation of the Cauchy criterion).` **Improper integral** $\int\limits_{\alpha}^{\to b} f(x)\,dx$ **diverges if and only if** $\exists \varepsilon_0 > 0$ **such that** $\forall \Delta \in [\alpha, b)$ $\exists \delta_{01}, \delta_{02} \in (\Delta, b)$ **the inequality is satisfied**

$$\left| \int\limits_{\delta_{01}}^{\delta_{02}} f(x)\,dx \right| \geq \varepsilon_0 \ .$$

A useful research tool may be called *main part selection method*,

`Theorem 6.4 (about identifying the main part).` **Let on the interval** $[a, b)$ **function** $f(x)$ **representable as** $f(x) = g(x) + R(x)$, **where the function** $R(x)$ — *is absolutely integrable*. **Then the improper integrals** $\int\limits_{\alpha}^{\to b} f(x)\,dx$ **and** $\int\limits_{\alpha}^{\to b} g(x)\,dx$

— **or,** *simultaneously* **converge absolutely,**
— **or,** *simultaneously* **converge conditionally,**
— **or,** *simultaneously* **diverge.**

In some cases, you can also use the following sufficient criteria.

`Theorem 6.5 (Dirichlet criterion).` **Let the function $f(x)$ on $[a,b)$ be continuous and have a bounded antiderivative, and the function $g(x)$ on this interval be continuously differentiable, monotone and, in addition, $\lim\limits_{x\to b} g(x) = 0$. Then an improper integral of the form $\int\limits_{\alpha}^{\to b} f(x)g(x)\,dx$ converges.**

`Theorem 6.6 (Abel's criterion).` **Let the function $f(x)$ be continuous on $[a,b)$ and the integral $\int\limits_{\alpha}^{\to b} f(x)\,dx$ converges, and the function $g(x)$ on this interval is bounded, continuously differentiable, and monotonic. Then the improper integral $\int\limits_{\alpha}^{\to b} f(x)g(x)\,dx$ converges.**

Note that in the Dirichlet and Abel criteria, three (out of five) conditions are the same: $f(x)$ is continuous on $[a, b)$, a $g(x)$ is monotone and continuously differentiable.

Different pairs of conditions can be compared in the following table:

In the criterion → ↓ **function**	*Dirichlet*	*Abel*
$f(x)$	has bounded antiderivative	integral $\int\limits_{\alpha}^{\to b} f(x)\,dx$ converges
$g(x)$	$\lim\limits_{x\to b} g(x) = 0$	bounded

`Example` 6.14. Examine for convergence for any values of the parameter λ improper integral

$$J = \int\limits_{1}^{\to +\infty} \frac{\sin x \, dx}{x^{\lambda}} \, .$$

`Solution`. 1) This integral from an alternating function and it converges absolutely for $\lambda > 1$. Really,

$$\int\limits_{1}^{\to +\infty} \left| \frac{\sin x}{x^{\lambda}} \right| dx = \int\limits_{1}^{\to +\infty} \frac{|\sin x| \, dx}{x^{\lambda}} \leq \int\limits_{1}^{\to +\infty} \frac{dx}{x^{\lambda}} ,$$

but the last integral converges (see Example 6.3). This means that by comparison the integral J converges absolutely.

2) For $\lambda \leq 0$ the integral diverges. Let us prove this by using the negation of the Cauchy criterion (Theorem 6.3).
For arbitrary $\Delta \in (1, +\infty)$ choose a natural number n such that $2\pi n > \Delta$. For integration limits $\delta_{01} = 2\pi n$ and $\delta_{02} = \dfrac{\pi}{2} + 2\pi n$ let us estimate «from below» $\forall \lambda \leq 0$ definite integral

$$\left| \int\limits_{\delta_{01}}^{\delta_{02}} \frac{\sin x}{x^{\lambda}} dx \right| \geq \left| \int\limits_{\delta_{01}}^{\delta_{02}} \sin x \, dx \right| =$$

$$= \left| - \cos x \Bigg|_{\delta_{01}}^{\delta_{02}} \right| = \left| (-0) - (-1) \right| = 1\,.$$

So there is $\varepsilon_0 = 1$ is such that for any $\Delta \in (1, +\infty)$ available $\delta_{01} = 2\pi n$ and $\delta_{02} = \dfrac{\pi}{2} + 2\pi n$, belonging interval $(\Delta, +\infty)$, for which the inequality is true

$$\left| \int\limits_{\delta_{01}}^{\delta_{02}} \frac{\sin x}{x^\lambda} dx \right| \geq \varepsilon_0 .$$

This means that the integral J according to the negation of the Cauchy criterion diverges as $\forall \lambda \leq 0$.

