RESEARCH OF FUNCTIONS OF MANY VARIABLES AT EXTREMUM

First, let's consider the case of searching for a local minimum of a function *two* variables, since already for $n=2$ all significant differences from a one-dimensional problem can be easily demonstrated.

Let us be given a twice continuously differentiable in some domain $\Omega \subseteq E^2$ with ONB function $f(x,y)$. Let us find out under what conditions this function has at the point $\begin{Vmatrix} x^* \\ y^* \end{Vmatrix} \in \Omega$ local minimum. Let's give

Definition 1. We will say that the function $f(x,y)$ has at the point $\begin{Vmatrix} x^* \\ y^* \end{Vmatrix}$ *strict local minimum*, if exists $\overset{\circ}{U}_\varepsilon$ - punctured surroundings, such that for *any* points $\begin{Vmatrix} x \\ y \end{Vmatrix} \in \overset{\circ}{U}_\varepsilon$ the inequality is satisfied

$$f(x,y) > f(x^*,y^*) \,. \tag{1}$$

It is clear that to check the fulfillment of the condition of Definition 1 for all points in the neighborhood $\overset{\circ}{U}_{\varepsilon}$ impossible. Therefore, it is necessary to obtain conditions for the existence of an extremum, the verification of which is practically feasible.

In some cases, it is possible to convert the function record $f(x,y)$ to a form in which the fulfillment of inequality (1) is obvious or easily verified. For example, the function $f(x,y)=x^4-x^2y+y^2$ can be written like this:

$$f(x,y)=x^4-x^2y+\frac{y^2}{4}+\frac{3y^2}{4}=\left(x^2-\frac{y}{2}\right)^2+\left(\frac{y\sqrt{3}}{2}\right)^2 .$$

Whence it follows that the point $\left\|\begin{matrix} x^* \\ y^* \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \end{matrix}\right\|$ there is a local minimum point. However, such a situation is the exception, not the rule.

A more convenient way to use Definition 1 is to estimate the sign of the difference $f(x,y)-f(x^*,y^*)$ for neighborhood points $\overset{\circ}{U}_\varepsilon$ with help *Taylor formulas.*

Taylor's formula for the function $f(x,y)$ at the point $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$, as is known, can be written in the form

$$f(x,y)=f(x^*,y^*)+d\,f+\frac{1}{2}d^2f+o(\rho^2)\,, \tag{2}$$

Where $\rho=\sqrt{dx^2+dy^2}$, $dx=x-x^*$ And $dy=y-y^*$, and the differentials $d\,f$ And d^2f respectively equal

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\,, \qquad d^2f=\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial y^2}dy^2.$$

Note that in the last two formulas the partial derivatives are calculated at a known point $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$.

From (2) we obtain that

$$f(x,y)-f(x^*,y^*)=d\,f+\frac{1}{2}d^2 f+o(\rho^2)\,, \tag{3}$$

Here we note that, according to Taylor's theorem (on the remainder term in Peano form), the first term on the right-hand side of (3) due to the twice continuous differentiability of the function $f(x,y)$, has the order of smallness $\rho=\sqrt{dx^2+dy^2}$. The second term on the right side is the order of smallness $\rho^2=dx^2+dy^2$.

This means that in a sufficiently small neighborhood of the point $\left\|\begin{matrix} x^* \\ y^* \end{matrix}\right\|$ increment sign $f(x,y)-f(x^*,y^*)$ determined by the sign of the quantity $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$, which may be *any* due to linear dependence df from dx And dy.

In other words, if grad $f(x,y) \neq o$ and for some vector $\begin{Vmatrix} dx \\ dy \end{Vmatrix}$ we have $f(x,y) - f(x^*, y^*) > 0,$ then for the vector $\left(- \begin{Vmatrix} dx \\ dy \end{Vmatrix} \right)$ we will definitely have that $f(x,y) - f(x^*, y^*) < 0,$ since derivatives $\frac{\partial f}{\partial x}$ And $\frac{\partial f}{\partial y}$ from values dx And dy they don't depend.

