Functional series

We are considering methods of approximation some functions others, which are simpler or more convenient. Here you can use as an approximating object *series*, each member of which is some known function.

It should be noted that many tasks the series turns out to be not only an approximation, but the only possible form presentation of solutions.

First let's give

Definition 9.1	Let each member of a certain series be a function $u_k(x)$, where $x \in X \subset \mathbb{R}$. Then we will say that *functional series* $$\sum_{k=1}^{\infty} u_k(x) \qquad (9.1)$$ is given with domain X.

Types of convergence of a functional series

The concepts of series sum, partial sum, as well as the concepts of pointwise and absolute convergence functional series, we introduce just as these concepts were defined for number series.

For example, the function $S_n(x) = \sum\limits_{k=1}^{n} u_k(x)$ we will call the n-th *partial sum of the series* (9.1),

Definition 9.2	The function $F(x)$ is called *limit function* for functional series $\sum\limits_{k=1}^{\infty} u_k(x)$, if $\forall x_0 \in X$ occurs $$\lim_{n\to\infty} S_n(x_0) = F(x_0)\,.$$

In some cases we will also use sum of a series $r_n(x) = \sum\limits_{k=n+1}^{\infty} u_k(x)$, called the n-th *remainder of the series* (9.1).

Definition 9.3	We will also say that if $$\lim_{n\to\infty} S_n(x_0) = F(x_0) \quad \forall x_0 \in X \,, \qquad (9.2)$$ then the functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ *converges pointwise* to $F(x)$ on the set X .

Fact of pointwise convergence of a functional series on the set X can be denoted as follows: $S_n(x) \underset{X}{\rightarrow} F(x)$ at $n \to \infty$. The absence of pointed convergence on this set is denoted by accordingly, as $S_n(x) \underset{X}{\not\rightarrow} F(x)$ at $n \to \infty$.

In quantifier form, condition (9.2) is formulated as follows:

$\forall x_0 \in X$ and $\forall \varepsilon > 0 \quad \exists N_{x_0,\,\varepsilon}$, such as $\forall n \geq N_{x_0,\,\varepsilon}$ inequality

$$\left| \, S_n(x_0) - F(x_0) \, \right| < \varepsilon \tag{9.3}$$

is true.

Note also that the functional series is called *absolutely convergent*, if series $\sum\limits_{k=1}^{n} \left| \, u_k(x) \, \right|$ converges pointwise.

As for functional sequences, pointwise convergence of a functional series does not guarantee the coincidence of function properties $u_k(x)$ and $F(x)$, which illustrates

Problem 9.1 *Find the limit function for functional series*

$$\sum_{k=1}^{\infty} \frac{x^2}{\big(1+(k-1)x^2\big)\big(1+kx^2\big)} \quad \textit{on set } x \in \mathbb{R}\,.$$

`Solution.` Note that the common member of this functional series decomposes into simple fractions

$$u_k(x) = \frac{1}{1+(k-1)x^2} - \frac{1}{1+kx^2}\,.$$

Whence it follows that

$$S_n(x) = \left(1 - \frac{1}{1+x^2}\right) + \left(\frac{1}{1+x^2} - \frac{1}{1+2x^2}\right) + \ldots$$
$$\ldots + \left(\frac{1}{1+(k-1)x^2} - \frac{1}{1+kx^2}\right)\,,$$

and we get

$$S_n(x) = 1 - \frac{1}{1+nx^2} = \frac{nx^2}{1+nx^2}\,.$$

It is easy to see that in this case

`Solution found.`

$$F(x) = \lim_{k\to\infty} S_n(x) = \lim_{k\to\infty} \frac{nx^2}{1+nx^2} = \operatorname{sgn}^2 x = \left|\operatorname{sgn} x\right| .$$

Comparison of the properties of the functions $u_k(x)$ and $F(x)$ (see Fig. 1) in Problem 9.1 leads us to a conclusion similar to that obtained for functional sequences. Namely:

> for pointwise convergence, the properties of the terms functional range may do not coincide with the properties of the limit function.

