DIFFERENTIATION OF COMPOSITE FUNCTIONS

Let the function be given $F(x_1, x_2, \dots , x_n)$ and n-dimensional vector function $\vec{x}(t)$, Where $t \in [a,b] \subset R.$

We will assume that all the functions under consideration are continuously differentiable. Then *full* derivative function of *complex* (compositions) functions of one variable $\Phi(t) = F(x_1(t), x_2(t), \dots , x_n(t), t)$ looks like

$$\frac{d\Phi}{dt} = \frac{\partial F}{\partial t} + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \frac{dx_j}{dt} . \tag{1}$$

Let there be more than one independent variables, then

$$\Phi(\vec{t}) = F[x_1(t_1, t_2, \dots , t_k), x_2(t_1, t_2, \dots , t_k), \dots , x_n(t_1, t_2, \dots , t_k), t_1, t_2, \dots , t_k]$$

And the formula generalizing (1) is valid. So as not to confuse this derivative with *private* derived from a non-composition function is usually written like this

$$\Phi'_{t_p} = \frac{\partial F}{\partial t_p} + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \frac{\partial x_j}{\partial t_p} \qquad \forall p \in [1,k]. \tag{2}$$

Formulas (1) and (2) are applicable in the simplest case, when the formula representation of functions $F(\boxtimes\ )$ And $\vec{x}(\boxtimes\ )$ known. However, it often turns out that these functions are given *implicitly*: let's say, as solutions to some other problems.

Let it be required to find all partial derivatives for the functions $u(x,y)$ And $v(x,y)$, whose values for a specific pair of arguments x And *and* are found from the system of equations

$$\begin{cases} F(u,v,x,y)=0, \\ G(u,v,x,y)=0 \end{cases} \text{ or, more precisely, } \begin{cases} F(u(x,y),v(x,y),x,y)=0, \\ G(u(x,y),v(x,y),x,y)=0. \end{cases} \qquad (3)$$

Applying formula (2), i.e. differentiated sequentially by x And *and* each of the equations of system (3), we obtain two systems of equations of the form:

$$\begin{cases} \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=-\frac{\partial F}{\partial x}, \\ \frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=-\frac{\partial G}{\partial x} \end{cases} \quad \text{And} \quad \begin{cases} \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=-\frac{\partial F}{\partial y}, \\ \frac{\partial G}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial y}=-\frac{\partial G}{\partial y}. \end{cases} \qquad (4)$$

Let us assume that for fixed x And *and* quantities $u, v, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}, \frac{\partial G}{\partial u}, \frac{\partial G}{\partial v}, \frac{\partial F}{\partial x}$ And $\frac{\partial G}{\partial y}$ exist.

Then systems of equations (4) will be linear with respect to $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ And $\frac{\partial v}{\partial y}$.

Note also that, despite the linearity and similarity of the form of the main matrices in systems (4), the greatest difficulty is not their solution, but their composition, which requires a preliminary solution of the nonlinear system (3).

Regarding *properties* functions specified implicitly, we recall that the condition *local* their existence and differentiability is given by the theorem *about the system of implicitly specified functions*. Let us give its formulation.

Let the functions $F_i(u_1,\ldots,u_m,x_1,\ldots,x_n) \quad \forall i=[1,m]$

continuously differentiable *in some neighborhood of the point* $\{u^*,x^*\}$.
Then if

1) $F_i(u^*,x^*)=0 \quad \forall i=[1,m]$,

2) *at the point* $\{u^*,x^*\}$ *Jacobian* $\dfrac{\partial(F_1,\ldots,F_m)}{\partial(u_1,\ldots,u_m)} \neq 0$,

then there are neighborhoods of points u^* *And* x^**, in which* ***will exist*** *continuously differentiable functions* $u(x)$,
which in these neighborhoods are solutions to the system of equations

$$F_i(u(x),x)=0 \quad \forall i=[1,m].$$

Recall that the Jacobian

$$\frac{\partial(F_1,\ldots,F_m)}{\partial(u_1,\ldots,u_m)} = \det\begin{Vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \ldots & \frac{\partial F_1}{\partial u_m} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \ldots & \frac{\partial F_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial u_1} & \frac{\partial F_m}{\partial u_2} & \ldots & \frac{\partial F_m}{\partial u_m} \end{Vmatrix}.$$

Let us illustrate the use of formulas (1) ... (4) with the following examples.

