Number series

In some of applied problems, there is a need to use mathematical objects that are formally a sum of numbers with an unlimited number of terms.

Such objects, for historical reasons, are called *number series*. They can be described using

Definition 7.1	Let some number sequence $\{a_n\}$ be given. Then, formally written, an expression of the form $$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$ is called *a number series*. The numbers $a_1, a_2, a_3, \ldots a_n, \ldots$ are *members of a number series*

and

Definition 7.2	Sums of the first n terms of the number series $\sum\limits_{k=1}^{\infty} a_k$ $$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$ are called *partial sums* of this series.

Finally

Definition 7.3	If *exists* $\lim\limits_{n\to+\infty} S_n = A$, then the number series $\sum\limits_{k=1}^{\infty} a_k$ is called *convergent,* and the number A is called *the sum of the series.* Otherwise, the number series is called *divergent.*

Problem 7.1 *Find the sum of the series* $\sum\limits_{k=1}^{\infty} \dfrac{1}{k+k^2}$.

Solution. Note that in this case $a_k = \dfrac{1}{k+k^2} = \dfrac{1}{k} - \dfrac{1}{k+1}$. That's why

$$S_n = \sum_{k=1}^{n} \frac{1}{k+k^2} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$= 1 - \frac{1}{n+1} \quad \to 1 \qquad \text{at} \quad n \to \infty \; .$$

Solution found. It means $\sum\limits_{k=1}^{\infty} \dfrac{1}{k+k^2} = 1$.

`Problem 7.2` *Find the sum of the series* $\sum\limits_{k=0}^{\infty} \frac{1}{3^k}$.

`Solution.` In this problem

$$S_n = \sum_{k=0}^{n} \frac{1}{3^k} = \frac{1 - \dfrac{1}{3^{n+1}}}{1 - \dfrac{1}{3}} \quad \to \quad \frac{3}{2} \qquad \text{at} \quad n \to \infty \; .$$

`Solution found.` That is, $\sum\limits_{k=0}^{\infty} \frac{1}{3^k} = \frac{3}{2}$.

`Theorem 7.1` **If $\lim\limits_{k\to\infty} a_k \neq 0$, then the series $\sum\limits_{k=0}^{\infty} a_k$ diverges.**

This is a *necessary* condition for the convergence of a number series. Illustrates it

`Problem 7.3` *Examine the series for convergence* $\sum\limits_{k=1}^{\infty} \left(\dfrac{3k^3-1}{3k^3+2}\right)^{k^3}$

`Solution.` We have

$$\lim_{k\to\infty}\left(\frac{3k^3-1}{3k^3+2}\right)^{k^3} = \lim_{k\to\infty}\left(1-\frac{3}{3k^3+2}\right)^{k^3} =$$

$$= \lim_{k\to\infty}\left(1-\frac{1}{k^3+\dfrac{2}{3}}\right)^{k^3+\frac{2}{3}} \cdot \lim_{k\to\infty}\left(1-\frac{1}{k^3+\dfrac{2}{3}}\right)^{-\frac{2}{3}} = \frac{1}{e} \neq 0\,.$$

`Solution found.` Therefore, this series diverges.

Essentially Definition 7.3 is a specific case of the definition limit of a numerical sequence. However, its practical application limited by the presence of the number A in it, which is usually unknown.

Therefore, the convergence criteria who don't use A are very useful. An example of such a criterion can serve, say, Theorem 7.1. Another such criterion is

`Theorem 7.2 (Cauchy criterion)` **Number series series $\sum\limits_{k=1}^{\infty} a_k$ converges *if and only if* the sequence partial sums of this series $\{S_n\}$ is *fundamental*, i.e.**

$\forall \varepsilon > 0 \;\; \exists N_\varepsilon \in \mathbb{N}$ **such that for all natural** $n \geq N_\varepsilon$ **and** $\forall p \in \mathbb{N}$ **the inequality holds**

$$\left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \, .$$

This criterion is both *necessary* and *sufficient* condition for the convergence of a number series. Therefore, to prove the divergence of a number series can be used

