CURVILINEAR AND SURFACE INTEGRALS FUNCTIONS OF MANY VARIABLES

In E^3 with ONB :	**Scalar field** $f(\vec{r}) = f(x,y,z)$	**Vector field** $\vec{F}(\vec{r}) = \begin{cases} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{cases}$
Line Γ $\vec{r} = \vec{r}(t)\ t \in [\alpha,\beta]$ or in coordinates $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ Functions $x(t)$, $y(t)$, $z(t)$ are continuously differentiable on $t \in [\alpha,\beta]$	*Curvilinear integral of the first kind* $\int_\Gamma f(x,y,z)\,dl =$ $= \int_\alpha^\beta f(x(t),y(t),z(t))\sqrt{x_t'^2 + y_t'^2 + z_t'^2}\,dt$	*Curvilinear integral of the second kind* $\int_\Gamma P\,dx + Q\,dy + R\,dz =$ $= \int_\alpha^\beta (P(x(t),y(t),z(t))x_t' + Q(x(t),y(t),z(t))y_t' + R(x(t),y(t),z(t))z_t')\,dt$
Surface S $\vec{r} = \vec{r}(u,v)\quad (u;v) \in \Omega \subseteq E^2$ or in coordinates $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ Functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are continuously differentiable on $(u;v) \in \Omega \subseteq E^2$	*Surface integral of the first kind* $\iint_S f(x,y,z)\,ds =$ $= \iint_\Omega f(x(u,v),\ y(u,v),\ z(u,v))\sqrt{EG - F^2}\,du\,dv$, where $E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$ $G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$ $F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$	*Surface integral of the second kind* $\iint_S P\,dydz + Q\,dzdx + R\,dxdy =$ $= \iint_\Omega \begin{vmatrix} P(x(u,v),y(u,v),z(u,v)) & Q(\ldots\ldots) & R(\ldots\ldots) \\ x_u' & y_u' & z_u' \\ x_v' & y_v' & z_v' \end{vmatrix} du\,dv$

Table 5.1

Properties of curvilinear and surface integrals

A curvilinear integral of the first kind does not change sign when changing the direction of traversal, but of the second kind it changes sign.

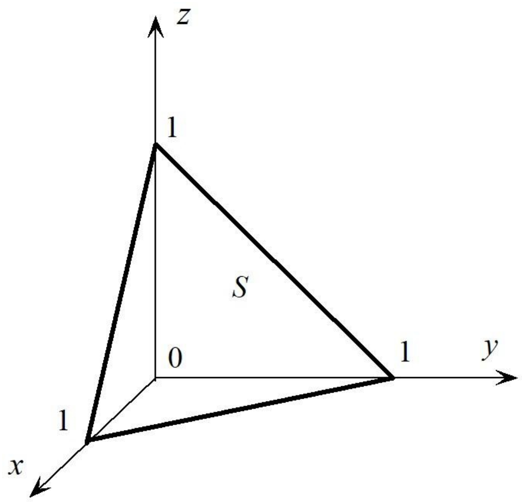
A surface integral of the first kind does not change sign when the surface orientation changes, but a surface integral of the second kind does not change sign.

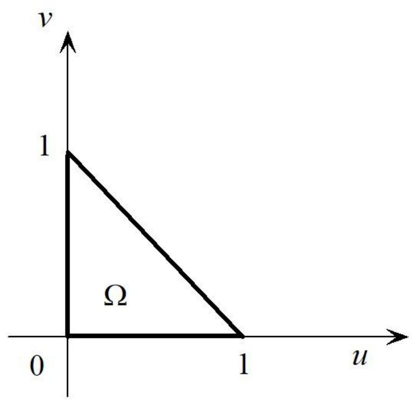
Properties of the second kind curvilinear integral of the total differential:

− depends only on the starting and ending points of the line and does not depend on the shape of the line,

− the value along a closed contour (within which there are no special points) is equal to zero.

Example 01 .Calculate $I = \iint\limits_S xyz\, ds$, Where $S: \begin{cases} x+y+z=1, \\ x \ge 0, \\ y \ge 0, \\ z \ge 0 \end{cases}$.





