Elements of field theory

Let Ω − area in E^3 with *right rectangular* Cartesian coordinate system. We will denote the radius vector of a point in this area as $\vec{r}$, with a coordinate representation of the form $\left\| \vec{r} \right\| = \left\| \begin{matrix} x \\ y \\ z \end{matrix} \right\|$.

Let in the area Ω given *scalar* field $f(x,y,z)$ And *vector* field $\vec{F}(x,y,z)$ with coordinate representation $\left\| \vec{F} \right\| = \left\| \begin{matrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{matrix} \right\|$.

Let us introduce (by definition) *vector differential operator* ∇ called "*nabla*"and having a coordinate representation in the ONB $\|\nabla\| = \left\| \begin{matrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{matrix} \right\|$.

When using this operator, keep in mind that

1°. ∇ behaves like a vector that satisfies all the rules for operations with vectors;

2°. ∇ is a differential operator that obeys the rules of differentiation. In this case, the subject of his action must be an object located *in the formula to the right of* ∇ .

In the case when the object of action of the nabla operator is not uniquely determined, it is necessary to indicate it explicitly, highlighting this object in the formula with an upper vertical arrow.

Example 1. Write the expression in coordinate form $(\nabla,\varphi\vec{W})$, Where $\varphi(x,y,z)$ is a continuously differentiable scalar field, and $\vec{W}(x,y,z)$ – vector field, also continuously differentiable, with coordinate representation $\left\| \vec{W} \right\| = \left\| \begin{matrix} W_x(x,y,z) \\ W_y(x,y,z) \\ W_z(x,y,z) \end{matrix} \right\|$.

Solution; Since nabla is a differential operator, using the properties of the scalar product of a vector and the orthonormality of the basis, we obtain

$$(\nabla,\varphi\vec{W}) = (\nabla,\overset{\downarrow}{\varphi}\vec{W}) + (\nabla,\varphi\overset{\downarrow}{\vec{W}}) = (\nabla\varphi,\vec{W}) + \varphi(\nabla,\vec{W}) =$$

$$= \frac{\partial\varphi}{\partial x}W_x + \frac{\partial\varphi}{\partial y}W_y + \frac{\partial\varphi}{\partial z}W_z + \varphi\left(\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z}\right).$$

One of the main characteristics used in the description *scalar* field is his *gradient*.

Definition: *Scalar field gradient* $f(x,y,z)$ is called a vector function of the form $\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$. This function has a coordinate representation

$$\left\| \operatorname{grad} f \right\| = \left\| \begin{matrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{matrix} \right\| .$$

Note that using the nabla operator, the gradient vector can be written as follows:

$$\operatorname{grad} f = \nabla f .$$

To describe the properties of vector fields in applications, the following two characteristics are also often used: scalar, called *divergence*, and vector, called *rotor* (Sometimes, *rotation* or *whirlwind*).

Definition: *Divergence* vector field $\vec{F}(x,y,z)$ called a scalar function $\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, which using Nabla is given by the formula $\operatorname{div}\vec{F} = (\nabla, \vec{F})$.

Definition: *with a rotor* vector field $\vec{F}(x,y,z)$ called a vector function

$$\operatorname{rot}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \det\begin{Vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{Vmatrix},$$

which, using nabla, is given by the formula $\operatorname{rot}\vec{F} = [\nabla, \vec{F}]$.

It is easy to verify that, due to the definitions made, the following equalities are true: $\operatorname{div} \vec{r} = 3$ And $\operatorname{rot} \vec{r} = \vec{o}$.

The introduced characteristics of scalar and vector fields allow the use of alternative formulations for:

1) theorems *Stokes*

$$\oint\limits_{\partial S} (\vec{F}, d\,\vec{r}\,) = \iint\limits_{S} (\vec{n}, \operatorname{rot} \vec{F}\,)\, ds$$

or

$$\oint\limits_{\partial S} (\vec{F}, d\,\vec{r}\,) = \iint\limits_{S} (\vec{n}, [\nabla, \vec{F}])\, ds = \iint\limits_{S} (\vec{n}, \nabla, \vec{F}\,)\, ds ,$$

2) theorems *Gauss-Ostrogradsky*

$$\oiint\limits_{\partial\Omega} (\vec{F}, \vec{n})\, ds = \iiint\limits_{\Omega} \operatorname{div} \vec{F}\, dV$$

or

$$\oiint\limits_{\partial\Omega} (\vec{F}, \vec{n})\, ds = \iiint\limits_{\Omega} (\nabla, \vec{F}\,)\, dV .$$

Classification of vector fields

Vector fields are usually classified using quantitative characteristics called *circulation* And *flow,* and also *potential*. We will assume that all fields are continuously differentiable, and lines and surfaces are piecewise smooth.

