MULTIPLE INTEGRALS OF FUNCTIONS OF MULTIPLE VARIABLES

Double integrals

First, for greater clarity, let's consider the two-dimensional case.

Let a doubly continuous in some Jordan measurable (with measure) be given μ), areas $\Omega \subseteq E^2$ with ONB function $f(x,y)$.

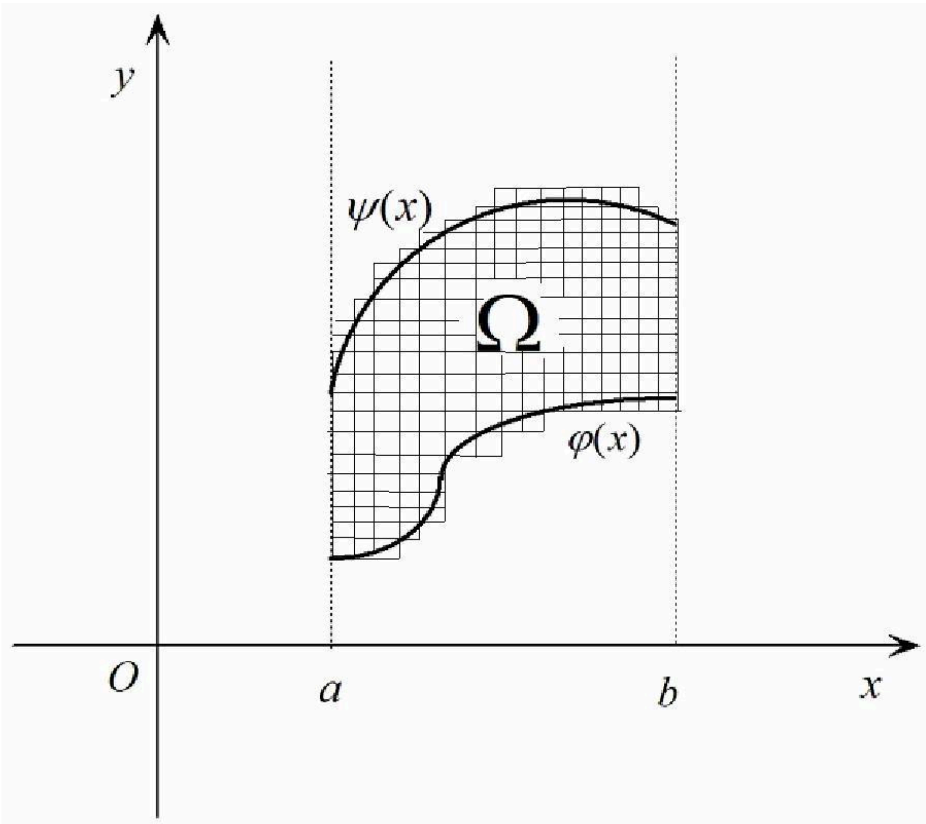
And let this set be divided into subsets $\{\Omega_k \; k=[1,N]\}$ with *fineness* this partition, determined, for example, by the value $\tau = \max\limits_{k=[1,N]} \{\mu(\Omega_k)\}$. Although it is more accurate to use as τ – "diameter" Ω , i.e. $\tau = \sup\limits_{\substack{x,y\in\Omega_k \\ k=[1,N]}} \{ \| x-y \| \}$.

Further, in each subset Ω_k arbitrary point selected $\left\| \begin{matrix} x_k \\ y_k \end{matrix} \right\|$, the collection of which we denote Θ_τ .

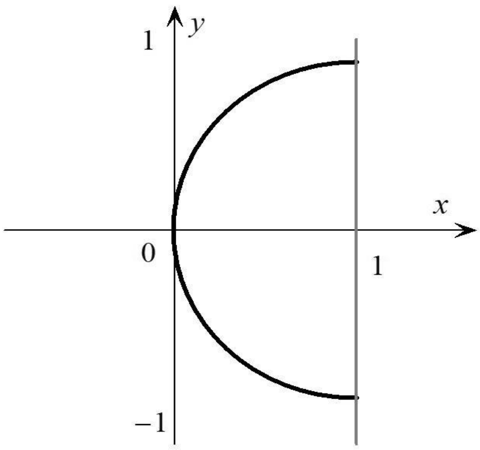
Then *Riemann sum* from function $f(x,y)$ by region Ω called an expression of the form $\sigma_\tau(f;\Theta_\tau) = \sum\limits_{k=1}^{N} f(x_k,y_k)\mu(\Omega_k)$.