This means that a continuously differentiable function has $f(x,y)$ at points where grad $f(x,y) \neq o$, there cannot be an extremum (i.e. minimum or maximum). Note that the points at which grad $f(x,y) \neq o$, usually called *stationary points* for function $f(x,y)$.

As a result, we come to the following *necessary* condition for the existence of an extremum:

> **If the function** $f(x,y)$ **has an extremum at the point** $\begin{Vmatrix} x^* \\ y^* \end{Vmatrix}$ **, That**
>
> **or** degree $f(x^*, y^*) = o$ **,**
> **or** *there is no gradient at this point***.**

This necessary condition is not sufficient. Example: $f(x,y) = xy$. Here at the origin the gradient is a zero vector, but there is no extremum.

Let us now consider only the points at which grad $f(x,y)=o$. In this case, the difference sign $f(x,y)-f(x^*,y^*)$ will coincide with the sign of the second term on the right side of (3), i.e.

$$d^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2. \qquad (4)$$

In formula (4), the derivative values are calculated at a fixed point $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$ and do not depend on the values dx And dy. For simplicity of notation, we will denote these values as follows: $A=\frac{\partial^2 f}{\partial x^2}$, $B=\frac{\partial^2 f}{\partial x \partial y}$, $C=\frac{\partial^2 f}{\partial y^2}$. Recall that the matrix $\left\| \begin{matrix} A & B \\ B & C \end{matrix} \right\|$ called *Hessian matrix*.

Then we can say that the sign of the difference $f(x,y)-f(x^*,y^*)$ coincides with the sign of the quadratic form

$$Adx^2 + 2Bdxdy + Cdy^2. \qquad (5)$$

If the quadratic form $Adx^2 + 2Bdxdy + Cdy^2$. *positive definite*, then at the point $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$ function $f(x,y)$ will have *strict local minimum*, and if this form *negatively defined*, That - *strict local maximum*.

Finally, if the form does not have sign certainty (both strict and non-strict), then there is guaranteed to be no extremum. The remaining possible cases will require additional investigation.

So, we have arrived at sufficient conditions of the form:

If a twice continuously differentiable function $f(x,y)$ V *stationary* point quadratic form (5) *positively* certain, it $f(x,y)$ has at this point *strict local minimum*.

If a twice continuously differentiable function $f(x,y)$ V *stationary* point quadratic form (5) *negatively defined*, That $f(x,y)$ has at this point *strict local maximum*.

If a twice continuously differentiable function $f(x,y)$ V *stationary* point, quadratic form (5) does not have *symbolic certainty*, That $f(x,y)$ does not have at this point *strict local extremum*.

Note that, for example, *the first sufficient condition is not necessary*. Counter example: $f(x,y) = x^4 + y^4$. Here at the origin there is a strict minimum, but there is no strict positivity.

Let us now recall the methods of studying the quadratic form (5) for the presence or absence of signed uncertainty, proven in the course of linear algebra.

1) *Lagrangian method*. It comes down to constructing a diagonal (or canonical) basis by selecting complete squares, i.e. basis in which the coefficient B for the quadratic form (5) is equal to zero.

2) *Sylvester criterion*. This criterion states that for the form (5) to be positive definite it is necessary and sufficient that the inequalities be simultaneously satisfied $A > 0$ And $\det \begin{Vmatrix} A & B \\ B & C \end{Vmatrix} > 0$.

   For form (5) to be negatively definite, it is necessary and sufficient that the inequalities be simultaneously satisfied $A < 0$ And $\det \begin{Vmatrix} A & B \\ B & C \end{Vmatrix} > 0$.

3) *Diagonalization of a matrix of quadratic form in Euclidean space*. This method is based on the theorem that in a basis of eigenvectors of a self-adjoint transformation, which has a matrix of the form in the ONB $\begin{Vmatrix} A & B \\ B & C \end{Vmatrix}$, the matrix of such a transformation is diagonal, and its main diagonal contains the eigenvalues of this transformation.