Definition: *Line integral* (*or work*) of the vector field $\vec{F}(x,y,z)$ along the line *G* called integral

$$\int_{\Gamma} (\vec{F}, d\,\vec{r}) = \int_{\Gamma} P\,dx + Q\,dy + R\,dz \quad .$$

If the line *closed*, then the integral $\int_{\Gamma} (\vec{F}, d\,\vec{r})$ called *circulation* fields $\vec{F}(x,y,z)$ along the line *G*

Using the properties of curvilinear and surface integrals allows us to obtain a description of the properties of scalar and vector fields in *integral* form.

1^0. For a scalar field f and lines G, coming from the point A to the point B, in some area Ω, the equality is true $\int\limits_{\Gamma}(\operatorname{grad} f, d\vec{r}) = f(B) - f(A)$. In particular, for the vector field $\operatorname{grad} f$ its circulation along any closed circuit is zero.

2^0. For a vector field $\vec{F}$ and surfaces Soriented by the unit normal vector $\vec{n}$, having a consistently oriented edge ∂S, Stokes' theorem gives $\oint\limits_{\partial S}(\vec{F}, d\vec{r}) = \iint\limits_{S}(\vec{n}, \operatorname{rot}\vec{F})\, ds$.

That is, *circulation* $\vec{F}$ *along the closed edge of the surface is equal to the flux of the field rotor through this surface*.

3^0. According to the Gauss-Ostrogradsky formula $\iint\limits_{\partial V}(\vec{F}, \vec{n})\, ds = \iiint\limits_{V}\operatorname{div}\vec{F}\, dV$.

In other words, *flow of a vector field* $\vec{F}$ *through the outside of a closed surface* ∂V*, bounding the body V, is equal to the triple integral of the field divergence over this body.*

Vector fields are also usually classified according to their properties. In particular, field theory uses the concepts *potential*, *vortex-free* And *solenoidal* vector field.

Definition: Vector field $\vec{F}(x,y,z)$ called *potential*, if such a function exists $f(x,y,z)$, What $\vec{F} = \operatorname{grad} f$. Function $f(x,y,z)$ in this case called *potential* vector field $\vec{F}(x,y,z)$.

A necessary and sufficient condition for the potentiality of the field $\vec{F}(x,y,z)$ in the area Ω is the equality to zero of its circulation along any closed contour G V Ω, i.e. $\oint\limits_{\Gamma}(\vec{F}, d\vec{r})$. Condition $\operatorname{rot}\vec{F} = \vec{o}$ is only necessary. It turns out to be sufficient in the case of a simply connected domain Ω.

Definition: Vector field $\vec{F}(x,y,z)$ called *irrotational*, If $\operatorname{rot}\vec{F} = \vec{o}$.

In any simply connected region, the irrotational field is potential.

Definition: Vector field $\vec{F}(x,y,z)$ called *solenoidal*, if for any closed region $V \subset \Omega$ with border ∂V The field flux across this boundary is zero, i.e. $\iint\limits_{\partial V} (\vec{F},\vec{n})\, ds = 0$.

For field solenoidality $\vec{F}$ in the area Ω it is necessary and sufficient that in this area $\operatorname{div}\vec{F} = 0$.

Let us note, in conclusion, an interesting fact (called the Helmholtz theorem in field theory) that any continuously differentiable field can be represented as the sum of irrotational and solenoidal fields.

Example 2. Find using the nabla operator,

1) $\text{rot}\,\text{grad}\, f$,

2) $\text{div}\,\text{rot}\,\vec{F}$,

3) $\text{div}\,\text{grad}\, f$,

Solution: Considering that nabla simultaneously has the properties of a vector and a differential operator, we obtain:

1) Let's take advantage of the fact that we can derive a scalar factor from the vector product, which we must write down *right* from the operator acting on it, and also by the fact that the vector product of a vector with itself is equal to the zero vector. Then we get

$$\text{rot}\,\text{grad}\, f = [\nabla, \nabla f] = [\nabla, \nabla] f = \vec{o} \quad .$$

2) In this case, we will use the possibility of cyclic rearrangement of factors in the mixed product and the commutativity of the scalar product to comply with the requirement for the location of the operator. As a result we have:

$$\text{div}\,\text{rot}\,\vec{F} = (\nabla, [\nabla, \vec{F}]) = ([\nabla, \nabla], \vec{F}) = (\vec{o}, \vec{F}) = 0 \quad .$$

3) We have

$$\operatorname{div}\operatorname{grad} f = \operatorname{div}\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) = (\nabla, \nabla f) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = \Delta f \ ,$$

Where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ - Laplace operator.

Example 3. For known vector fields $\vec{A}(x,y,z)$ And $\vec{B}(x,y,z)$ find $\operatorname{div}[\vec{A},\vec{B}]$.