`Theorem 7.3 (negation of the Cauchy criterion)` **Number series series** $\sum\limits_{k=1}^{\infty} a_k$ **diverges** ***if and only if***
$\exists \varepsilon_0 > 0$ **such that** $\forall N \in \mathbb{N}$ **there are** $n_0 \geq N$ **and** $p_0 \in \mathbb{N}$, **for which the inequality holds**

$$\left| \sum_{k=n_0+1}^{n_0+p_0} a_k \right| \geq \varepsilon_0 .$$

The use of the Cauchy criterion and its negation illustrates

Problem 7.4 *Prove that*:

1) *row* $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$ *converges* ;

2) *row* $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ *diverges.*

Solution. 1) We have an assessment

$$\sum_{k=n+1}^{n+p}\frac{1}{k^2} \le \sum_{k=n+1}^{n+p}\frac{1}{(k-1)k} = \sum_{k=n+1}^{n+p}\left(\frac{1}{k-1}-\frac{1}{k}\right) =$$

$$= \frac{1}{n}-\frac{1}{n+p} < \frac{1}{n}.$$

Whence it follows that

$$\forall\varepsilon>0 \quad \exists N_\varepsilon = \left[\frac{1}{\varepsilon}\right]+1 \quad : \qquad \forall n \ge N_\varepsilon \text{ and } \forall p\in\mathbb{N} \quad \longrightarrow$$

$$\longrightarrow \quad \left|\sum_{k=n+1}^{n+p}\frac{1}{k^2}\right| < \varepsilon\,.$$

Therefore, this number series converges.

2) Here we use a different estimate

$$\sum_{k=n_0+1}^{n_0+p_0} \left| \frac{1}{n_0+1} + \frac{1}{n_0+2} + \ldots + \frac{1}{n_0+p_0} \right| \geq \frac{p_0}{n_0+p_0} .$$

Since p_0 is any natural number, we take $p_0 = n_0$, and this gives $\quad \dfrac{p_0}{n_0+p_0} = \dfrac{1}{2}$.

As a result, we get that

$$\forall N \in \mathbb{N} \quad \exists \varepsilon_0 = \frac{1}{2} \quad : \qquad \exists n_0 = N+1 \text{ and } p_0 = n_0 \quad \longrightarrow$$

$$\longrightarrow \quad \left| \sum_{k=n_0+1}^{n_0+p_0} \frac{1}{k} \right| \geq \frac{1}{2} = \varepsilon_0 .$$

`Solution found.`

Therefore, this number series diverges.

Let's consider another example of using the Cauchy criterion

Problem 7.5 *Prove that row* $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ *converges.*

Solution. 1) We have

$$\left| \sum_{k=n+1}^{n+p} \frac{(-1)^k}{k+1} \right| =$$

$$= \left| \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+4} - \frac{1}{n+5} + \dots \right.$$

$$\left. \dots - \frac{(-1)^{n+(p-2)}}{n+(p-1)} + \frac{(-1)^{n+(p-1)}}{n+p} \right| \leq$$

(let's reduce the denominator of each positive term by 1)

$$\leq \left| \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+3} - \frac{1}{n+5} + \frac{1}{n+5} - \frac{1}{n+7} + \dots \right|$$

Note that if the last term has a positive sign, then it is destroyed with the previous one. Otherwise we simply discard it, increasing the estimate «from above».

As a result, we get that

$$\left| \sum_{k=n+1}^{n+p} \frac{(-1)^k}{k+1} \right| \leq \frac{1}{n+1} \quad \forall p \in \mathbb{N}\,.$$

Solution found.

Carrying out reasoning as in 1) problem 7.4, we come to the conclusion that this number series converges according to the Cauchy criterion.

Let us pay attention to the following property of number series. The sum with an unlimited number of terms does not, generally speaking, have the property *associativity.*

We illustrate this fact with the following examples.

Problem 7.6 *Show the inapplicability of the combining property for a sum of the form*

$$A = 1+2-3+1+2-3+1+2-3+1+2-3+1+2-3+\dots$$

Solution. 1) Let us first note that if all terms of a series are zeros, then it converges and its sum is zero. This is true because in this case the sequence of partial sums tends to zero.