Example 1. IN E^2 examine the function for extremum $f(x,y) = x^3 + y^3 - 3xy$.

Solution: 1) Let us first find for an infinitely differentiable function $f(x,y)$ all its stationary points, i.e. points suspected of extremum. These points are from *necessary* conditions

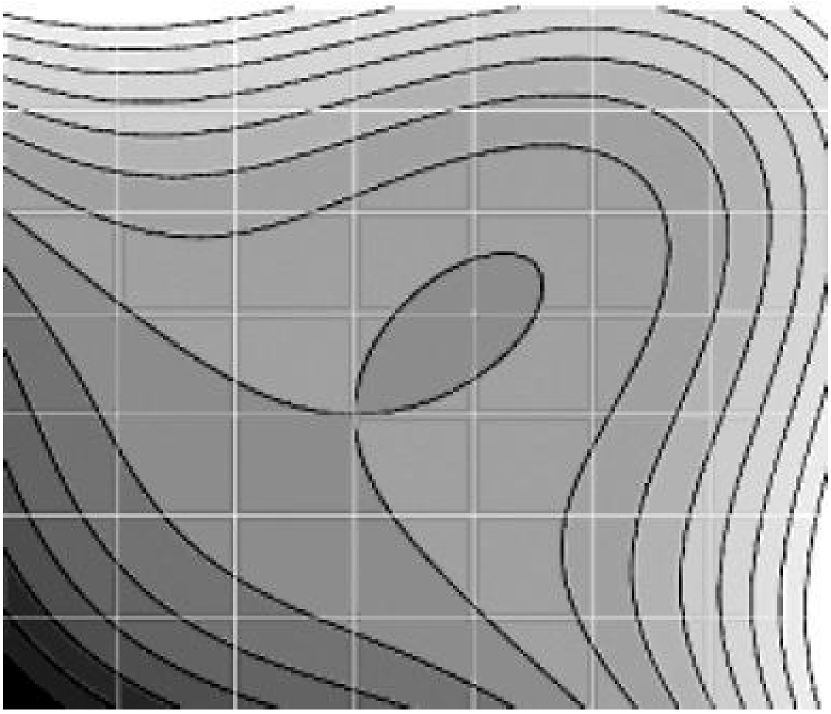
$$\operatorname{grad} f(x,y) = o \quad \text{or} \quad \begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 - 3y = 0, \\ \dfrac{\partial f}{\partial y} = 3y^2 - 3x = 0 \end{cases} \Rightarrow \left[\begin{array}{l} \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\| \\ \\ \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\| \end{array} \right. .$$

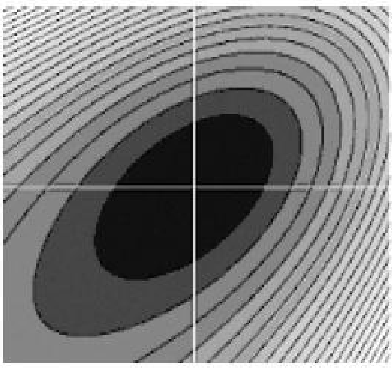
2) Now check the execution *sufficient* conditions at stationary points. Building the Hessian matrix $\left\|\begin{matrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{matrix}\right\| = \left\|\begin{matrix} 6x & -3 \\ -3 & 6y \end{matrix}\right\|$.

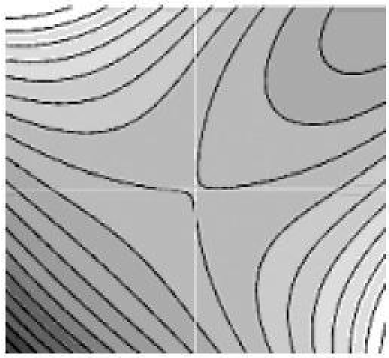
At the first stationary point the Hessian matrix will be $\left\|\begin{matrix} 6 & -3 \\ -3 & 6 \end{matrix}\right\|$. For it, the Sylvester criterion for positive definiteness of a quadratic form (5) is satisfied. So at the point $\left\|\begin{matrix} 1 \\ 1 \end{matrix}\right\|$ the function has a strict local minimum.