Solution: 1) Parameterize the surface $S: \begin{cases} x(u,v) = u, \\ y(u,v) = v, \\ z(u,v) = 1-u-v \end{cases}$,
Where $(u,v) \in \Omega$.

2) In our case $f(x(u,v), y(u,v), z(u,v)) = uv - u^2 v - uv^2$,.

3) Find $E = 2, G = 2, F = 1$ $\sqrt{EG - F^2} = \sqrt{3}$.

4) Formulate and calculate the integral $I = \iint\limits_\Omega (uv - u^2 v - uv^2)\sqrt{3}\, dudv = \dfrac{\sqrt{3}}{120}$.

Example 02 (1 method) .Calculate $I = \iint\limits_S z\, dxdy$, Where S - ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, Where $a > 0, b > 0, c > 0$.

Solution: 1) Parameterize the surface $S: \begin{cases} x(u,v) = a\cos u \cos v, \\ y(u,v) = b \sin u \cos v, \\ z(u,v) = c \sin v, \end{cases}$

Where $(u,v) \in \Omega = \begin{cases} 0 \le u < 2\pi \\ -\frac{\pi}{2} \le v \le \frac{\pi}{2} \end{cases}$.

2) Formulate and calculate the integral

$$I = \iint\limits_{\Omega} \det \begin{Vmatrix} 0 & 0 & c\sin v \\ -a\sin u \cos v & b\cos u \cos v & 0 \\ -a\cos u \sin v & -b \sin u \sin v & c\cos v \end{Vmatrix} dudv = \iint\limits_{\Omega} Hc \sin v\, dudv,$$

Where $H = ab \sin v \cos v$.

Thus, $I = \iint\limits_{\Omega} abc \sin^2 v \cos v\, dudv = \frac{4\pi}{3} abc$.

Example 02 (2nd method) .Calculate $I = \iint\limits_S z\,dxdy$, Where S - ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, Where $a > 0, b > 0, c > 0$.

Solution: 1) In this case, the surface over which the integral is taken is formed by the graphs of the functions $z(x,y) = \pm c\sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}$, and we represent the integral itself in the form

$$I = \pm c \iint\limits_D \sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}\,dxdy$$

Where D − domain of definition of the function $z(x,y)$.

The sign in front of the integral is taken "+" if the normal vector forms an acute angle with the axis *With*, and "−" if this angle is obtuse.

2) Let us first take the integral over the upper half of the ellipsoid, moving to generalized polar coordinates:

$$\begin{cases} x = ar\cos\varphi, \\ y = br\sin\varphi. \end{cases}$$

Check for yourself that the module of the Jacobian with such a replacement is equal to abr, and the area D^* there is a rectangle $\{0 \le r \le 1,\, 0 \le \varphi < 2\pi\}$. Then we get

$$I^+ = abc\iint\limits_{D^*} \sqrt{1-r^2}\, r\, dr d\varphi = abc\int\limits_0^{2\pi} d\varphi \int\limits_0^1 \sqrt{1-r^2}\, r\, dr = \pi abc\int\limits_0^1 \sqrt{1-r^2}\, dr^2 =$$

$$= \pi abc\left(-\frac{2\left(1-r^2\right)^{3/2}}{3} \Bigg|_0^1 \right) = \frac{2\pi}{3} abc.$$

For the integral over the lower part of the ellipsoid we have $z(x,y) \le 0$ and the obtuse angle between the outer normal vector and the positive direction of the axis Oz. This means that the integral over the lower half of the surface I^- will be equal to the integral over the upper I^+.

Finally we get $I = I^+ + I^- = \dfrac{4\pi}{3} abc$.

GREEN'S FORMULA

Definition and properties

Let ∂G there is a piecewise smooth contour, which is the boundary of a flat bounded region G.

