STUDYING THE FUNCTIONS OF MANY VARIABLES TO THE CONDITIONAL EXTREME

Let us be given a twice continuously differentiable in some domain $\Omega \subseteq E^2$ with ONB function $f(x,y)$. Let us find out under what conditions this function has at the point $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\| \in \Omega$ local minimum *given that* $g(x,y)=0$. Let's give

Definition 1. We will say that the function $f(x,y)$ has at point $\left\| \begin{matrix} x^* \\ y^* \end{matrix} \right\|$ *conditional minimum*, if exists $\overset{\circ}{U}_\varepsilon$ - punctured surroundings, such that for *any* points $\left\| \begin{matrix} x \\ y \end{matrix} \right\| \in \overset{\circ}{U}_\varepsilon$ takes place

$$g(x,y)=0$$

and the inequality holds

$$f(x,y) > f(x^*,y^*) . \qquad (1)$$

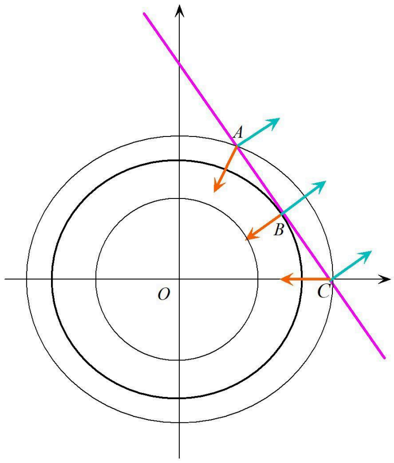
Let's look at the specific one first. task:

Example 1. IN E^2 examine the function for extremum $f(x,y) = -x^2 - y^2$ given that $g(x,y) = x + y - 2 = 0$.

Solution: 1) Obviously, this problem can be solved *by elimination*, expressing y from the condition of communication through x and substituting $y = 2 - x$ to the target function.
Then we come to the problem of maximizing the function $\Phi(x) = -x^2 - (2-x)^2$ *no restrictions*, whose solution $x^* = 1,\ y^* = 1$ And $\Phi^* = f^* = -2$.

Lagrange function and its uses

Let us first consider the geometric interpretation of the problem, obtaining the necessary conditions for its solution.



This figure shows the isolines of the objective function in gray (and black)

$$f(x,y) = -x^2 - y^2,$$

purple - points of a straight line $g(x,y) = x + y - 2 = 0$.

Gradients are shown in orange. *objective function* $\left\| \begin{matrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} -2x \\ -2y \end{matrix} \right\|$.

Finally, in blue - gradients *functions-constraints* $\left\| \begin{matrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\|$.

Obviously, if the projection of the gradient of the objective function onto the set of points satisfying the constraint, *not zero*, then this is not an extremum and *necessary* the extremum condition is that the vectors are collinear $\left\|\begin{matrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{matrix}\right\|$ And $\left\|\begin{matrix}\frac{\partial g}{\partial x}\\ \frac{\partial g}{\partial y}\end{matrix}\right\|$ at many points $g(x,y)=x+y-2=0$.

In other words, $\left\|\begin{matrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{matrix}\right\| = -\lambda \left\|\begin{matrix}\frac{\partial g}{\partial x}\\ \frac{\partial g}{\partial y}\end{matrix}\right\|$ or, in vector form, $\operatorname{grad} f + \lambda \operatorname{grad} g = o$, Where λ – some constant.

Whence, due to the linearity property of the differentiation operation, it follows that there is a function

$$L(x,y,\lambda)=f(x,y)+\lambda\, g(x,y)\,,$$

in terms of which the necessary condition for the existence of a conditional extremum in this problem takes the form:

$$\begin{cases} \left\|\begin{matrix} \dfrac{\partial L}{\partial x} \\ \dfrac{\partial L}{\partial y} \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \end{matrix}\right\|, \\ g(x,y)=0. \end{cases}$$

In the problem under consideration $L(x,y,\lambda) = -x^2 - y^2 + \lambda(x + y - 2)$. This means that the necessary condition for an extremum has the form

$$\begin{cases} -2x + \lambda = 0, \\ -2y + \lambda = 0, \\ x + y - 2 = 0. \end{cases}$$

The obvious solution to this system is a triple of numbers

$$x^* = 1, \;\; y^* = 1, \;\; \lambda^* = 2, \quad f^* = -2 \, .$$

Let's generalize this approach

For the task:

examine the function for extremum $f(x_1, x_2, \ldots, x_n)$ (2)
under conditions: $g_i(x_1, x_2, \ldots, x_n) = 0 \quad i = [1, m].$

we will assume that

$$m < n \quad \text{And} \quad \text{rg}\left\| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(g_1, g_2, \ldots, g_m)} \right\| = m .$$

Functions $f(x_1, x_2, \ldots, x_n)$ And $g_i(x_1, x_2, \ldots, x_n) \quad i = [1, m]$ − continuously differentiable.