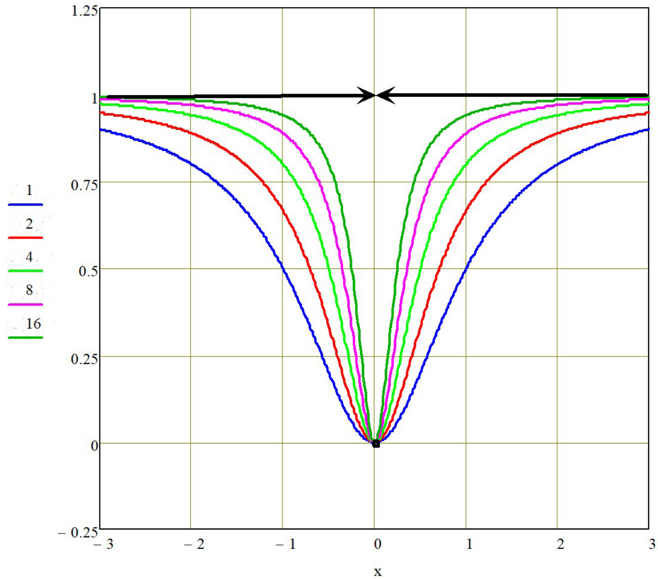

Fig.1. Graphs of functions $S_n(x)$ for $n = 1, 2, 4, 8, 16$ to Problem 9.1.

Uniform convergence of a functional series

In the definitions made, the convergence of a functional series is identified with convergence functional sequence consisting of partial sums of the series.

Therefore it is natural to expect that conditions ensuring the coincidence of properties of members functional series and its limit function, can be formulated using the concept uniform convergence.

By making appropriate changes to the Definition 9.3 (pointwise convergence on the set X function sequence $\{S_n(x)\}$) we get

Definition 9.4	Let's say that the functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ *converges uniformly on the set* X to the function $F(x)$, if $\forall \varepsilon > 0 \quad \exists N_\varepsilon \ : \ \forall x \in X$ and $\forall n \geq N_\varepsilon$, such as inequality $\Big| S_n(x) - F(x) \Big| < \varepsilon$ is true.

This property, as for functional sequences, we will denote by the symbol $\sum\limits_{k=1}^{\infty} u_k(x) \underset{X}{\rightrightarrows} F(x)$ at $n \to \infty$.

Let us emphasize once again that the difference between definitions 9.4 and 9.3 is that in the case of uniform convergence of the functional series, number N_ε is found (selected) according to the same rule *for all* points of the set of arguments X . While for pointwise convergence the choice of $N_{x_0,\varepsilon}$ can be done for each x individually.

As in the case of functional sequence from the uniform convergence of the functional series should be pointwise, but not vice versa.

Accordingly, properties of absence of pointwise or uniform convergence We will denote the functional series by symbols $\sum\limits_{k=1}^{\infty} u_k(x) \underset{X}{\not\to} F(x)$ and $\sum\limits_{k=1}^{\infty} u_k(x)) \underset{X}{\not\rightrightarrows} F(x)$.

Then the property of non-uniform convergence of a functional series can be described by

Definition 9.5	We will say that the series $\sum\limits_{k=1}^{\infty} u_k(x)$ *converges not uniformly on the set* X , if at $n \to \infty$ $$S_n(x) \underset{X}{\to} F(x)\,, \quad \text{but} \quad S_n(x) \underset{X}{\not\rightrightarrows} F(x)\,.$$

Let us now formulate conditions for the coincidence of properties of members of a functional series and its limit function.

Theorem 9.1 **If all terms of functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ are continuous functions on $[a,b]$, and the series converges uniformly on $[a,b]$ to the limit function $F(x)$, then $F(x)$ is continuous on $[a,b]$.**

Theorem 9.2 **If all terms of functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ are continuous functions on $[a,b]$, and the series converges uniformly on $[a,b]$ to the limit function $F(x)$, then the functional series $\sum\limits_{k=1}^{\infty} \int\limits_{x_0}^{x} u_k(u)\,du$ converges uniformly to the function $\int\limits_{x_0}^{x} F(u)\,du$. where $x_0 \in [a,b]$.**

Theorem 9.3 **If all terms of functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ are continuously differentiable functions on $[a,b]$,**
— the series itself converges at some point on $[a,b]$, and
— the series of $u'_k(x)$ converges uniformly on $[a,b]$, then

$$F'(x) = \sum_{k=1}^{\infty} u'_k(x)\,,$$

where $F(x)$ is continuously differentiable limit function of original functional series. Moreover, the series $\sum\limits_{k=1}^{\infty} u_k(x)$ also converges uniformly.

Conditions for uniform convergence of functional series

Theorems 9.1 — 9.3 are based on the concept of uniform convergence for functional series. They give *sufficient* conditions, at which the limit function will have the same properties as the terms of the series.

Therefore, the conditions for uniform convergence of functional series are of practical interest.