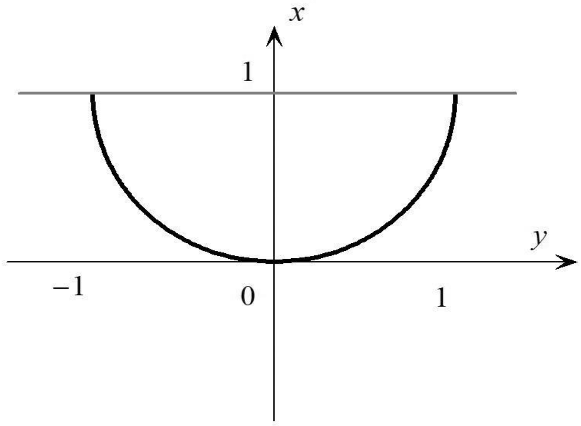
Double integral functions $f(x,y)$ by region Ω called number $I = \lim\limits_{\tau \to 0} \sigma_\tau$ *-limit* its Riemann sum when the partition is fine $\tau \to 0$.

Region Ω called *elementary relative to the axis* Oy, if functions exist $\varphi(x)$ And $\psi(x)$ $x \in [a,b]$, such that $\Omega = \begin{cases} \varphi(x) \le y \le \psi(x), \\ x \in [a,b]. \end{cases}$



If the area Ω elementary relative to the axis Oy, then the double integral, denoted as $J = \iint\limits_{\Omega} f(x,y)\,dxdy$, equal to $J = \int\limits_a^b dx \int\limits_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$.





Example 1: Find $I = \iint\limits_{\Omega} xy^2\, dxdy\,,$ If $\Omega = \left\{ \begin{array}{c} 0 \le x \le 1, \\ y^2 \le x. \end{array} \right\}$

Solution: 1) Note that the area Ω elementary relative to the axis Oy And $\varphi(x) = -\sqrt{x}, \psi(x) = \sqrt{x}$. That's why

$$I = \iint\limits_{\Omega} xy^2\, dxdy = \int\limits_0^1 xdx \int\limits_{-\sqrt{x}}^{\sqrt{x}} y^2 dy = \int\limits_0^1 x \left(\frac{y^3}{3} \Bigg|_{-\sqrt{x}}^{\sqrt{x}} \right) dx = \frac{2}{3} \int\limits_0^1 x^{\frac{5}{2}} dx = \frac{2}{3} \frac{x^{\frac{7}{2}}}{\frac{7}{2}} \Bigg|_0^1 = \frac{4}{21}.$$

2) You can also notice that the area Ω elementary relative to the axis Ox with $\varphi(y) = y^2, \psi(y) = 1$. This means that the double integral can be calculated differently

$$I = \iint\limits_{\Omega} xy^2\, dxdy = \int\limits_{-1}^1 y^2 dy \int\limits_{y^2}^1 xdx = \int\limits_0^1 y^2 \left(\frac{x^2}{2} \Bigg|_{y^2}^1 \right) dy = \frac{1}{2} \int\limits_0^1 y^2 (1 - y^4) dy =$$

$$= \int\limits_0^1 y^2 (1 - y^4) dy = \left(\frac{y^3}{3} - \frac{y^7}{7} \right) \Bigg|_0^1 = \frac{1}{3} - \frac{1}{7} = \frac{4}{21}.$$

3) The results were the same. This is a theorem!

Example 2: Find the double integral $I = \iint\limits_{\Omega} (x + 2y)\, dxdy$, if the area looks like $\Omega = \left\{ \begin{array}{c} 2 \le x \le 3, \\ x \le y \le 2x. \end{array} \right\}$, choosing the most convenient order of sequential integration.

Solution: 1) Note that the area Ω elementary relative to the axis Oy c $\varphi(x) = x, \psi(x) = 2x$.

Relative to axis Ox it is not elementary, but it can be divided into three parts, each of which is elementary relative to the axis Ox. However, you will have to calculate three double integrals over areas I, II, III. The aditivity property of an integral.

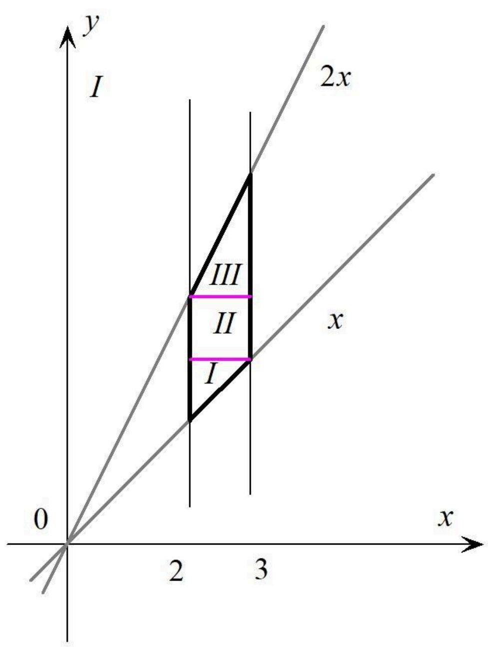
2) Therefore, it is better to think like this:

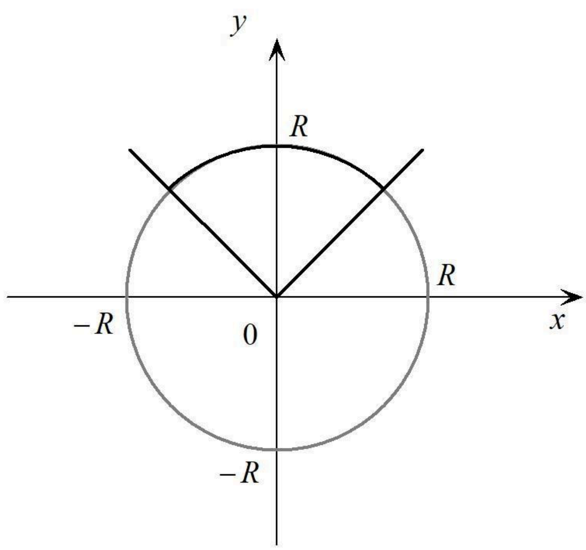
$$I = \iint\limits_{\Omega} (x + 2y)\, dxdy = \int\limits_{2}^{3} dx \int\limits_{x}^{2x} (x + 2y) dy = \int\limits_{2}^{3} [(xy + y^2)\Big|_{x}^{2x}] dx =$$

$$= \int\limits_{2}^{3} [(2x^2 + 4x^2) - (x^2 + x^2)] dx = 4\int\limits_{2}^{3} x^2 dy = \frac{76}{3}.$$

Often, the calculation of double integrals can be simplified by making a suitable nonlinear change of variables.

Example 3: Find the double integral $I = \iint\limits_{\Omega} (x+y)\,dxdy$, if the area looks like $\Omega = \left\{ \begin{array}{c} x^2 + y^2 \le R^2, \\ |x| \le y. \end{array} \right\}$, going to *polar* coordinate system.





Solution: 1) Region Ω elementary relative to the axes Oy And Ox with borders $\varphi(x) = |x|, \quad \psi(x) = \sqrt{R^2 - x^2}$, but the integrals turn out to be quite complex.

2) The calculations turn out to be much simpler if we switch to the polar coordinate system using the formulas $\begin{cases} x = r\cos\alpha, \\ y = r\sin\alpha. \end{cases}$.

When moving from variables $\{x, y\}$ to variables $\{r, \alpha\}$ the equality will be true (this is a theorem!) $dxdy = |J|\, drd\alpha$, Where J – this *Jacobian* (determinant of the Jacobian matrix).

We'll find *module of the Jacobian* in the case under consideration. We have

$$|J| = \left|\frac{\partial(x,y)}{\partial(r,\alpha)}\right| = |\det \begin{Vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \alpha} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \alpha} \end{Vmatrix}| = |\det \begin{Vmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{Vmatrix}| = r\left(\cos^2\alpha + \sin^2\alpha\right) = r .$$

That is, with the replacement used $dx\,dy = r\,dr\,d\alpha$, and the problem reduces to finding a double integral of the form

$$I = \iint\limits_{\Omega^*} (r\cos\alpha + r\sin\alpha)\, r\, dr\, d\alpha ,$$

where is the area Ω^* There is *rectangle* $\begin{cases} 0 \le r \le R, \\ \dfrac{\pi}{4} \le \alpha \le \dfrac{3\pi}{4} \end{cases}$ in variables r And α
.

3) Since the integrals over polar coordinates end up on a rectangle Ω^* independent, then the calculations are greatly simplified:

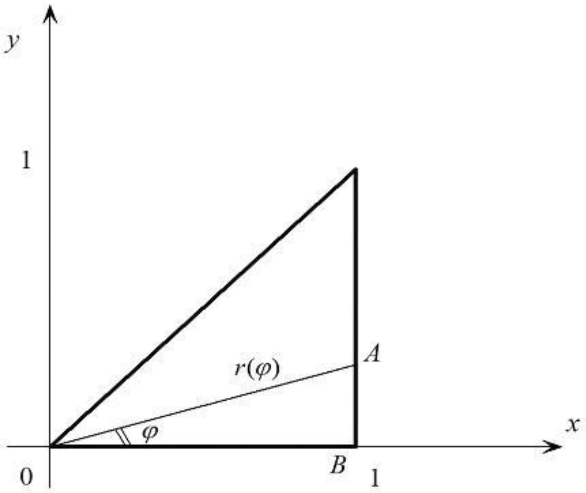
$$I = \iint\limits_{\Omega^*} (r\cos\alpha + r\sin\alpha)\, r\, dr\, d\alpha = \int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\alpha + \sin\alpha) d\alpha \int\limits_0^R r^2 dr =$$

$$= \frac{R^3}{3}(\sin\alpha - \cos\alpha)\Bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{R^3\sqrt{2}}{3}.$$