2) Let's try to calculate the sum of the series under consideration, grouping the terms first as

$$A = (1{+}2{-}3){+}(1{+}2{-}3){+}(1{+}2{-}3){+}(1{+}2{-}3){+}(1{+}2{-}3){+}\dots\,.$$

In this case, we conclude that $A = 0$, since each sum in parentheses equals zero.

However, with another grouping method

$$A = 1{+}(2{-}3{+}1){+}(2{-}3{+}1){+}(2{-}3{+}1){+}(2{-}3{+}1){+}(2{-}3{+}\dots$$

Solution found.

we get $A = 1$. This means that the association rule is inapplicable for sums with an unlimited number of terms.

Problem 7.6 may not seem like a very convincing example, since the series considered in it is not convergent. Indeed, the *necessary* condition for the convergence of the series is not satisfied for it: $\lim\limits_{k\to\infty} a_k \neq 0$. Therefore, consider again the number series from Problem 7.5.

Problem 7.7 *Show that the sum of the series* $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ *depends on the way its members are grouped.*

Solution. 1) The series converges (moreover, it can be shown that that it converges to $\ln 2$). We have

$$A = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} =$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots + \frac{1}{2n-1} - \frac{1}{2n} + \ldots$$

Let us rearrange the terms of the series as follows

$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)+\ldots$$

$$=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\ldots+\left(\frac{1}{4n-2}-\frac{1}{4n}\right)+\ldots=$$

$$=\frac{1}{2}\left(1-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\ldots=$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}+\ldots\right)=\frac{A}{2}.$$

This means that changing the way terms are grouped in this case does not violate convergence, but changes the sum of the series.

Solution found.

Number series with non-negative terms

If $a_k \geq 0 \quad \forall k \in \mathbb{N}$, then for the number series $\sum\limits_{k=1}^{\infty} a_k$ there are additional possibilities for studying it for convergence. Let's list the main ones.

Theorem 7.4 **A series with non-negative terms converges if and only if when the sequence of its partial sums is limited.**

Sequence of partial sums of a number series with non-negative terms obviously monotonically increasing, since

$$\sum_{k=1}^{n} a_k = S_n \leq S_{n+1} = \sum_{k=1}^{n+1} a_k = S_n + a_{n+1} \,.$$

A monotonically increasing numerical sequence bounded from above converges by Weierstrass's theorem.

`Theorem 7.5 (1st comparison criterion)` **Let for number series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ there is a number N such that $\forall k \geq N \quad 0 \leq a_k \leq b_k$.**
In this case the series $\sum_{k=1}^{\infty} a_k$ converges by virtue of the convergence of the series $\sum_{k=1}^{\infty} b_k$. Also from divergence $\sum_{k=1}^{\infty} a_k$ follows divergence $\sum_{k=1}^{\infty} b_k$.

`Theorem 7.6 (2nd comparison criterion)` **Number series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge or diverge simultaneously if exist:**

1) **the number N is such that $\forall k \geq N : \; 0 < a_k$ and $0 \leq b_k$,**

2) **finite non-zero limit $\lim\limits_{k\to\infty} \dfrac{b_k}{a_k}$.**

To study the convergence of series with non-negative terms method *selecting the main part* can also be used.

Theorem 7.7 (integral criterion) **If the function $f(x)$ is non-negative and decreasing on $[1, +\infty)$, then the number series $\sum\limits_{k=1}^{\infty} f(k)$ and improper integral $\int\limits_{1}^{+\infty} f(x)\,dx$ converge or diverge simultaneously.**

Problem 7.8 *Examine the series for convergence* $\sum\limits_{k=1}^{\infty} \frac{1}{k^{\lambda}}$.

Solution. From the integral feature it follows that this series converges and diverges simultaneously with the improper integral

$$\int\limits_{1}^{+\infty} \frac{dx}{x^{\lambda}}.$$

Solution found. Therefore, the series under study converges for $\lambda > 1$ and diverges as $\lambda \leq 1$.

Finally, let us describe criteria Dalembert and Cauchy. They are quite important for practice.