At the second stationary point the Hessian matrix is equal to $\left\|\begin{matrix} 0 & -3 \\ -3 & 0 \end{matrix}\right\|$. For it, the sufficient condition for the sign definiteness of the quadratic form (5) is not satisfied.

At the point $\left\|\begin{matrix} 0 \\ 0 \end{matrix}\right\|$ there is no extremum, since in a small neighborhood of the origin $f(x,y) = -3dxdy + dx^3 + dy^3$ and at $dx = dy$ we have $\Delta f = -3dx^2 + 2dx^3 < 0$, and at $dx = -dy$ will $\Delta f = 3dx^2 - 2dx^3 > 0$.







Example 2. IN E^2 examine the function for extremum

$$f(x,y) = 3x^2y + y^3 - 12x - 15y \, .$$

Solution: 1) For an infinitely differentiable function $f(x,y)$ let's find all its stationary points, i.e. points suspected of extremum. These points are from *necessary* conditions

$$\operatorname{grad} f(x,y) = o \quad \text{or} \quad \begin{cases} \dfrac{\partial f}{\partial x} = 6xy - 12 = 0, \\ \dfrac{\partial f}{\partial y} = 3x^2 + 3y^2 - 15 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} xy = 2, \\ x^2 + y^2 = 5 \end{cases}$$

Where do we get four points that are suspicious of extremeness?

$$\left\|\begin{matrix} 1 \\ 2 \end{matrix}\right\|, \left\|\begin{matrix} -1 \\ -2 \end{matrix}\right\|, \left\|\begin{matrix} 2 \\ 1 \end{matrix}\right\|, \left\|\begin{matrix} -2 \\ -1 \end{matrix}\right\|.$$

2) Let's check the execution *sufficient* conditions at stationary points. The Hessian matrix has the form

$$\left\|\begin{matrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{matrix}\right\| = \left\|\begin{matrix} 6y & 6x \\ 6x & 6y \end{matrix}\right\| .$$

This matrix is symmetric and in the ONB can be considered as a self-adjoint transformation matrix. Which, in turn, in the ONB of eigenvectors has a diagonal form. Moreover, on the main diagonal there are eigenvalues.

The results of applying the method are given in the following table

Stac. point	**Matrix Hesse**	**Iconic certainty**	**Character eq. and own values**	**Extreme?**
$\begin{Vmatrix} 1 \\ 2 \end{Vmatrix}$	$\begin{Vmatrix} 12 & 6 \\ 6 & 12 \end{Vmatrix}$	Positive	$(12-\lambda)^2-36=0$ $\lambda_1=6, \quad \lambda_2=18$	Strict minimum
$\begin{Vmatrix} -1 \\ -2 \end{Vmatrix}$	$\begin{Vmatrix} -12 & -6 \\ -6 & -12 \end{Vmatrix}$	Negative	$(-12-\lambda)^2-36=0$ $\lambda_1=-6, \quad \lambda_2=-18$	Strict maximum
$\begin{Vmatrix} 2 \\ 1 \end{Vmatrix}$	$\begin{Vmatrix} 6 & 12 \\ 12 & 6 \end{Vmatrix}$	No	$(6-\lambda)^2-144=0$ $\lambda_1=-6, \quad \lambda_2=18$	No
$\begin{Vmatrix} -2 \\ -1 \end{Vmatrix}$	$\begin{Vmatrix} -6 & -12 \\ -12 & -6 \end{Vmatrix}$	No	$(-6-\lambda)^2-144=0$ $\lambda_1=6, \quad \lambda_2=-18$	No

Example 3. IN E^2 explore to the extreme $f(x,y) = x^4 + y^4 - 2(x-y)^2$.