Problem 3: in the integral $I = \iint\limits_{\Omega} f(x,y)\,dxdy$ go to polar coordinates and set the limits of integration in the corresponding iterated integral in two ways, if

1) $\Omega: \begin{cases} 0 \le x \le 1, \\ y \ge 0, \\ x \ge y, \end{cases}$ 2) $\Omega: \begin{cases} x \ge 0, \\ y \ge 0, \\ x + y \le 1. \end{cases}$

Solution: 1)

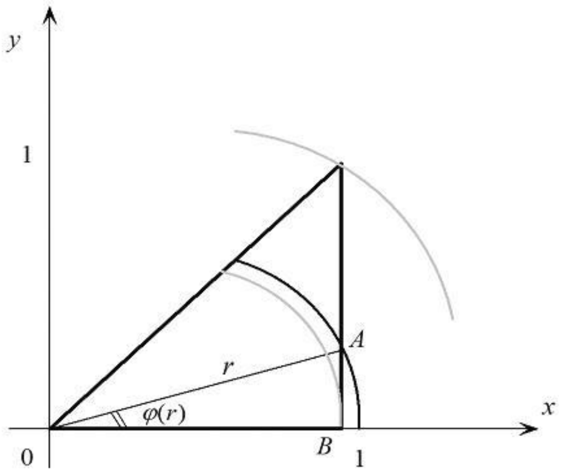


A) Let φ is an external variable, and r - internal. Then $I = \int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{r(\varphi)} f(r\cos\varphi, r\sin\varphi)\,r\,dr$,

Where $r(\varphi) = |0A(\varphi)|$.

In our case, from $\Delta 0AB$ it follows that $\cos\varphi = \dfrac{1}{r(\varphi)}$. That's why $r(\varphi) = \dfrac{1}{\cos\varphi}$ and, finally,

$I = \int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{\frac{1}{\cos\varphi}} f(r\cos\varphi, r\sin\varphi)\,r\,dr$.

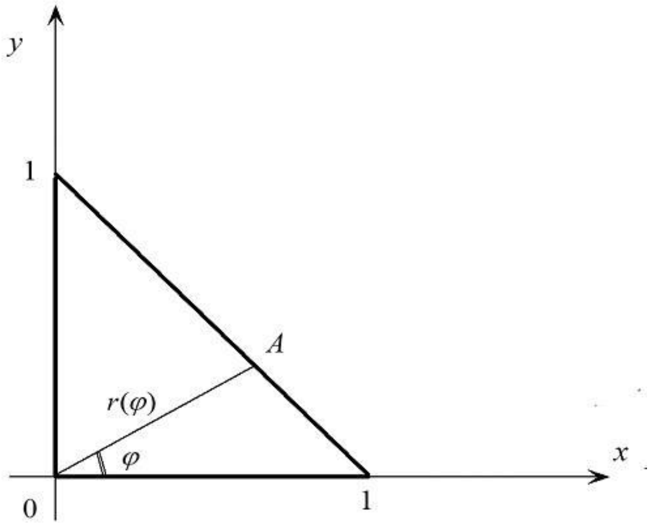


b) Let r is an external variable, and φ - internal. Then

$$I=\int\limits_0^1 rdr\int\limits_0^{\frac{\pi}{4}} f(r\cos\varphi, r\sin\varphi)\,d\varphi+\int\limits_1^{\sqrt{2}} rdr\int\limits_{\varphi(r)}^{\frac{\pi}{4}} f(r\cos\varphi, r\sin\varphi)\,d\varphi,$$

where from $\Delta 0AB$ we find that $\cos\varphi(r)=\dfrac{1}{r}$ or $\varphi(r)=\arccos\dfrac{1}{r}$. That is,

$$I=\int\limits_0^1 rdr\int\limits_0^{\frac{\pi}{4}} f(r\cos\varphi, r\sin\varphi)\,d\varphi+\int\limits_1^{\sqrt{2}} rdr\int\limits_{\arccos\frac{1}{r}}^{\frac{\pi}{4}} f(r\cos\varphi, r\sin\varphi)\,d\varphi.$$