`Theorem 7.9 (Dalembert criterion)` **If** $a_k > 0 \ \ \forall k \in \mathbb{N}$ **and exists** $\lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lambda$, **then series** $\sum\limits_{k=1}^{\infty} a_k$ **converges for** $\lambda < 1$ **and diverges for** $\lambda > 1$. **In the case** $\lambda = 1$ **the series can either converge or diverge.**

`Theorem 7.10 (Cauchy test)` **If** $a_k \geq 0 \ \ \forall k \in \mathbb{N}$ **and exists** $\lim\limits_{k\to\infty} \sqrt[k]{a_k} = \lambda$, **then series** $\sum\limits_{k=1}^{\infty} a_k$ **converges for** $\lambda < 1$ **and diverges for** $\lambda > 1$. **In the case** $\lambda = 1$ **the series can either converge or diverge.**

Problem 7.9 *Examine series for convergence*

1) $\sum\limits_{k=1}^{\infty} \frac{k!}{k^k}$,

2) $\sum\limits_{k=2}^{\infty} \frac{1}{(\ln k)^{\ln k}}$.

Solution. 1) Apply the 1st comparison criterion

$$\frac{k!}{k^k} = \frac{1\cdot 2\cdot 3\cdot 4\ldots k}{k\cdot k\cdot k\cdot k\ldots k} \le \frac{1\cdot 2\cdot k\cdot k\ldots k}{k\cdot k\cdot k\cdot k\ldots k} = \frac{1\cdot 2}{k\cdot k} = \frac{2}{k^2}$$

Using the solution to problem 7.8, we find that the series converges.

2) Let's use the equality $\ln k = k^{\log_k(\ln k)} = k^{\frac{\ln\ln k}{\ln k}}$. Then the general term of the series will take the form $\frac{1}{k^{\ln\ln k}}$. Since for sufficiently large k it is true that $\ln\ln k > 2$, we get an estimate

$$\frac{1}{(\ln k)^{\ln k}} = \frac{1}{k^{\ln\ln k}} < \frac{1}{k^2}.$$

Solution found. So, according to the 1st criterion of comparison the series converges.

`Problem 7.10` *Examine series for convergence*

$$1) \qquad \sum_{k=1}^{\infty} \frac{k^k}{\lambda^k k!}\,, \quad \lambda > 0\,,$$

$$2) \qquad \sum_{k=1}^{\infty} k^3 \left(\frac{2k+1}{3k+2} \right)^k \,.$$

`Solution.` 1) Let us apply Dalembert criterion, we have for $k \to \infty$

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1} \cdot \lambda^k \cdot k!}{\lambda^{k+1} \cdot (k+1)! \cdot k^k} = \frac{1}{\lambda} \left(1 + \frac{1}{k} \right)^k \longrightarrow \frac{e}{\lambda}.$$

we find that the series diverges for $\lambda < e$ and converges for $\lambda > e$.

2) Let us use the Cauchy test, we have for $k \to \infty$

$$\sqrt[k]{a_k} = \sqrt[k]{k^3\left(\frac{2k+1}{3k+2}\right)^k} = \sqrt[k]{k^3}\,\frac{2k+1}{3k+2} =$$

$$= \left(\sqrt[k]{k}\right)^3\,\frac{2k+1}{3k+2} \Longrightarrow \frac{2}{3} < 1\,.$$

This means that, according to Cauchy test, the series under consideration converges.

3) In conclusion let's try to use Cauchy test for the point 1).

In other words, let's try to evaluate $\lim\limits_{k\to\infty}\sqrt[k]{\dfrac{k^k}{\lambda^k k!}}$.
Here you can apply the method of selecting the main part, using Stirling's formula $k! \sim \sqrt{2\pi k}\left(\dfrac{k}{e}\right)^k$, which gives the expected

Solution found.

$$\sqrt[k]{\frac{k^k}{\lambda^k k!}} = \frac{k}{\lambda\sqrt[k]{k!}} \sim \frac{k}{\lambda}\cdot\frac{e}{k}\sqrt{\frac{1}{\sqrt[k]{2\pi k}}} \underset{k\to\infty}{\longrightarrow} \frac{e}{\lambda}\,.$$

To conclude the topic, we will give another example of using the method of selecting the main part.