Then, if the functions $P(x,y)$ And $Q(x,y)$ continuously differentiable in G, then it is fair *Green's formula*

$$\iint\limits_{G}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint\limits_{\partial G}Pdx+Qdy$$

The direction of bypassing the contour is such that during the bypass the area G remains *left*.

Important: continuous differentiability of functions $P(x,y)$ And $Q(x,y)$ required not only on a piecewise smooth contour ∂G, but also *inside the whole* region G.

Let Γ_{AB} some piecewise smooth line entirely lying in the region G, and A is its beginning, and B is its end. Then the value of the integral $\int\limits_{\Gamma_{AB}} Pdx + Qdy$ *does not depend on the shape of the integration trajectory* then and only when $\exists u(x,y)$ such that $du = Pdx + Qdy$.

In this case

$$\int\limits_{\Gamma_{AB}} Pdx + Qdy = u(B) - u(A)$$
.

Necessary condition for the independence of the value of the integral from the path of integration There is $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. And if the area G *simply connected*, then this condition is sufficient.

Finally, from Green's formula it follows that the area of the region G may be
according to the formula

$$S = \iint\limits_G dxdy = \frac{1}{2}\int\limits_{\partial G} xdy - ydx$$
.

Example 03. Calculate $\oint\limits_{\partial G}(1-x^2)y\,dx + x(1+y^2)\,dy$, Where $G: \ \{x^2+y^2 \le R^2\}$.

Solution: We have

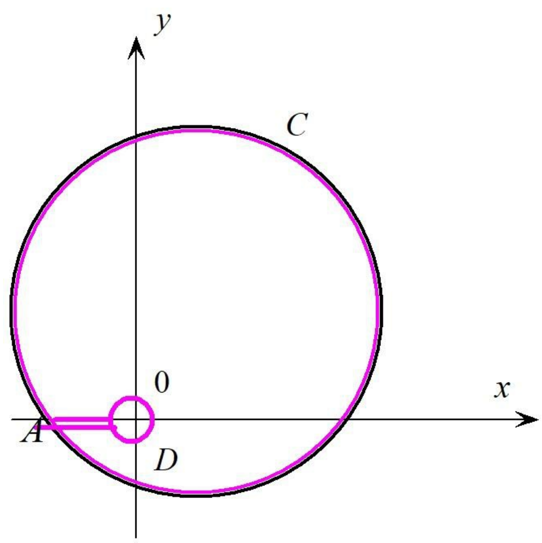
$$\oint\limits_{\partial G}(1-x^2)y\,dx + x(1+y^2)\,dy = \iint\limits_{G}(1+y^2-1+x^2)\,dxdy = \iint\limits_{G}(x^2+y^2)\,dxdy =$$

moving to polar coordinates with $J = r\,drd\varphi$,

$$= \iint\limits_{G^*} r^2 r\,drd\varphi = \int\limits_0^{2\pi} d\varphi \int\limits_0^R r^3 dr = \frac{\pi R^4}{2}.$$

Example 04. Calculate $I_C = \oint\limits_{\partial G} \dfrac{xdy - ydx}{x^2 + y^2}$, Where ∂G - a simple contour that does not pass through the origin and goes around the region *G*, leaving her on the left..

Solution: We have $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{x^2 + y^2} = \dfrac{\partial P}{\partial y}$. Does this mean that $I_C = 0$?
\ Not necessary!



The point is that the original ("black") integration contour may have *internal singular point* 0.

If the circuit C does not cover this point, then the answer will be $I_C = 0$.

If the special point is inside the “black” contour, then we will construct a “purple” contour (as shown in the figure) by adding a cut $A0$ and a sufficiently small circle enclosing the singular point.

Let the integrals be equal:

I_+ - along the “upper bank” of the section,
I_- - along the “lower bank” of the section,
I_D - in a “small circle” around 0,
I_L - along the "purple" contour.

We have $I_- = -I_+$.

I_D Let's calculate it directly. Let the parameterization of the "small circle" be chosen as follows: $\begin{cases} x(t) = a\cos t, \\ y(t) = a\sin t \end{cases} \quad t \in [0, 2\pi)$,
Where a is a fairly small positive number.