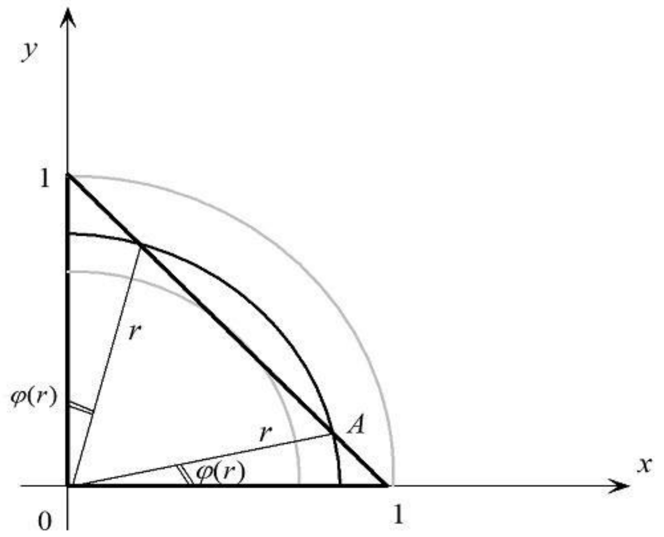


2)

a) Let φ is an external variable, and r - internal. Then $I = \int\limits_{0}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{r(\varphi)} f(r\cos\varphi, r\sin\varphi)\, r dr$, Where $r(\varphi) = |0A(\varphi)|$.

In our case, from the point belonging to *A* hypotenuse it follows that $x_A + y_A = 1$ or, in polar coordinates, $r(\varphi)\cos\varphi + r(\varphi)\sin\varphi = 1 \quad \Rightarrow \quad r(\varphi)\sqrt{2}\sin\left(\varphi + \frac{\pi}{4}\right) = 1$

That's why $r(\varphi) = \dfrac{1}{\sqrt{2}\sin(\varphi + \frac{\pi}{4})}$ and finally $I = \int\limits_{0}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{\frac{1}{\sqrt{2}\sin(\varphi+\frac{\pi}{4})}} f(r\cos\varphi, r\sin\varphi)\, r dr.$



b) Let r is an external variable, and φ - internal. Then it is convenient to split the interval of change r by two: $r \in [0, \frac{1}{\sqrt{2}}]$ And $r \in [\frac{1}{\sqrt{2}}, 1]$. Then we find that the integral over the entire region can be represented in the form

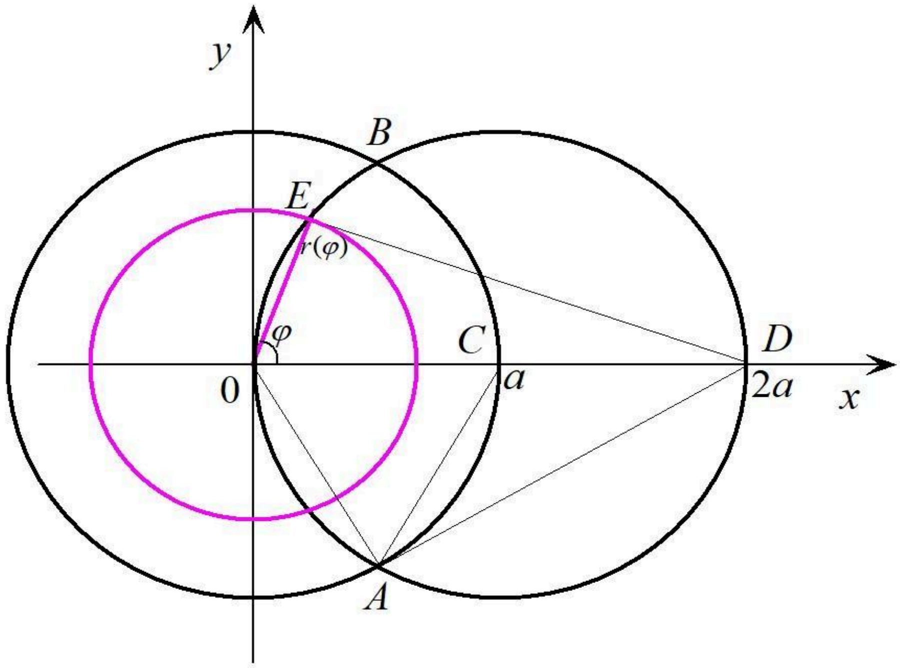
$$I = \int\limits_0^{\frac{1}{\sqrt{2}}} r dr \int\limits_0^{\frac{\pi}{2}} f(r\cos\varphi, r\sin\varphi)\, d\varphi + \int\limits_{\frac{1}{\sqrt{2}}}^{1} r dr \int\limits_0^{\varphi(r)} f(r\cos\varphi, r\sin\varphi)\, d\varphi + \int\limits_{\frac{1}{\sqrt{2}}}^{1} r dr \int\limits_{\frac{\pi}{2}-\varphi(r)}^{\frac{\pi}{2}} f(r\cos\varphi, r\sin\varphi)\, d\varphi ,$$

where is the function $\varphi(r)$ is found from the condition of the point belonging *A* hypotenuse:

$$r\cos\varphi(r) + r\sin\varphi(r) = 1 \quad \Rightarrow \quad r\sqrt{2}\sin\left(\varphi(r) + \frac{\pi}{4}\right) = 1 \quad \Rightarrow \quad \sin\left(\varphi(r) + \frac{\pi}{4}\right) = \frac{1}{r\sqrt{2}} .$$

That is $\varphi(r) = \arcsin\frac{1}{r\sqrt{2}} - \frac{\pi}{4}$.