Problem 7.11 *Examine the series for convergence*

$$\sum_{k=1}^{\infty}\left(1-\sqrt{\frac{k^2-1}{k^2+1}}\right)^{\lambda}.$$

Solution. 1) We apply Taylor's formulas for the binomial function twice, to select the main part of the expression in parentheses:

$$1-\sqrt{\frac{k^2-1}{k^2+1}}=1-\sqrt{\frac{1-\dfrac{1}{k^2}}{1+\dfrac{1}{k^2}}}=1-\sqrt{\left(1-\frac{1}{k^2}\right)\left(1+\frac{1}{k^2}\right)^{-1}}=$$

$$=1-\sqrt{1-\frac{2}{k^2}+o\left(\frac{1}{k^2}\right)}=1-\left(1-\frac{1}{k^2}+o\left(\frac{1}{k^2}\right)\right)\sim\frac{1}{k^2}.$$

2) According to the 2nd criterion of comparison, the series under study will be converge and diverge at the same time

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\lambda}}.$$

Solution found.

Using the solution to Problem 7.8 again, we find that the series under study converges at $2\lambda > 1 \implies \lambda > \frac{1}{2}$.

Alternating number series

Let's consider some types of number series, when the sign of a member of the number series may change when its number changes,

Absolutely convergent series

First let's give

Definition 7.4	Number series $\sum\limits_{k=1}^{\infty} a_k$ is called *absolutely convergent*, if the series converges $\sum\limits_{k=1}^{\infty} \left| a_k \right|$.

The importance of the class of absolutely convergent series is, first of all, follows from the statement

Theorem 7.11 **Let the row $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely. Then the series $\sum\limits_{k=1}^{\infty} \left(a_k + b_k \right)$ converges (or diverges) simultaneously with the series $\sum\limits_{k=1}^{\infty} b_k$.**

Let us indicate the main properties of absolutely convergent series.

Theorem 7.12 **Let the row $\sum\limits_{k=1}^{\infty} |\, a_k|$ converges and its sum is equal to S. Then the series $\sum\limits_{k=1}^{\infty} a_k$ also converges. If it has the sum σ, then the estimate $|\sigma| \leq S$.**

Theorem 7.13 **Let the row $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely and its sum is equal to S. Then the series obtained from $\sum\limits_{k=1}^{\infty} a_k$ by arbitrary rearrangement of its terms, also converges absolutely and its sum is equal to S.**

Note that rearranging terms in a number series gives *a new* series because a different number sequence generates a new series. However, under the conditions of Theorem 7.13, *the sum of the series* does not change.

Theorem 7.14 **If the rows $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ converge absolutely, then for any λ and μ a series of the form**

$$\sum_{k=1}^{\infty} \Big(\lambda a_k + \mu b_k\Big)$$

also converges absolutely.

Alternating series

Definition 7.5	*Alternating* is called number series of the form $\sum\limits_{k=1}^{\infty}(-1)^{k-1}a_k$, in which $a_k \geq 0 \quad \forall k \in \mathbb{N}$.

Let us describe some properties of alternating series.

Theorem 7.15 (Leibniz test)

If $\lim\limits_{k\to\infty} a_k = 0$ **And** $\forall k \in \mathbb{N} \;:\; a_k \geq a_{k+1} \geq 0$, **then a series** $\sum\limits_{k=1}^{\infty}(-1)^{k-1}a_k$ **converges.**

Theorem 7.16

If all the conditions of Theorem 7.17 are satisfied and $\sum\limits_{k=1}^{\infty}(-1)^{k-1}a_k = S$, **A** $\sum\limits_{k=1}^{n}(-1)^{k-1}a_k = S_n$, **then the assessment is fair** $\left| S - S_n \right| \leq a_{n+1}$.

`Problem 7.12` *Examine the series for convergence* $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\lambda}}$.