Let us formulate one of the criteria for the n-th remainder of the functional series $\sum\limits_{k=1}^{\infty} u_k(x)$

$$r_n(x) = \sum_{k=n+1}^{\infty} u_k(x) = F(x) - S_n(x)\,.$$

Theorem 9.4 **In order for the functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ defined on the set X converged uniformly on this set, necessary and sufficient to**

$$\sup_{x\in X} \bigl| r_n(x) \bigr| \to 0 \qquad \textbf{at } n \to \infty\,.$$

`Corollary 9.1` **For uniform convergence of a functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ defined on the set X, it is necessary that $u_k(x) \underset{X}{\rightrightarrows} 0$.**

`Theorem 9.5 (Weyerstrass sign)` **If for the functional series $\sum\limits_{k=1}^{\infty} u_k(x)$ defined on the set X, there is a** *convergent numerical* **series $\sum\limits_{k=1}^{\infty} a_k$ such that $\forall k \geq k_0$ and $\forall x \in X$ the inequalities**

$$\left| u_k(x) \right| \leq a_k$$

are true,
then the original functional series converges uniformly on the set X.

`Theorem 9.6 (Cauchy criterion)` **For** *uniform convergence* **of a function series** $\sum\limits_{k=1}^{\infty} u_k(x)$ **defined on the set** X **, necessary and sufficient,**

to

$\forall \varepsilon > 0 \quad \exists\, N_\varepsilon$ **:**

$\forall k \geq N_\varepsilon\,, \quad \forall p \in \mathbb{N}, \quad \forall x \in X,$

such that inequality $\left| \sum\limits_{k=n+1}^{n+p} u_k(x) \right| < \varepsilon$ **is true.**

`Theorem 9.7 (negation of the Cauchy criterion)` **In order for the functional series** $\sum\limits_{k=1}^{\infty} u_k(x)$ **, defined on the set** X **,** *did not converge uniformly***, necessary and sufficient,**

so that for any $N \in \mathbb{N}$ **found**

$\varepsilon_0 > 0\,, \quad k_0 \geq N\,, \quad p_0 \in \mathbb{N}$ **and** $x_0 \in X$

such that the inequality holds

$$\left| \sum_{k=n_0+1}^{n_0+p_0} u_k(x_0) \right| \geq \varepsilon_0\,.$$

It is advisable to use the Cauchy criterion (as well as its negation) in cases where the limit function $F(x)$ is unknown or cannot be represented in a form convenient for use.

`Theorem 9.8 (Dirichlet test)` **Functional series** $\sum\limits_{k=1}^{\infty} a_k(x)b_k(x)$, **defined on the set** X, *converges uniformly* **on this set, if**

1) **sequence of partial sums of a series** $\sum\limits_{k=1}^{\infty} a_k(x)$ **is limited to** X,
2) **functional sequence** $\{\, b_k(x)\,\}$ **is monotonic** $\forall x \in X$ **and** $b_k(x) \underset{X}{\rightrightarrows} 0$.

`Theorem 9.9 (Abel test)` **Functional series** $\sum\limits_{k=1}^{\infty} a_k(x)b_k(x)$, **defined on the set** X, *converges uniformly* **on this set, if**

1) **functional series** $\sum\limits_{k=1}^{\infty} a_k(x)$ **converges uniformly on** X,
2) **functional sequence** $\{\, b_k(x)\,\}$ **is monotonic** $\forall x \in X$ **and it is bounded on the set** X.

Examples of studying functional series for convergence

`Problem 9.2` *Investigate the uniform convergence of a functional series*

$$\sum_{k=1}^{\infty} \frac{\operatorname{arctg}(kx)}{x^4 + k\sqrt[3]{k}} \qquad x \in (-\infty, +\infty) \; .$$

`Solution.` 1) For any fixed x this series converges pointwise by comparison criteria (check it!).

2) Due to inequalities $\left|\operatorname{arctg} x\right| \leq \dfrac{\pi}{2}$ and $x^4 + k\sqrt[3]{k} \geq k\sqrt[3]{k}$, which are true when $\forall x \in \mathbb{R}$ and $\forall k \in \mathbb{N}$, for the general term of the series the following estimate is valid:

$$\left| \frac{\operatorname{arctg}(kx)}{x^4 + k\sqrt[3]{k}} \right| \leq \frac{\pi}{2} \frac{1}{k\sqrt[3]{k}} ,$$

Since the majorizing number series $\dfrac{\pi}{2} \sum\limits_{k=1}^{\infty} \dfrac{1}{k\sqrt[3]{k}}$ converges, then by Weierstrass's sign (Theorem 9.5) the functional series under study will converge uniformly.