Then, taking into account that when going around the circle D, it remains on the right, we get

$$I_D = \oint_D \frac{xdy - ydx}{x^2 + y^2} = \int_{2\pi}^{0} \frac{a\cos t \cdot a\cos t - a\sin t \cdot (-a\sin t)}{a^2(\cos^2 t + \sin^2 t)} dt = -\int_0^{2\pi} dt = -2\pi .$$

Finally, $I_L = 0$, since there are no special points inside the "purple" contour.

From the additivity property of the integral it follows that $I_L = I_+ + I_D + I_- + I_C = 0$.
Then the required integral will be equal to

$$I_C = 2\pi .$$

SELECTION OF SURFACE SIDE
(*Coordinate system right rectangular!*)

Parameterization of a smooth surface $S: \quad \vec{r} = \vec{r}(u,v) \quad (u;v) \in \Omega \subseteq E^2$, or in a right-handed rectangular coordinate system: $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$, determines not only *surface*, but also *her side* direction of the normal vector $\vec{n}$.

Indeed, let the functions $x(u,v), y(u,v), z(u,v)$ continuously differentiable in Ω and let in Ω point selected (u_0, v_0).

Let us introduce the vectors $\vec{r}_u' = \left\| \begin{matrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{matrix} \right\|$ And $\vec{r}_v' = \left\| \begin{matrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{matrix} \right\|$. If they are non-collinear, then their cross product is non-zero *normal vector* $\vec{n} = [\vec{r}_u', \vec{r}_v']$. Such a point (u_0, v_0) usually called *unremarkable*. The normalized normal vector is called *orientation* surface at a point (u_0, v_0).

At each nonsingular point, the surface can have only two orientations: *positive* $+\frac{\vec{n}}{|\vec{n}|}$ And *negative* $-\frac{\vec{n}}{|\vec{n}|}$. The choice of one of the orientations determines *side* surfaces.

Note, finally, that the orientation can be *continuous* may or may not be a function of a surface point.

An example of the second case is the well-known *Möbius strip*, the parametric form of which can, for example, have the following form:

$$\begin{cases} x(u,v) = (1 + v\cos u)\cos 2u, \\ y(u,v) = (1 + v\cos u)\sin 2u, \\ z(u,v) = v\sin u, \end{cases}$$ with the area $$\Omega = \left\{ \begin{array}{c} 0 \le u < \pi, \\ -\frac{1}{2} \le v \le \frac{1}{2}. \end{array} \right\}$$

Direction cosines

From the course of linear algebra it is known that in ONB $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for vector coordinates $\vec{r}$ there are equalities $x_i = (\vec{r}, \vec{e}_i) \quad i = 1,2.3$.

From this formula it follows that if $\vec{n}$ is a normalized (unit length) vector, then $\vec{n} = \begin{Vmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{Vmatrix}$, Where α, β, γ − angles between the vector $\vec{n}$ and orts $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Example 05: Cfera S : radius R and with the center at the origin can be parametrically specified as follows:

$$\begin{cases} x = R\cos u\cos v \\ y = R\sin u\cos v \\ z = R\sin v \end{cases} \quad \Omega = \begin{cases} 0 \le u \le 2\pi \\ -\dfrac{\pi}{2} \le v \le \dfrac{\pi}{2} \end{cases}.$$

At the same time $\vec{n} = R^2 \begin{Vmatrix} \cos u\cos^2 v \\ \sin u\cos^2 v \\ \sin v\cos v \end{Vmatrix}$ and it is clear that when $v \ne \pm\dfrac{\pi}{2}$ will $\vec{r} = k\vec{n}$, Where $k = R\cos v > 0$. So this is *positive* orientation and *external* normal

Please note that this parameterization is not *one-to-one* display $\Omega \to S$.

A surface integral of the second kind, in the case where the orientation of the surface (that is, a continuous vector function $\vec{n}(\vec{r})$), given, can be expressed through a surface integral of the first kind.