In the next problem we will indicate only the idea of the solution and the answer. Check it out yourself.



Problem 4: In integral $I = \iint\limits_{\Omega} f(x,y)\,dxdy$ go to polar coordinates and set the limits of integration in the corresponding iterated integral in two ways, if

$$\Omega: \begin{cases} x^2 + y^2 \le a^2, \quad a > 0, \\ x^2 + y^2 \le 2ax. \end{cases}$$

Solution: 1) Note that the triangle $0AC$ correct, and the triangle $0ED$ rectangular. Where can you get that $r(\varphi) = 2a\cos\varphi$.

2) The area in this problem is a double segment $0BCA$. In this case, the region is elementary for both sequences of one-dimensional integration, but for the external variable φ the upper bound consists of three functions, and with an external variable r - from one.

Answer: 1) if the external variable of sequential integration φ, That

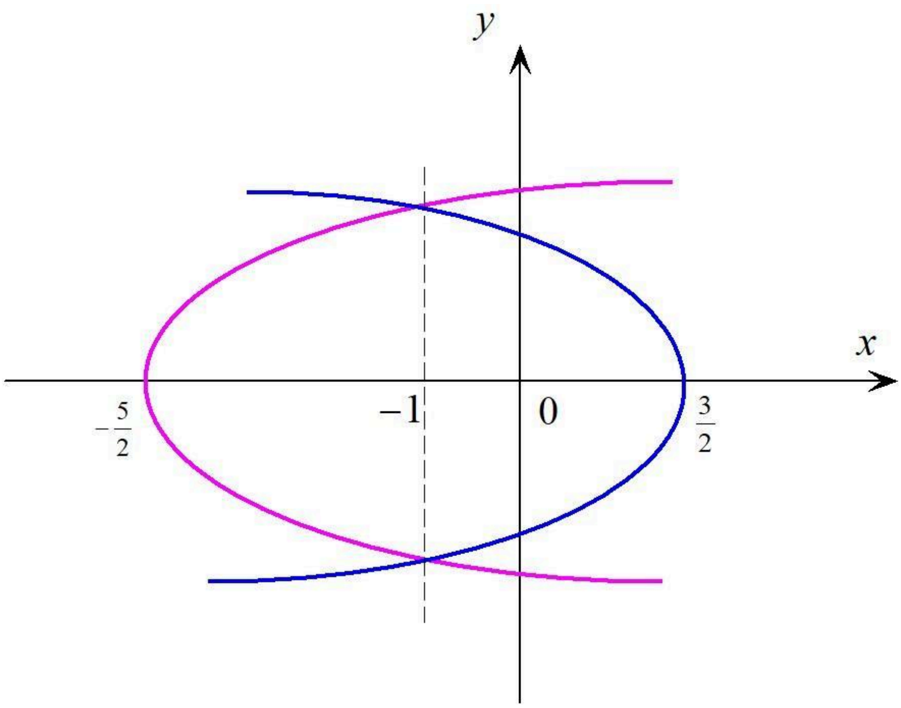
$$I = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} d\varphi \int_{0}^{2a\cos\varphi} f(r\cos\varphi, r\sin\varphi) r dr +$$

$$+ \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\varphi \int_{0}^{a} f(r\cos\varphi, r\sin\varphi) r dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_{0}^{2a\cos\varphi} f(r\cos\varphi, r\sin\varphi) r dr.$$

2) if the external variable of sequential integration r, That

$$I = \int_{0}^{a} r dr \int_{-\arccos\frac{r}{2a}}^{\arccos\frac{r}{2a}} f(r\cos\varphi, r\sin\varphi) d\varphi.$$

Problem 5: On a plane Oxy in a rectangular coordinate system, find the area of the figure bounded by the lines $y^2 = 9 - 6x$ And $y^2 = 10x + 25$.