`Solution found.`

Problem 9.3 *Investigate the uniform convergence of a functional series*

$$\sum_{k=1}^{\infty} 3^k \sin\frac{x}{4^k} \qquad x \in [\,0, +\infty)\ .$$

Solution. 1) The series under study converges pointwise, since $\sin\dfrac{x}{4^k} \sim \dfrac{x}{4^k}$ and the inequality is true

$$\left| 3^k \sin\frac{x}{4^k} \right| \le x \left(\frac{3}{4}\right)^k \ .$$

2) For the n_0-th remainder of a given functional series at some point $x_0 \in (0, +\infty)$ we have

$$r_{n_0}(x_0) = \sum_{k=n_0+1}^{\infty} 3^k \sin\frac{x_0}{4^k} \ge 3^{n_0+1} \sin\frac{x_0}{4^{n_0+1}} \ge 3 \sin\frac{x_0}{4^{n_0+1}}\,.$$

3) Then for any $N \in \mathbb{N}$ on $(0, +\infty)$ exist

$$n_0 = N, \quad x_0 = 4^{n_0+1}, \quad \varepsilon_0 = 3\sin 1$$

such that $\quad r_{n_0}(x_0) \geq 3\sin 1 = \varepsilon_0\,.$

It means that

$$\sup_{x \in X} \big| \, r_n(x) \, \big| \not\to 0 \qquad \text{at } n \to \infty$$

Solution found. and by virtue of Theorem 9.4, the series under study converges nonuniformly.

`Problem 9.4` *Investigate the uniform convergence of a functional series*

$$\sum_{k=1}^{\infty} \frac{x}{1+k^3x^3}$$

on sets: 1) $E_1 : \ x \in [\,0, 1\,]$,

2) $E_2 : \ x \in [\,1, +\infty)$.

`Solution.` 1) The series under study converges pointwise on E_1 and E_2, due to the comparison criterion and the convergence of the number series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^3}$.

2) Derivative of the common term of the series

$$u'_k(x) = \frac{1 - 2k^3x^3}{(1 + k^3x^3)^2} < 0 \qquad \forall x \in E_2 \,.$$

Therefore, the continuous $u_k(x)$ has a maximum on E_2 at the point $x = 1$ and the inequality will be true

$$\frac{x}{1 + k^3x^3} \leq \frac{1}{1 + k^3} \qquad \forall x \in E_2 \,.$$

From the convergence of the majorizing series $\sum\limits_{k=1}^{\infty} \frac{1}{1 + k^3}$. and the Weierstrass sign we obtain uniform convergence series under study on E_2 .

3) Study on uniform convergence at $x \in E_1$. Let's do it using the negation of the Cauchy criterion. For any natural number N we can take

$$k_0 = N \geq N\,, \quad p_0 = N \in \mathbb{N} \quad \text{and} \quad x_0 = \frac{1}{N} \in E_1\,,$$

such that the inequality holds

$$\left| \sum_{k=n_0+1}^{n_0+p_0} \frac{x_0}{1+k^3x_0^3} \right| = \sum_{k=N+1}^{2N} \frac{\dfrac{1}{N}}{1+\dfrac{k^3}{N^3}} \geq$$

(here we'll make all the terms the same, replacing each of them with the smallest term)

$$\geq \sum_{k=N+1}^{2N} \frac{1}{N} \frac{1}{1+\dfrac{(2N)^3}{N^3}} = N\frac{1}{N}\frac{1}{1+8} = \frac{1}{9} = \varepsilon_0\,,$$

that is,

$$\left| \sum_{k=n_0+1}^{n_0+p_0} \frac{x_0}{1+k^3x_0^3} \right| \geq \varepsilon_0\,.$$

The last inequality means that on E_1 by virtue of 1) and Theorem 9.7, the functional series under study converges unevenly.

Solution found.

Problem 9.5 *Investigate the uniform convergence of a functional series*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+x^2}} \operatorname{arctg} x^k$$

on the set $X : \; x \in [\,1, +\infty)\,.$

Solution. 1) Let us first consider the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\sqrt{k+x^2}}$. Let the sequences $a_k(x) = (-1)^k$ and $b_k(x) = \dfrac{1}{\sqrt{k+x^2}}$.

It is easy to see that the sequence of partial sums of the series $\sum\limits_{k=1}^{\infty} (-1)^k$ is bounded, and a sequence with a common term $\{b_k(x)\}$ is monotonic in k.
It converges uniformly on $X : \; x \in [\,1, +\infty)\,.$ to the zero function due to the inequality

$$\frac{1}{\sqrt{k+x^2}} \le \frac{1}{\sqrt{k}} \to 0 \qquad \text{at } \; k \to \infty\,.$$

Then the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\sqrt{k+x^2}}$. converges on X uniformly according to the Dirichlet criterion (Theorem 9.8).