Solution: 1) Necessary condition for an extremum $\operatorname{grad} f(x,y) = o$ or

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 - 4(x-y) = 0, \\ \dfrac{\partial f}{\partial y} = 4y^3 + 4(x-y) = 0 \end{cases} \Rightarrow \quad 4x^3 + 4y^3 = 0 \quad \Rightarrow \quad y = -x$$

From where, substituting into the first equation of the system, we get $4x^3 - 8x = 0$. This means there are three stationary points: $\left\| \begin{matrix} \sqrt{2} \\ -\sqrt{2} \end{matrix} \right\|, \left\| \begin{matrix} -\sqrt{2} \\ \sqrt{2} \end{matrix} \right\|, \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\|$.

2) We check sufficient conditions. We have a Hessian matrix of the following form $\begin{Vmatrix} 12x^2-4 & 4 \\ 4 & 12y^2-4 \end{Vmatrix}$. At the first two stationary points the equality $x^2 = y^2 = 2$ and therefore at these points the Hess matrix $\begin{Vmatrix} 20 & 4 \\ 4 & 20 \end{Vmatrix}$. Consequently, at the first two points we have strict local minima.

At the third stationary point - at the origin - the Hessian matrix is equal to $\begin{Vmatrix} -4 & 4 \\ 4 & -4 \end{Vmatrix}$ and the quadratic form (5) is negative semidefinite. This means that none of the three sufficient conditions *not applicable*.

Let us take advantage of the fact that the change in the value of the function under study at the origin of coordinates is determined by the following formula:

$$\Delta f(x,y) = dx^4 + dy^4 - 2(dx-dy)^2 .$$

In this case, leaving the origin in the direction $dx = dy$, we get that $\Delta f(x,y) > 0$, and going out in the direction $dx = -dy$, we will have $\Delta f(x,y) = 2dx^4 - 8dx^2 < 0$ at sufficiently small $dx = dy$
This means that the origin is not an extreme point for the function under study.

Example 4. IN E^2 explore to the extreme $f(x,y)=4x^2-4xy+y^2+4x-2y+1$.

Solution: 1) Necessary condition for an extremum $\operatorname{grad} f(x,y)=o$ or

$$\begin{cases} \dfrac{\partial f}{\partial x}=8x-4y+4=0, \\ \dfrac{\partial f}{\partial y}=-4x+2y-2=0 \end{cases} \quad\Rightarrow\quad 2x-y+1=0$$

The set of stationary points is unlimited and there is a straight line in E^2.

2) We check sufficient conditions. We have a Hessian matrix that is the same for all stationary points $\left\|\begin{matrix} 8 & -4 \\ -4 & 2 \end{matrix}\right\|$. This means that at each stationary point the quadratic form (5) is positive semidefinite and none of the three sufficient conditions *not applicable*.

In this situation, you can take advantage of the fact that the function under study can be written like this:

$$f(x,y)=(2x-y)^2+2(2x-y)+1=(2x-y+1)^2.$$

Whence it follows that every point is a straight line $2x-y+1=0$ is the minimum point of the function $f(x,y)$, but not strict, since in any neighborhood there are other points with the same value $f(x,y)$.

Example 5. IN E^2 explore to the extreme $f(x,y) = 1 - \sqrt{x^2 + y^2}$.

Solution: 1) A necessary condition for an extremum states that either $\operatorname{grad} f(x,y) = o$, or it doesn't exist.

It is easy to see that at any point other than the origin, the vector $\operatorname{grad} f(x,y)$ looks like

$$\left\| \begin{matrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} -\dfrac{x}{\sqrt{x^2 + y^2}} \\ -\dfrac{y}{\sqrt{x^2 + y^2}} \end{matrix} \right\| . \qquad (6)$$

Among such points there are no stationary ones, since the length of the gradient vector for them is equal to one.