3) Consider on the half-interval $\lambda \in (0, 1]$ integral of an alternating function

$$J = \int\limits_{1}^{\to+\infty} \frac{\sin x}{x^{\lambda}}\, dx\,.$$

Let $f(x) = \sin x$ and $g(x) = \dfrac{1}{x^{\lambda}}$. Then on the interval $x \in [1, +\infty)$ we have that:

— $f(x)$ is continuous,

— $f(x)$ has a bounded antiderivative $-\cos x$,

— $g'(x) = -\dfrac{\lambda}{x^{\lambda+1}}$, i.e., continuous,

— $g(x)$ is monotonic, because $g'(x) < 0$

— $\lim\limits_{x\to+\infty} g(x) = 0\,.$

Then, by the Dirichlet criterion, the integral J converges.

4) For $0 < \lambda \leq 1$ integral

$$I = \int\limits_{1}^{\to +\infty} \frac{|\sin x|}{x^\lambda}\, dx\,,$$

diverges.

Let us first note that

$$|\sin x| \geq |\sin x| \cdot |\sin x| = \sin^2 x$$

and prove the divergence of the integral

$$\int\limits_{1}^{\to +\infty} \frac{\sin^2 x}{x^\lambda}\, dx\,,$$

again using the negation of the Cauchy criterion.

For arbitrary $\Delta \in (1, +\infty)$ choose a natural number n such that $\pi n > \Delta$ and for the limits of integration $\delta_{01} = \pi n$ and $\delta_{02} = 2\pi n$ let us estimate «from below» for $0 < \lambda \leq 1$ the integral

$$\left| \int\limits_{\delta_{01}}^{\delta_{02}} \frac{\sin^2 x}{x^\lambda} dx \right| \geq \int\limits_{\pi n}^{2\pi n} \frac{\sin^2 x}{x} dx \geq \frac{1}{2\pi n} \int\limits_{\pi n}^{2\pi n} \sin^2 x \, dx =$$

$$= \frac{1}{2\pi n} \int\limits_{\pi n}^{2\pi n} \frac{1 - \cos 2x}{2} dx = \frac{1}{2\pi n} \frac{\pi n}{2} = \frac{1}{4}.$$

So there is $\varepsilon_0 = \frac{1}{4}$ is like this that for any $\Delta \in (1, +\infty)$ available $\delta_{01} = \pi n$ and $\delta_{02} = 2\pi n$, belonging interval $(\Delta, +\infty)$, for which the inequality is true

$$\left| \int\limits_{\delta_{01}}^{\delta_{02}} \frac{\sin x}{x^\lambda} dx \right| \geq \varepsilon_0 \,.$$

Then the integral I diverges by the negation of the Cauchy criterion for $0 < \lambda \leq 1$.

Finally, combining results 3) and 4), according to definition 6.4 we conclude that the integral

$$J = \int\limits_{1}^{\to +\infty} \frac{\sin x}{x^\lambda} dx \,.$$

converges conditionally for $0 < \lambda \leq 1$.

`Example` 6.15. Investigate the convergence of an improper integral for $\lambda > 0$

$$J = \int\limits_{1}^{\to +\infty} \frac{\sin x \cdot \operatorname{th} x \, dx}{x^{\lambda}} .$$

`Solution.` 1) Let $f(x) = \dfrac{\sin x}{x^{\lambda}}$, $g(x) = \operatorname{th} x$ and make sure that this integral converges according to Abel's criterion.

Indeed, we have that on $[1, +\infty)$ the function $f(x)$ is continuous and its integral (as was shown in Example 6.14) converges for $\lambda > 0$.

2) Function $g(x) = \operatorname{th} x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ obviously limited by $|g(x)| < 1$, and, in addition (check it yourself!), due to $g'(x) = \dfrac{1}{\operatorname{ch}^2 x} > 0$, is monotonic.

In cases where Dirichlet's or Abel's criteria are not applicable, you can try to use the method of isolating the main part.

`Example` 6.16. Examine the improper integral for convergence

$$J = \int\limits_{1}^{\to +\infty} \operatorname{tg}\left(\frac{\sin x}{\sqrt[3]{x^2}}\right) dx\ .$$

`Solution.` 1) Note that By virtue of Maclaurin's formula, the function under the integral is written in the form

$$\operatorname{tg}\left(\frac{\sin x}{\sqrt[3]{x^2}}\right) = \frac{\sin x}{\sqrt[3]{x^2}} + R(x)\,,$$

Where $|R(x)| \leq \dfrac{1}{x^2}$ for $x > 1$.

2) Since the integral of $R(x)$ converges absolutely, then by Theorem 6.4 the integral J has the form of convergence what and

$$I = \int\limits_{1}^{\to +\infty} \frac{\sin x}{\sqrt[3]{x^2}}\, dx\ ,$$

which, as was shown in solving Example 6.14, converges conditionally.

Let's consider another example of applying Theorem 6.4.