Example 1. Find partial derivatives of functions $u(x,y)$ And $v(x,y)$, specified implicitly by the system of equations

$$\begin{cases} xe^{u+v} + 2uv = 1, \\ ye^{u-v} - \dfrac{u}{v+1} = 2x, \end{cases}$$

at the point $\{1;2\}$ for which the equalities are satisfied

$$u(1,2) = v(1,2) = 0. \tag{5}$$

Solution: In this problem for system (3) we have

$$\begin{cases} F(u,v,x,y) = xe^{u+v} + 2uv - 1, \\ G(u,v,x,y) = ye^{u-v} - \dfrac{u}{v+1} - 2x. \end{cases}$$

There is no need to solve system (3) here, because it is easy to check that the equalities are true $F(0,0,1,2) = 0$ And $G(0,0,1,2) = 0.$

Therefore at the point $\{1;2\}$ by virtue of (5) we have:

$$\frac{\partial F}{\partial u} = xe^{u+v} + 2v = 1e^0 + 0 = 1,$$
$$\frac{\partial F}{\partial v} = xe^{u+v} + 2u = 1e^0 + 0 = 1,$$
$$\frac{\partial F}{\partial x} = e^{u+v} = 1e^0 = 1, \quad \frac{\partial F}{\partial y} = 0,$$

$$\frac{\partial G}{\partial u} = ye^{u-v} - \frac{1}{v+1} = 2e^0 - \frac{1}{0+1} = 1,$$

$$\frac{\partial G}{\partial v} = -ye^{u-v} + \frac{u}{(v+1)^2} = -2e^0 + \frac{0}{(0+1)^2} = -2,$$

$$\frac{\partial G}{\partial x} = -2, \quad \frac{\partial G}{\partial y} = e^{u-v} = e^0 = 1.$$

And systems (4) will be $\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} = -1 , \\ \dfrac{\partial u}{\partial x} - 2\dfrac{\partial v}{\partial x} = 2 \end{cases}$ And $\begin{cases} \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} = 0 , \\ \dfrac{\partial u}{\partial y} - 2\dfrac{\partial v}{\partial y} = -1 . \end{cases}$

From them, we get $\dfrac{\partial u}{\partial x} = 0, \ \dfrac{\partial v}{\partial x} = -1, \ \dfrac{\partial u}{\partial y} = -\dfrac{1}{3}, \ \dfrac{\partial v}{\partial y} = \dfrac{1}{3},$ which is the solution to the problem.

REPLACING VARIABLES

Example 1. Find $dz(x,y),$ If $z = u^3 + v^3 \quad u \neq v,$ and functions $u(x,y)$ And $v(x,y)$ are determined by a system of equations $\begin{cases} u + v = x, \\ u^2 + v^2 = y. \end{cases}$. (6)

Solution: 1) The required differential has the form $dz(x,y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$ therefore, the problem comes down to finding two partial derivatives, which can be represented here as:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{и} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

2) Derivatives $\dfrac{\partial z}{\partial u} = 3u^2$ и $\dfrac{\partial z}{\partial v} = 3v^2$ are found trivially, and to calculate the remaining four we will have to use systems (3) and (4).
Nonlinear system (3) of the form

$$\begin{cases} F(u,v,x,y) = u + v - x = 0, \\ G(u,v,x,y) = u^2 + v^2 - y = 0\ , \end{cases}$$

can be easily solved using “school” methods, but here it is more convenient to create two linear systems (4). Check for yourself that they have the following form and solutions

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} = 1\ , \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{v}{v-u} \\ \dfrac{\partial v}{\partial x} = \dfrac{u}{u-v} \end{array}$$

And

$$\begin{cases} \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} = 0\ , \\ u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y} = \dfrac{1}{2} \end{cases} \Rightarrow \begin{array}{l} \dfrac{\partial u}{\partial y} = \dfrac{1}{2(u-v)} \\ \dfrac{\partial v}{\partial y} = \dfrac{1}{2(u-v)} \end{array} .$$

3) Thus, we find

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=3u^2\frac{v}{v-u}+3v^2\frac{u}{u-v}=-3uv\ ,$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=3u^2\frac{1}{2(u-v)}+3v^2\frac{1}{2(v-u)}=\frac{3}{2}(u+v).$$

4) Note that from (6) it follows that $u+v=x$ And $uv=\frac{x^2-y}{2}$, which allows the desired derivatives to be written as functions of x And *and*

$$\frac{\partial z}{\partial x}=-\frac{3}{2}(x^2-y)\quad \text{и}\quad \frac{\partial z}{\partial y}=\frac{3}{2}x\ .$$

Therefore we finally get that $dz(x,y)=\frac{3}{2}(y-x^2)\,dx+\frac{3}{2}x\,dy$.