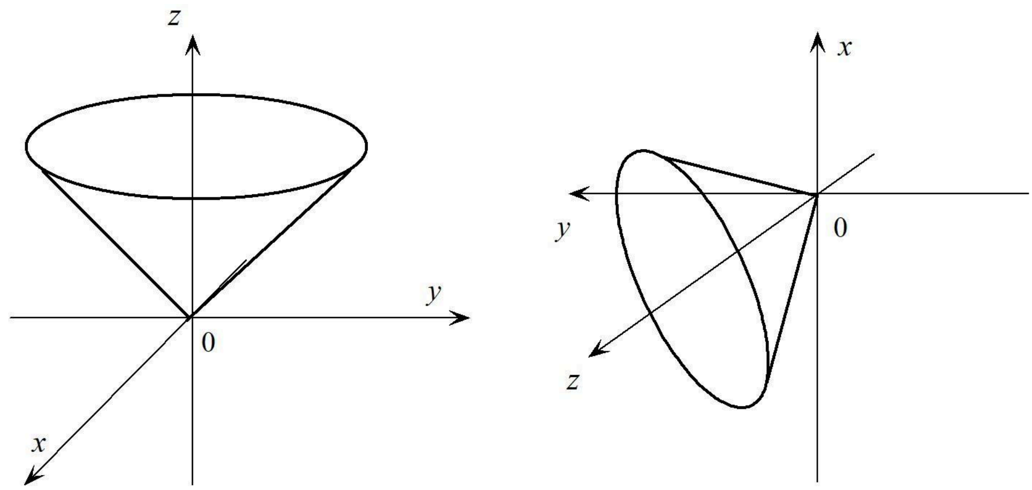


Solution: 1) Let us use the theorem that the area of the region Ω can be found by the formula $S = \iint\limits_{\Omega} dxdy$. Note that the coordinates of the points of intersection of the parabolas $\left\| \begin{matrix} -1 \\ \pm\sqrt{15} \end{matrix} \right\|$.

2) In this case, it is more convenient to take advantage of the elementary nature of the region Ω relative to the axis Ox . In this case we have

$$S = \iint\limits_{\Omega} dxdy = \int\limits_{-\sqrt{15}}^{\sqrt{15}} dy \int\limits_{\frac{y^2-25}{10}}^{\frac{9-y^2}{6}} dx = \int\limits_{-\sqrt{15}}^{\sqrt{15}} x \Bigg|_{\frac{y^2-25}{10}}^{\frac{9-y^2}{6}} dy = \int\limits_{-\sqrt{15}}^{\sqrt{15}} \left(\frac{9-y^2}{6} - \frac{y^2-25}{10} \right) dy = \frac{16\sqrt{15}}{3}.$$

Let us now consider examples of stereometric problems.



Problem 6: Rearrange the re-integration limits in order $\{z, y, x\}$ for the integral

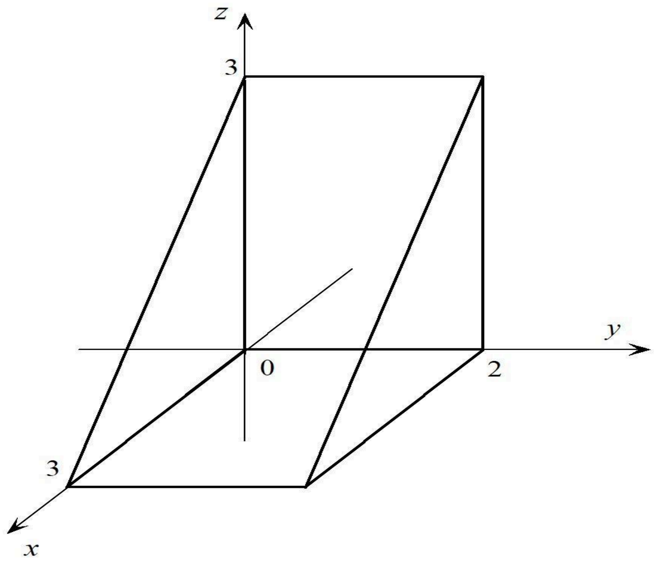
$$I = \iiint\limits_V f(x,y,z)\,dxdydz = \int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz .$$

Solution: 1) Note that the body V bounded below by part of the conical surface $z^2 = x^2 + y^2$, and above the plane $z = 1$.

2) After the indicated change in the order of integration, applying to the area V cutting plane method, we get

$$I = \iiint\limits_V f(x,y,z)\,dxdydz = \int\limits_{0}^{1} dz \int\limits_{-z}^{z} dy \int\limits_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\,dx .$$

Problem 7: Calculate integral $I = \iiint\limits_V \frac{dxdydz}{(x+y+z+1)^3}$, where the area V lies in the positive orthant and is bounded by planes $y=2$ And $x+z=3$.