`Example` 6.17. Examine the improper integral for convergence

$$J = \int\limits_{1}^{\to+\infty} \frac{\sin x\, dx}{\sqrt{x}+\sin x}\,.$$

`Solution`. 1) Using Maclaurin formulas, transform the integrand function in the following way

$$\frac{\sin x}{\sqrt{x}+\sin x} = \frac{\dfrac{\sin x}{\sqrt{x}}}{1+\dfrac{\sin x}{\sqrt{x}}} = \frac{\sin x}{\sqrt{x}}\left(1-\frac{\sin x}{\sqrt{x}}+Q(x)\right) =$$

$$= \frac{\sin x}{\sqrt{x}} - \frac{\sin^2 x}{x} + R(x)\,,$$

Where $|R(x)| \le \dfrac{1}{x\sqrt{x}}$ for $x > 1$.

2) Since the integral of $R(x)$ converges absolutely, then by Theorem 6.4 the integral J has the form of convergence what and

$$I = \int\limits_{1}^{\to +\infty} \left(\frac{\sin x}{\sqrt{x}} - \frac{\sin^2 x}{x} \right) dx \ ,$$

where the first term in parentheses is the integrable function, but the second one is not. Therefore, J diverges

3) Note that although the integral $\int\limits_{1}^{\to +\infty} \frac{\sin x}{\sqrt{x}} dx$ converges and the function $\sin x$ is infinitesimal compared to function $\sqrt{x}$ in a neighborhood of the singular point $\to +\infty$,

It is impossible to apply the Dirichlet criterion in this problem, since the condition of *monotonic* (see Fig. 3) tendency to zero is not valid for function $G(x)$.

$$g(x) := \frac{1}{\sqrt{x}} \qquad G(x) := \frac{1}{\sqrt{x} + \sin(x)}$$

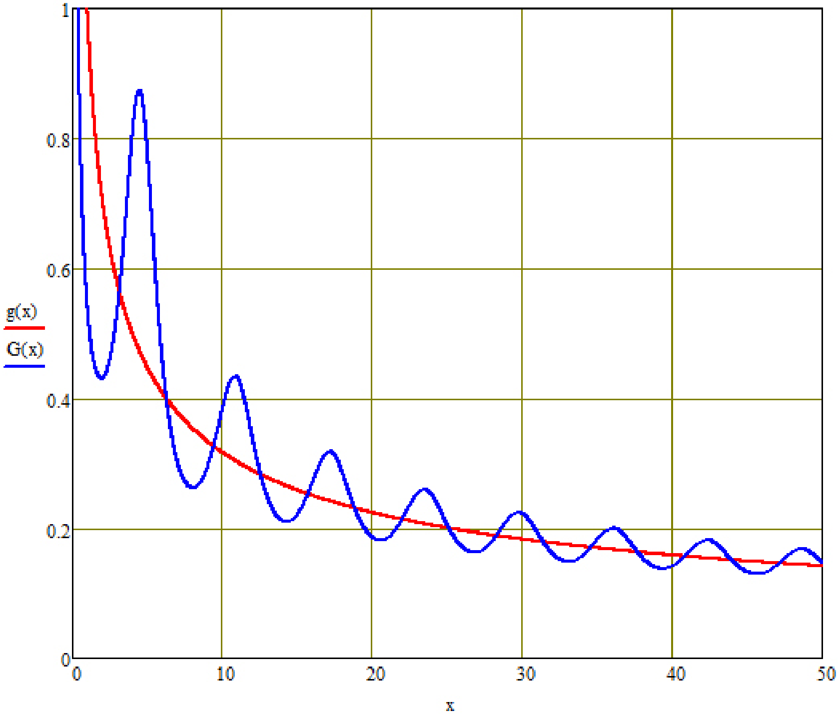

Fig. 3

`Example` 6.18. Examine the improper integral for convergence

$$J = \int\limits_{1}^{\to +\infty} \cos x^2 \, dx \; .$$

`Solution.` 1) Let's change the integration variable $u = x^2$ so that the argument of the trigonometric function becomes linear. Then $x = \sqrt{u}$ and $dx = \dfrac{du}{2\sqrt{u}}$. So,

$$J = \frac{1}{2} \int\limits_{1}^{\to +\infty} \frac{\cos u}{\sqrt{u}} \, du \, ,$$

that is, the integral converges conditionally according to the Dirichlet criterion.

Note that this example shows that for convergence improper integral of the continuous $f(x)$ in the neighborhood of the singular point $+\infty$ condition $\lim\limits_{x\to+\infty} f(x) = 0$ is not necessary. Having examined the convergence of the integral

$$J = \int\limits_{0}^{\to+\infty} x\cos x^4\,dx\,,$$

show yourself that the boundedness of the integrand is also not a necessary condition for convergence of the integral.