`Solution.` In this example $a_k = \frac{1}{k^{\lambda}}$. This number sequence with positive terms and it will be infinitesimal, and monotonically decreasing, for $\lambda > 0$.

`Solution found.` Then, according to Leibniz's criterion, the series under consideration will converge at $\lambda > 0$.

Monotonicity condition $a_k \geq a_{k+1}$ in Leibniz's criterion is essential, which illustrates

Problem 7.13 *Examine the series for convergence* $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k} - (-1)^k}$.

Solution. Let's use the method of selecting the main part

$$\frac{(-1)^k}{\sqrt[3]{k} - (-1)^k} = \frac{\frac{(-1)^k}{\sqrt[3]{k}}}{1 - \frac{(-1)^k}{\sqrt[3]{k}}} =$$

$$= \frac{(-1)^k}{\sqrt[3]{k}} \left(1 + \frac{(-1)^k}{\sqrt[3]{k}} + \frac{1}{\sqrt[3]{k^2}} + o\left(\frac{1}{\sqrt[3]{k^2}} \right) \right) =$$

$$= \frac{(-1)^k}{\sqrt[3]{k}} + \frac{1}{\sqrt[3]{k^2}} + \frac{(-1)^k}{k\sqrt[3]{k}} + o\left(\frac{1}{\sqrt[3]{k^2}} \right) .$$

Then the series under study will take the form

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[3]{k}}+\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k^2}}+\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{k\sqrt[3]{k}}+o\left(\frac{1}{\sqrt[3]{k^2}}\right)\right] .$$

A series with a common term in square brackets converges absolutely. Therefore (by virtue of Theorem 7.11) the conditions for the convergence of the series under study will be the same as for the sum of the series

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[3]{k}}+\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k^2}} ,$$

the first of which converges according to the Leibniz criterion, and the second diverges (see solution to Problem 7.8).

This means that the series under study diverges, although the denominator of its common term is equivalent to $\sqrt[3]{k}$ at $k\to\infty$.

Solution found.

To study the convergence of series of a more complex structure you can try to use sufficient Dirichlet and Abel tests.

Theorem 7.17 (Dirichlet test) **If the partial sums of the series $\sum\limits_{k=1}^{\infty} a_k$ are bounded, and the sequence $\{b_k\}$ monotonic and infinitesimal, then a series $\sum\limits_{k=1}^{\infty} a_k b_k$ converges.**

Theorem 7.18 (Abel's test) **If the sequence $\{b_k\}$ monotonous and limited and a series $\sum\limits_{k=1}^{\infty} a_k$ converges, then the series $\sum\limits_{k=1}^{\infty} a_k b_k$ converges.**

For a more accurate description of the conditions for the convergence of number series may also be useful

Definition 7.6	Number series $\sum\limits_{k=1}^{\infty} a_k$ is called *conditionally convergent* if it converges, but at the same time the series $\sum\limits_{k=1}^{\infty} \lvert a_k \rvert$ diverges.

Problem 7.14 *Examine the series for convergence*

$$\sum_{k=1}^{\infty} \frac{\cos\dfrac{\pi k}{4}}{(k+2)\sqrt{\ln^3(k+3)}} .$$

Solution. We use estimate

$$\left| \frac{\cos\dfrac{\pi k}{4}}{(k+2)\sqrt{\ln^3(k+3)}} \right| \leq \frac{1}{k\ln^{3/2}k} .$$

Consequently, the series under study converges absolutely on the basis of comparison and integral criteria, since the improper integral of a function of constant sign converges $\int\limits_{1}^{+\infty} \frac{dx}{x\,\ln^{3/2} x}$.

Solution found.

Problem 7.15 *For $\forall\lambda \in \mathbb{R}$ examine the series for convergence*

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^{\lambda}} .$$

Solution. 1) For $\lambda \leq 0$ the series diverges, since

$$\lim_{k\to\infty} \frac{\sin k}{k^{\lambda}} \neq 0 ,$$

that is, the necessary condition for the convergence of the number series is not satisfied.