Indeed, let $\vec{n} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix} = \vec{e}_1 \cos\alpha + \vec{e}_2 \cos\beta + \vec{e}_3 \cos\gamma$, then the surface integral of the second kind is equal to (this is a definition consistent with the formula in table 5.1!)

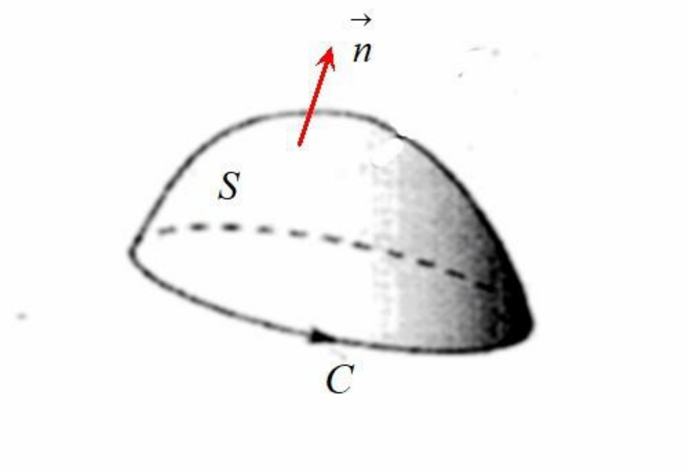
$$\iint_S P\,dydz + Q\,dzdx + R\,dxdy = \iint_S \left(P\cos\alpha + Q\cos\beta + R\cos\gamma\right)ds$$,

since it is geometrically clear that in an orthonormal coordinate system for a plane figure S, having area ds (and in the limit, for smooth S) the equalities are valid

$$\begin{cases} dxdy = \dfrac{\partial(x,y)}{\partial(u,v)}dudv = \cos\gamma\, ds, \\ dydz = \dfrac{\partial(y,z)}{\partial(u,v)}dudv = \cos\alpha\, ds, \\ dzdx = \dfrac{\partial(z,x)}{\partial(u,v)}dudv = \cos\beta\, ds. \end{cases}$$

Stokes formula

$$\oint_C Pdx + Qdy + Rdz = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx$$



In the Stokes formula, the direction of traversal of the contour C and the direction of the normal to the surface S must be *agreed upon*.

Agreement means that:

> observer moving in the direction of traversing the contour C so that in the direction of the normal S from "feet to head", sees the surface S to your left.

Note that when $dz = 0$ Stokes' formula turns into Green's formula.

Another way to write the Stokes formula

$$\oint_C Pdx + Qdy + Rdz = \iint_S \det \begin{Vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{Vmatrix} ds \, ,$$

Where the first row of the determinant matrix contains the direction cosines of the normalized vector $\vec{n}$, ensuring coordination.

Example 06. Calculate $I_C = \oint\limits_C y\,dx + z\,dy + x\,dz$, Where C - circumference $\begin{cases} x^2 + y^2 + z^2 = a^2,\ a > 0 \\ x + y + z = 0 \end{cases}$,

oriented counterclockwise when viewed from the end of the axis $0x$.

Solution: 1) Integral I_C can be calculated simply by the definition of a curvilinear integral of the second kind. However, in this case it is necessary to find a parametric description of the circle *C*.

Using the Stokes formula makes it easier to solve the problem. Indeed, since the surface S can be any, then as S let's take part of the plane $x + y + z = 0$, limited by contour C.

According to the agreement rule, in our case the normalized vector $\vec{n} = \frac{1}{\sqrt{3}} \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}$ is the same for all points S, and the vector field has the form $\vec{F} = \begin{Vmatrix} y \\ z \\ x \end{Vmatrix}$.