It remains to consider the point of origin. in it the gradient vector does not exist, since there are no limits that are partial derivatives with respect to x And y . Indeed, from the course of mathematical analysis it is known that the limits of the form

$$\lim_{dx\to 0}\frac{f(0+dx,0)-f(0,0)}{dx}=\lim_{dx\to 0}\frac{-\sqrt{dx^2}}{dx}=\lim_{dx\to 0}\frac{-|dx|}{dx}$$

And

$$\lim_{dy\to 0}\frac{f(0,0+dy)-f(0,0)}{dy}=\lim_{dy\to 0}\frac{-\sqrt{dy^2}}{dy}=\lim_{dy\to 0}\frac{-|dy|}{dy}$$

do not exist.
Note that the use of formulas (6) to justify the non-differentiability of the function under study at the origin is incorrect.

Since it is impossible to apply extremum criteria to this problem, we will use the definition directly. We have the following estimate of the sign of the change in the value of the function under study for an arbitrary deviation of the arguments from the origin:

$$f(0+dx,0+dy)-f(0,0)=1-\sqrt{dx^2+dy^2}-1=-\sqrt{dx^2+dy^2}<0\quad \forall dx,dy,$$

from which it follows that the origin is a strictly maximum point.

Example 6. IN E^2 examine the function for extremum $u(x,y)$, given *implicitly* condition

$$F(x,y,u) = 1 - x^2 - y^2 - u^2 = 0 \,. \tag{7}$$

Solution: 1) Find the derivatives of the function $u(x,y)$, given implicitly: we have $u'_x = -\dfrac{F'_x}{F'_u}, \quad u'_y = -\dfrac{F'_y}{F'_u}$, Where

$$F'_x = -2x,\ F'_y = -2y,\ F'_u = -2u.$$

Then, due to the necessary conditions of extremality

from $\begin{cases} u'_x = -\dfrac{x}{u} = 0 \\ u'_y = -\dfrac{y}{u} = 0 \end{cases}$ it follows that the point $\begin{cases} x = 0 \\ y = 0 \end{cases}$ suspicious of extremes.

2) To apply sufficient conditions, we find the Hessian matrix at a suspicious point, we get

$$\left\| \begin{matrix} u''_{xx} & u''_{xy} \\ u''_{yx} & u''_{yy} \end{matrix} \right\| = \left\| \begin{matrix} -\frac{1}{u} + x\frac{u'_x}{u^2} & x\frac{u'_y}{u^2} \\ y\frac{u'_x}{u^2} & -\frac{1}{u} + y\frac{u'_y}{u^2} \end{matrix} \right\| = \left\| \begin{matrix} -\frac{1}{u} & 0 \\ 0 & -\frac{1}{u} \end{matrix} \right\| . \quad (8)$$

On the other hand, from (7) we have that at the point $\begin{cases} x = 0 \\ y = 0 \end{cases}$ variable $u = \pm 1,$ that is, equation (7) in the vicinity of this point specifies two different functions.

Using sufficient conditions for the extremum, from (8) we obtain that one of these functions with $u = -1$ we have a minimum, and at $u = 1$ - maximum.

This conclusion could also be reached by noting that in the vicinity of the point $\begin{cases} x = 0 \\ y = 0 \end{cases}$ from equation (7) it follows that either $u = \sqrt{1 - x^2 - y^2}$, or $u = -\sqrt{1 - x^2 - y^2}$.

Example 7. Let the continuous function $F(x)$ has at the point x^* local extremum. Is the statement true: in this case there is such a neighborhood of the point x^*, for which $F(x)$ monotonic in both the right and left half-neighborhood of this point? Justify the answer.

Solution: No, that's not true. Example: continuous function

$$F(x)=\begin{cases} 0, & \text{если } x=0, \\ x^2\left(2+\cos\dfrac{1}{x}\right), & \text{если } x\neq 0 \end{cases}$$

has at point $x^*=0$ minimum, but is not monotonic in any half-neighborhood of this point.