2) For $\lambda > 1$ the series converges according to the comparison criterion, because the assessment is fair

$$\left| \frac{\sin k}{k^{\lambda}} \right| \leq \frac{1}{k^{\lambda}} ,$$

3) For $0 < \lambda \leq 1$ the series diverges non-absolutely according to the Dirichlet criterion.

Indeed, let $a_k = \sin k$ and $b_k = \dfrac{1}{k^\lambda}$. In this case sequence $\{b_k\}$ is infinitesimal and monotonic, and a sequence of partial sums of the form (according to formulas known from trigonometry)

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \sin k = \frac{\sin\dfrac{n}{2}\sin\dfrac{n+1}{2}}{\sin\dfrac{1}{2}} \leq \frac{1}{\sin\dfrac{1}{2}}$$

limited.

4) For $0 < \lambda \leq 1$ the series does not converge at all by comparison due to estimates

$$\left|\frac{\sin k}{k^\lambda}\right| \geq \frac{\sin^2 k}{k^\lambda} = \frac{1}{2k^\lambda} + \frac{\cos 2k}{2k^\lambda},$$

since the series $\sum\limits_{k=1}^{n} \dfrac{\cos 2k}{2k^\lambda}$ converges conditionally (see above, item 3)), and a row $\sum\limits_{k=1}^{n} \dfrac{1}{2k^\lambda}$ diverges.

Solution found.

In conclusion, let's look at examples of using the main part selection method, Leibniz's test and Abel's test.

Problem 7.16 *Examine the series for convergence*

$$\sum_{k=1}^{\infty}(-1)^k\left(1-\cos\frac{\pi}{\sqrt{k}}\right).$$

Solution. Applying Taylor's formula, we get

$$(-1)^k\left(1-\cos\frac{\pi}{\sqrt{k}}\right)=(-1)^k\frac{\pi^2}{k}-\left[\frac{1}{2}\frac{\pi^4}{k^2}+o\left(\frac{1}{k^3}\right)\right].$$

A series with a common term in square brackets converges absolutely, while the series $\sum\limits_{k=1}^{\infty}(-1)^k\frac{\pi^2}{k}$ converges conditionally according to Leibniz's criterion. This means that the original series also converges, but not absolutely.

Solution found.

Problem 7.17 *Examine the series for convergence*

$$\sum_{k=1}^{\infty} \frac{\cos k \cdot \operatorname{arctg} k}{\ln(k+1)} .$$

Solution. Let's denote $a_k = \dfrac{\cos k}{\ln(k+1)}$ and $b_k = \operatorname{arctg} k$. Wherein row $\sum\limits_{k=1}^{\infty} a_k$ converges according to the Dirichlet criterion, and the sequence $\{b_k\}$ — is monotonic and limited.

Solution found. Then the series under study converges according to Abel's criterion.

`Problem 7.18` *Examine the series for convergence*

$$\sum_{k=1}^{\infty}\left(\sqrt{k^2+\frac{k}{2}}\cdot\ln\left(1+\frac{1}{k}\right)\right)^{k^2}.$$

`Solution.` Let's try to apply the Cauchy test for a series of constant sign. That is, we need to evaluate $\lim\limits_{k\to\infty}\sqrt[k]{a_k}$. We have

$$\sqrt[k]{a_k}=\left[\sqrt{k^2+\frac{k}{2}}\cdot\ln\left(1+\frac{1}{k}\right)\right]^k=$$

$$=\left[k\sqrt{1+\frac{1}{2k}}\cdot\ln\left(1+\frac{1}{k}\right)\right]^k=$$

$$=\left[k\left(1+\frac{1}{4k}+o\left(\frac{1}{k}\right)\right)\cdot\left(\frac{1}{k}-\frac{1}{2k^2}+o\left(\frac{1}{k^2}\right)\right)\right]^k=$$

$$=\left[1-\frac{1}{4k}+o\left(\frac{1}{k}\right)\right]^k\to\quad e^{-\frac{1}{4}}<1\quad\text{at}\quad k\to\infty\,.$$

This means that the series under study converges according to the Cauchy test.

`Solution found.` Pay attention to what was required when selecting the main part, number of terms in the expansion of a logarithm.