2) On the other hand, the sequence $\operatorname{arctg} x^k$ is monotone in k and is limited to X.

Solution found. Therefore, the original functional series will be converge uniformly on X according to Abel's criterion (Theorem 9.9).

`Problem 9.6` *Is it true that*

A) *row* $\sum\limits_{k=1}^{\infty} \operatorname{arctg} \dfrac{x}{k^2}$ *can be differentiated term by term by* $\mathbb{R}$?

B) *This row converges uniformly on* $\mathbb{R}$?

`Solution.` 1) The series under study obviously converges at the point $x = 0$.

2) Consider a series whose common term is the derivative from the common member of the series under study. We have

$$u'_k(x) = \left(\operatorname{arctg} \frac{x}{k^2}\right)' = \frac{k^2}{k^4 + x^2} \leq \frac{1}{k^2} \quad \forall x \in (-\infty, +\infty) .$$

Whence it follows that a series composed of derivatives will converge uniformly on any segment $[-C, C]$ $C \in (0, +\infty)$ according to the Weierstrass sign.

Then, by Theorem 9.3, the original functional series it is possible to differentiate $\forall C \in (0, +\infty)$. *On the segment* $[-C, C]$ this series will converge uniformly to differentiable function (due to the arbitrariness of $C \in (0, +\infty)$) on the entire real axis.

Therefore, the answer to question A) is yes.

3) Consider question B).

Functional sequence $\{u_k(x)\}$ converges pointwise to the function, identically equal to zero on the entire real axis.

On the other hand, we have

$$\sup_{x\in X} \Big| u_k(x) \Big| \not\to 0 \quad \text{at } k\to\infty\,.$$

Indeed, for the functional sequence $\left\{ \operatorname{arctg} \dfrac{x}{k^2} \right\}$ can always be found a pair of numbers $k_0 \in \mathbb{N}$ and $x_0 \in \mathbb{R}$ such that $x_0 = k_0^2$. This gives

$$u_{k_0}(x_0) = \operatorname{arctg} 1 = \frac{\pi}{4} > 0\,.$$

That is, it is not fulfilled Corollary 9.1 — a necessary condition uniform convergence of the functional series $\sum\limits_{k=1}^{\infty} u_k(x)$.

`Solution found.` Hence the answer to question B) is negative.

Problem 9.7 *Calculate*

$$I = \int\limits_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{\sin^2 kx}{k(k+1)} \right) dx\,.$$

Solution. 1) The terms of the integrand are continuous on $[0, 2\pi]$ functions. In this case $\forall x \in [0, 2\pi]$ inequality is true

$$\left| \frac{\sin^2 kx}{k(k+1)} \right| \le \frac{1}{k^2}\,.$$

Then the integrand converges on $[0, 2\pi]$ uniformly and can be integrated term by term.

2) Because

$$\int\limits_0^{2\pi} \sin^2 kx\,dx = \frac{1}{2}\int\limits_0^{2\pi} (1 - \cos 2kx)\;dx = \pi\,,$$

then (check it out for yourself!)

Solution found.

$$I = \pi \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \pi \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \pi\,.$$

Problem 9.8 *Represent function* $\operatorname{arctg} x$ *in the form of a functional series and evaluate the segment, at which it converges.*

Solution. 1) The function $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$ is representable on $x \in (-1, 1)$ as the sum of an infinitely decreasing geometric progression

$$\frac{1}{1+x^2} = \sum_{k=1}^{\infty} (-1)^{k-1} x^{2(k-1)} . \qquad (9.4)$$

2) The functional series on the right side of equality (9.4) is majorized by a number series of the form $\sum\limits_{k=1}^{\infty} (-1)^{k-1} r^{2(k-1)}$. This series is convergent $\forall r \in (-1, 1)$. Then the functional series (9.4) will converge uniformly on the segment $[-r, r]$ and equality (9.4) can be integrated term by term.

3) As a result, we come to equality

$$\operatorname{arctg} x + C = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{2k-1} .$$

Integration constant $C = 0$, since $\operatorname{arctg} 0 = 0$.

So the function $\operatorname{arctg} x$ is appeared as functional series

$$\operatorname{arctg} x = \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k+1}}{2k+1} .$$

`Solution found.`

This series converges uniformly on $[-r, r] \ \forall r \in [0, 1)$.