5) A condition should be added to the received answer that guarantees the solvability of system (6).

This condition can be obtained in different ways: from the geometric interpretation of system (6) in a rectangular coordinate system Ouv or from the following relations:

$$0 \le (u-v)^2 = u^2 + v^2 - 2uv = y - 2\frac{x^2 - y}{2} = 2y - x^2,$$

what with $u \ne v$ gives $y > \frac{x^2}{2}$.

A natural question arises: is it possible *due to choice* special coordinate system (not necessarily Cartesian) *simplify* type of differential equation?

Example 2. Find all functions $u(x,y)$, satisfying in the interior of the right Cartesian coordinate half-plane an equation of the form

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1 \quad . \qquad (7)$$

Solution: 1) Let us transform equation (7) by moving to a polar coordinate system, that is, by making a change of variables $\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi \end{cases}$, the inverse of which

$$\begin{cases} r = \sqrt{x^2+y^2}, \\ \varphi = \operatorname{arctg}\dfrac{y}{x} \end{cases} .$$

Note that these formulas are correct, since by condition $x > 0$.

2) We have formulas expressing Cartesian variables through polar ones (and vice versa). We need to obtain formulas in polar coordinates for $\frac{\partial u}{\partial x}$ и $\frac{\partial u}{\partial y}$.
According to the rule (2) of differentiation of a complex function, we have

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x} \quad \text{And} \quad \frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial y}.$$

Let's compose systems (4) with $F(r,\varphi,x,y)=r\cos\varphi-x$ And $G(r,\varphi,x,y)=r\sin\varphi-y$ for derivatives with respect to x And *and* and solve them:

$$\begin{cases}\cos\varphi\cdot\frac{\partial r}{\partial x}+r(-\sin\varphi)\frac{\partial \varphi}{\partial x}=1, \\ \sin\varphi\frac{\partial r}{\partial x}+r\cos\varphi\frac{\partial \varphi}{\partial x}=0\end{cases} \Rightarrow \quad \begin{array}{l}\frac{\partial r}{\partial x}=\cos\varphi \\ \frac{\partial \varphi}{\partial x}=-\frac{1}{r}\sin\varphi\end{array}$$

$$\begin{cases}\cos\varphi\cdot\frac{\partial r}{\partial y}+r(-\sin\varphi)\frac{\partial \varphi}{\partial y}=0, \\ \sin\varphi\frac{\partial r}{\partial y}+r\cos\varphi\frac{\partial \varphi}{\partial y}=1\end{cases} \Rightarrow \quad \begin{array}{l}\frac{\partial r}{\partial y}=\sin\varphi \\ \frac{\partial \varphi}{\partial y}=\frac{1}{r}\cos\varphi\end{array}.$$

3) We substitute the found expressions in polar coordinates into the left side of equation (7), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=r\cos\varphi\left[\frac{\partial u}{\partial r}\cos\varphi+\frac{\partial u}{\partial\varphi}\left(-\frac{\sin\varphi}{r}\right)\right]+r\sin\varphi\left[\frac{\partial u}{\partial r}\sin\varphi+\frac{\partial u}{\partial\varphi}\frac{\cos\varphi}{r}\right]=$$

$$=r(\cos^2\varphi+\sin^2\varphi)\frac{\partial u}{\partial r}+(-\sin\varphi\cos\varphi+\sin\varphi\cos\varphi)\frac{\partial u}{\partial\varphi}=r\frac{\partial u}{\partial r}.$$

4) As a result, the original equation, taking into account the condition $x>0$, will be equivalent to the equation $\frac{\partial u}{\partial r}=\frac{1}{r}$, which is simpler than the original one.

Solving this equation means that we must find all continuously differentiable functions $u(r,\varphi),$ partial derivative of which by r equal $\frac{1}{r}$.

Based on the known rules of differentiation, we can conclude that

$$u(r,\varphi) = \ln r + C(\varphi), \quad \text{Where} \quad C(\varphi),$$

There is *free* continuously differentiable function *one* the argument φ .

5) We obtain the solution to the original problem by going back to the variables: $u(x,y) = \ln\sqrt{x^2+y^2} + C\left(\operatorname{arctg}\frac{y}{x}\right)$, and, given the condition $x > 0$ and the continuity of the superposition of continuous functions, this formula can be written as:

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2) + D\left(\frac{y}{x}\right),$$

Where $D(s)$ an arbitrary continuously differentiable function of one argument.