If we introduce the function *Lagrange* type:

$$L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \dots, x_n),$$

That *necessary* conditions for the extremum of the function $f(x_1, x_2, \dots, x_n)$, under conditions

$$g_i(x_1, x_2, \dots, x_n) = 0 \quad i = [1, m].$$

will look like

$$\begin{cases} \dfrac{\partial L}{\partial x_j}(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = 0 & j = [1, n] \\ g_i(x_1, x_2, \dots, x_n) = 0 & i = [1, m]. \end{cases} \tag{3}$$

Let $x_1^*, x_2^*, \ldots, x_n^*, \lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ satisfy (3). Then *sufficient* the extremum condition will have the formulation

> If the quadratic form $d_x^2 L$ positive (negative) definite provided $dg_i(x_1^*, x_2^*, \ldots, x_n^*) = 0 \quad i = [1, m]$, then problem (3) at the point $x_1^*, x_2^*, \ldots, x_n^*, \lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ has local *minimum* (*maximum*).

Example 2. IN E^3 examine the function for extremum $u(x,y,z) = x - y + 2z$ given that $x^2 + y^2 + 2z^2 = 16$.

Solution: 1) The Lagrange function for this problem will be

$$L = x - y + 2z + \lambda(x^2 + y^2 + 2z^2 - 16).$$

The conditions for its stationarity, together with the coupling equation, form the system

$$\begin{cases} \dfrac{\partial L}{\partial x} = \ \ 1 + 2\lambda x = 0, \\ \dfrac{\partial L}{\partial y} = -1 + 2\lambda y = 0, \\ \dfrac{\partial L}{\partial z} = \ \ 2 + 4\lambda z = 0, \\ x^2 + y^2 + 2z^2 = 16. \end{cases}$$

If from the first three equations we express $x = -\frac{1}{2\lambda}, y = \frac{1}{2\lambda}, z = -\frac{1}{2\lambda}$ and substitute into the fourth, we get from $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{2}{4\lambda^2} = 16$, What $\lambda = \pm\frac{1}{4}$.

2) We get two stationary points that are suspicious of an extremum:

$$\lambda^* = \frac{1}{4} \quad \Rightarrow \quad \begin{cases} x^* = -2 \\ y^* = \ \ 2 \\ z^* = -2 \end{cases}$$

And

$$\lambda^* = -\frac{1}{4} \quad \Rightarrow \quad \begin{cases} x^* = \ \ 2 \\ y^* = -2 \\ z^* = \ \ 2 \end{cases}$$

3) We check sufficient conditions. We find

$$\frac{\partial^2 L}{\partial x^2} = 2\lambda \quad \frac{\partial^2 L}{\partial y^2} = 2\lambda \quad \frac{\partial^2 L}{\partial z^2} = 4\lambda$$
$$\frac{\partial^2 L}{\partial x \partial y} = \frac{\partial^2 L}{\partial x \partial z} = \frac{\partial^2 L}{\partial z \partial y} = 0$$

Where $d^2 L = 2\lambda (dx)^2 + 2\lambda (dy)^2 + 4\lambda (dz)^2$. This quadratic form is positive definite when $\lambda = \frac{1}{4}$ and is negative definite at $\lambda = -\frac{1}{4}$.

In addition, the equality must be satisfied $2xdx + 2ydy + 4zdz = 0$. However, the last equality is signed definite $d^2 L$ will have no effect and can be ignored. So

$$\lambda_1^* = \frac{1}{4} \quad \begin{cases} x_1^* = -2 \\ y_1^* = \ \ 2 \\ z_1^* = -2 \end{cases} \quad f_1^* = -8$$

- point *minimum*

And

$$\lambda_2^* = -\frac{1}{4} \quad \begin{cases} x_2^* = \ \ 2 \\ y_2^* = -2 \\ z_2^* = \ \ 2 \end{cases} \quad f_2^* = \ \ 8$$

....−.point *maximum*.

Example 3. IN E^2 examine the function for extremum $u(x,y) = x^2 + xy + y^2$ given that $x^2 + y^2 = 1$.

Solution: 1) Lagrange function in this problem

$$L = x^2 + xy + y^2 + \lambda(x^2 + y^2 - 1).$$

2) The conditions for its stationarity will be

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2x + y + 2\lambda x = 0 \\ \dfrac{\partial L}{\partial y} = x + 2y + 2\lambda y = 0 \end{cases} \quad \text{or} \quad \begin{cases} (2+2\lambda)x + y = 0 \\ x + (2+2\lambda)y = 0 \end{cases}$$

(4)

2) We have (by Cramer's theorem) that if $\det \begin{Vmatrix} 2+2\lambda & 1 \\ 1 & 2+2\lambda \end{Vmatrix} \neq 0$,

That $x = y = 0$, and this is obvious (due to $x^2 + y^2 = 1$) is not a solution.