2) For derivatives of a vector field

$$\begin{cases} \dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z}=-1 \\ \dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x}=-1 \\ \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}=-1 \end{cases}.$$

Therefore, finally, using the Stokes formula, we find

$$I_C=\iint\limits_S\left((-1)\frac{1}{\sqrt{3}}+(-1)\frac{1}{\sqrt{3}}+(-1)\frac{1}{\sqrt{3}}\right)ds=-\frac{3}{\sqrt{3}}\iint\limits_S ds=-\pi\, a^2\sqrt{3}.$$

Gauss-Ostrogradsky formula

Let S piecewise smooth boundary of a closed region V with *continuously differentiable* vector field, then the formula is valid *Gauss-Ostrogradsky*

$$\iint\limits_{S} P\,dydz + Q\,dzdx + R\,dxdy = \iiint\limits_{V} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz \quad ,$$

If S $-$ *external* side of the area border V .

Example 02 (3rd method) Calculate $I = \iint\limits_S z\,dxdy$, Where S - ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, Where $a>0, b>0, c>0$.

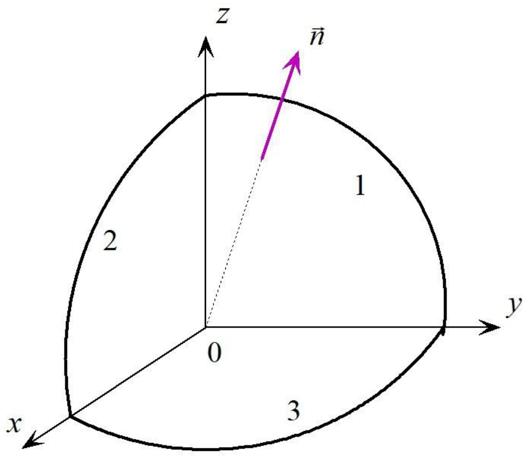
Solution: Since the surface of the ellipsoid is closed and smooth, we apply the Gauss-Ostrogradsky formula. For a vector field with $\vec{F}(x,y,z) = \begin{Vmatrix} 0 \\ 0 \\ z \end{Vmatrix}$ we have $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=1$. That's why $I = \oiint\limits_S z\,dxdy = \iiint\limits_V dxdydz$.

The last integral is equal to the volume of the body bounded by the surface *S*. We know that the volume of a body bounded by an ellipsoid is equal to $\frac{4\pi}{3}abc$. So, finally,

$$I = \frac{4\pi}{3}abc.$$

Example 07. Find $I = \iint_S x^2 y\, dydz + xy^2 dzdx + xyz\, dxdy$,
Where S - part of a sphere $x^2 + y^2 + z^2 = R^2, R > 0$,

located in the positive octant.



Solution: 1) Since the surface S open, then we will make it closed by adding parts of coordinate planes 1, 2 and 3 to it, as shown in the figure.

2) Note that on coordinate planes 1 and 2 the surface integral *null*, since the vector field is zero, because Here $P = Q = R = 0$.

On a flat boundary 3 the surface integral is also *equal to zero*, in force $\cos\alpha = \cos\beta = 0$ And $R = 0$.

3) This means that we can apply the Gauss-Ostrogradsky formula for a closed region V. We have

$$\begin{cases} P(x,y,z) = x^2 y \quad \Rightarrow \quad \dfrac{\partial P}{\partial x} = 2xy, \\ Q(x,y,z) = xy^2 \quad \Rightarrow \quad \dfrac{\partial Q}{\partial y} = 2xy, \\ R(x,y,z) = xyz \quad \Rightarrow \quad \dfrac{\partial R}{\partial z} = xy. \end{cases}$$

From where, passing in the triple integral to spherical coordinates

$$\begin{cases} x = r\cos\varphi\cos\psi, \\ y = r\sin\varphi\cos\psi, \\ z = r\sin\psi, \end{cases}$$

get

$$I = \iiint\limits_V 5xy\,dxdydz = 5\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{\frac{\pi}{2}} d\psi \int\limits_0^{R} (r\cos\varphi\cos\psi)\cdot(r\sin\varphi\cos\psi)\,r^2\cos\psi\,dr =$$

$$= R^5 \int\limits_0^{\frac{\pi}{2}} \cos\varphi\sin\varphi\,d\varphi \int\limits_0^{\frac{\pi}{2}} \cos^3\psi\,d\psi = \frac{1}{3}R^5.$$