Solution: 1) In this problem V there is a right triangular prism. Coordinates of points in V satisfy the system of inequalities $\begin{cases} 0 \le z \le 3-x, \\ 0 \le y \le 2, \\ 0 \le x \le 3. \end{cases}$

2) Therefore (check the result yourself)

$$I = \int\limits_0^3 dx \int\limits_0^2 dy \int\limits_0^{3-x} \frac{dz}{(x+y+z+1)^3} = \ln\sqrt{2} - \frac{1}{8}.$$

Problem 8: Calculate integral $I = \iiint\limits_V xy\,dxdydz$, where the area V given by the system of inequalities $\begin{Bmatrix} 0 \le z \le xy, \\ 0 \le x + y \le 1 \end{Bmatrix}$.

Solution: 1) Here V bounded below by a plane $z = 0$, and on top with a fragment of a hyperbolic paraboloid $z = xy$, located in the first quarter of the coordinate plane Oxy, for which the sum of coordinates *x* And *and* does not exceed 1. Sketch of the area V do it yourself as an exercise.

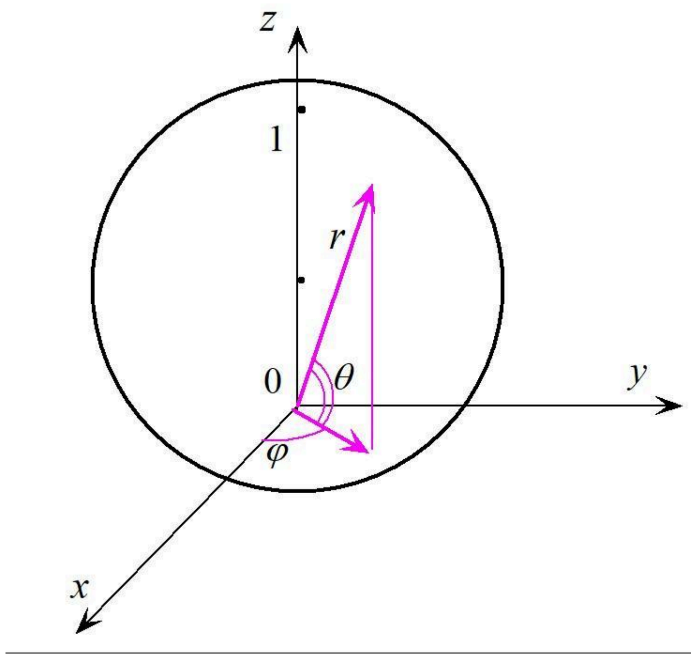
Region V is elementary with respect to all three coordinate axes, therefore the required integral is reduced to repeated ones as follows

$$I = \iiint\limits_V xy\,dxdydz = \int\limits_0^1 x\,dx \int\limits_0^{1-x} y\,dy \int\limits_0^{xy} dz = \int\limits_0^1 x^2\,dx \int\limits_0^{1-x} y^2\,dy = \int\limits_0^1 x^2 \left(\frac{y^3}{3} \Big|_0^{1-x} \right) dx .$$

Where do we get that

$$I = -\frac{1}{3}\int\limits_0^1 (x-1)^3 x^2 dx = -\frac{1}{3}\int\limits_0^1 \left(x^5 - 3x^4 + 3x^3 - x^2\right) dx = -\frac{1}{3}\left(\frac{1}{6} - \frac{3}{5} + \frac{3}{4} - \frac{1}{3}\right) = \frac{1}{180}.$$

Problem 9: Calculate integral $I = \iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz$, where the area V limited by surface $x^2+y^2+z^2=z$.



Solution: 1) Region V there is a ball of radius $\frac{1}{2}$ centered at a point $\left\| \begin{matrix} 0 \\ 0 \\ \frac{1}{2} \end{matrix} \right\|$, and the canonical equation of its surface $x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$.

2) Let's go to *spherical coordinate system* according to formulas

$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\sin\varphi\cos\theta \\ z = r\sin\theta \end{cases}$$

with the modulus of the Jacobian $|J| = r^2\cos\theta$.

3) Since on the border of the region V we have $r^2 = r\sin\theta$, then it will be here $r = \sin\theta$.

Where do we get that in spherical coordinates $V^* = \left\{ \begin{array}{l} 0 \le \varphi \le 2\pi, \\ 0 \le \theta \le \dfrac{\pi}{2}, \\ 0 \le r \le \sin\theta. \end{array} \right\}$.

4) In the end

$$I = \iiint\limits_{V} \sqrt{x^2 + y^2 + z^2}\, dxdydz = \iiint\limits_{V^*} r^3 \cos\theta \, dr\, d\varphi\, d\theta =$$

$$= \int\limits_0^{\frac{\pi}{2}} \cos\theta \, d\theta \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sin\theta} r^3 dr = \frac{1}{4} \int\limits_0^{\frac{\pi}{2}} \sin^4\theta \cos\theta \, d\theta \int\limits_0^{2\pi} d\varphi = \frac{\pi}{10}.$$