If $\det \begin{Vmatrix} 2+2\lambda & 1 \\ 1 & 2+2\lambda \end{Vmatrix} = 0 \quad \Rightarrow \quad \left[\begin{array}{l} \lambda_1 = -\dfrac{1}{2} \\ \lambda_2 = -\dfrac{3}{2} \end{array} \right.$, then two cases are possible.

3) $\lambda_1^* = -\frac{1}{2}$, in this case, due to (4)

$$\begin{cases} x + y = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \left[\begin{array}{l} \begin{cases} x_{11}^* = \frac{1}{\sqrt{2}} \\ y_{11}^* = -\frac{1}{\sqrt{2}} \end{cases} - \text{точка } D \\ \begin{cases} x_{12}^* = -\frac{1}{\sqrt{2}} \\ y_{12}^* = \frac{1}{\sqrt{2}} \end{cases} - \text{точка } C \end{array} \right. .$$

The Hessian matrix of the Lagrange function will have the form

$$\begin{Vmatrix} 2+2\lambda_1^* & 1 \\ 1 & 2+2\lambda_1^* \end{Vmatrix} \Rightarrow \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} .$$

She *positive semidefinite* and further research is required here.

Here at the points *C* And *D,*

$$d^2L = (dx)^2 + 2dxdy + (dy)^2 = (dx+dy)^2 ,$$

in this case the equality must be satisfied $xdx + ydy = 0.$

And since at these points $x = -y,$ That $dy = dx$. Means, $d^2L = 4(dx)^2$ and dots *C* And *D* − this *minimum* with $f^* = \frac{1}{2}$.

4) $\lambda_2^* = -\dfrac{3}{2}$, in this case (due to (4))

$$\begin{cases} x - y = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \left[\begin{array}{l} \begin{cases} x_{21}^* = \dfrac{1}{\sqrt{2}} \\ y_{21}^* = \dfrac{1}{\sqrt{2}} \end{cases} - \text{точка } A \\ \begin{cases} x_{22}^* = -\dfrac{1}{\sqrt{2}} \\ y_{22}^* = -\dfrac{1}{\sqrt{2}} \end{cases} - \text{точка } B \end{array} \right. .$$

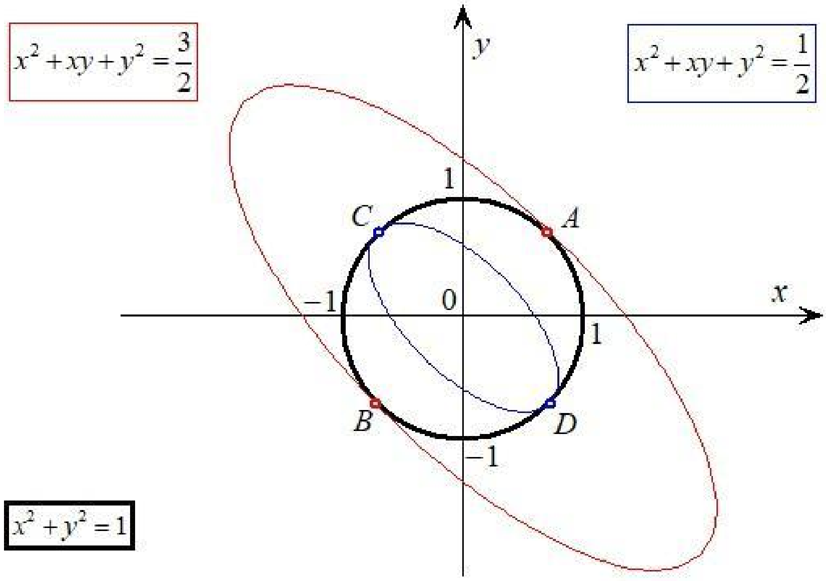
The Hessian matrix here will look like $\left\| \begin{matrix} 2+2\lambda_2^* & 1 \\ 1 & 2+2\lambda_2^* \end{matrix} \right\| \Rightarrow \left\| \begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix} \right\|$.
She *negative semidefinite* Here too additional research is required.

We have, at points *A* And *B,*

$$d^2L = -(dx)^2 + 2dxdy - (dy)^2 = -(dx - dy)^2 ,$$

in this case the equality must be satisfied $xdx + ydy = 0.$

And since at these points $x = y,$ That $dx = -dy$. Means, $d^2L = -4(dx)^2$ and dots *A* And *B*− this *maximum* with $f^* = \dfrac{3}{2}$.


$x^2+xy+y^2=\frac{3}{2}$
$x^2+xy+y^2=\frac{1}{2}$
$x^2+y^2=1$
y
x
1
−1
0
1
−1
C
A
B
D