Solution: We use the top arrow to indicate the object on which the nabla operator acts. In this case, you can write:

$$\operatorname{div}[\vec{A},\vec{B}] = (\nabla,\overset{\downarrow}{[\vec{A},\vec{B}]}) = (\nabla,[\overset{\downarrow}{\vec{A}},\vec{B}]) + (\nabla,[\vec{A},\overset{\downarrow}{\vec{B}}]) =$$

$$= (\vec{B},[\nabla,\overset{\downarrow}{\vec{A}}]) + (\vec{A},[\overset{\downarrow}{\vec{B}},\nabla]) = (\vec{B},[\nabla,\overset{\downarrow}{\vec{A}},]) - (\vec{A},[\nabla,\overset{\downarrow}{\vec{B}}]) =$$

$$= (\vec{B},\operatorname{rot}\vec{A}) - (\vec{A},\operatorname{rot}\vec{B}).$$

Example 4. For known fields $f(x,y,z)$ And $\vec{F}(x,y,z)$ find $\operatorname{rot}(f\,\vec{F})$.

Solution: Again we use the top arrow to indicate the object on which the nabla operator is acting. Then we have:

$$\operatorname{rot}(f\,\vec{F}) = [\nabla,\overset{\downarrow}{(f\,\vec{F})}] = [\nabla,(\overset{\downarrow}{f}\,\vec{F})] + [\nabla,(f\,\overset{\downarrow}{\vec{F}})] =$$

$$= [\nabla f,\vec{F}] + f[\nabla,\vec{F}] = [\operatorname{grad} f,\vec{F}] + f\operatorname{rot}\vec{F}\ .$$

Example 5. 1) Find $\operatorname{div}\operatorname{grad} f(r)$, Where $r = \sqrt{x^2 + y^2 + z^2}$.

2) For what scalar field will $\operatorname{div}\operatorname{grad} f(r) = 0$?

Solution: `1) Let us introduce the notation $P = \frac{\partial f(r)}{\partial x}; \quad Q = \frac{\partial f(r)}{\partial y}; \quad R = \frac{\partial f(r)}{\partial z}$. Then we have

$$P = \frac{\partial f(r)}{\partial x} = \frac{df}{dr} \cdot r'_x = f'(r) \cdot \frac{x}{r}$$

$$Q = \frac{\partial f(r)}{\partial y} = f'(r) \cdot \frac{y}{r};$$

$$R = \frac{\partial f(r)}{\partial z} = f'(r) \cdot \frac{z}{r}.$$

Where $\operatorname{grad} f(r) = \frac{f'(r)}{r} \vec{r}$.

On the other side,

$$\frac{\partial P}{\partial x}=\frac{f'(r)}{r}+x\frac{\partial}{\partial x}(\frac{f'(r)}{r})=\frac{f'(r)}{r}+x\frac{d}{dr}(\frac{f'(r)}{r})\cdot r'_x=$$

$$=\frac{f'(r)}{r}+x\frac{rf''(r)-f'(r)}{r^2}\cdot\frac{x}{r}=\frac{f'(r)}{r}+x^2\frac{rf''(r)-f'(r)}{r^3}.$$

Similarly we obtain that

$$\frac{\partial Q}{\partial y}=\frac{f'(r)}{r}+y^2\frac{rf''(r)-f'(r)}{r^3}$$

And

$$\frac{\partial R}{\partial z}=\frac{f'(r)}{r}+z^2\frac{rf''(r)-f'(r)}{r^3}$$

Whence it follows that

$$\operatorname{div}\operatorname{grad} f(r)=\frac{3f'(r)}{r}+\frac{rf''(r)-f'(r)}{r^3}r^2=f''+\frac{2f'(r)}{r}$$

2) Now let’s find for which field $\operatorname{div}\operatorname{grad} f(r) = 0$.

Let's solve the equation $f'' + \frac{2f'(r)}{r} = 0$.

Let's lower the order by replacing $u(r) = f'(r)$. We obtain an equation with separable variables $u' + \frac{2u}{r} = 0$. This gives, for example, by the method of separation of variables, $u(r) = \frac{C_1}{r^2}$. Where, finally,

$$f(r) = \frac{C_1}{r} + C_2 \quad \forall C_1, C_2 .$$

Example 6. Show that the flow of a vector field $\vec{F} = \vec{r}$ through any piecewise smooth closed surface is equal to three times the volume of the body bounded by this surface.

Solution: `The validity of the statement follows from the Gauss-Ostrogradsky formula and the equality $\operatorname{div} \vec{r} = 3$.

Example 7. Show that from Maxwell's equations for an electromagnetic field in a vacuum it follows that the strengths of both electric and magnetic fields satisfy a uniform *harmonic equation*.

Solution: `1) Maxwell's equations in this case can be written in the form

$$\begin{cases} \operatorname{rot} \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \operatorname{div} \vec{E} = 0, \\ \operatorname{rot} \vec{B} = \dfrac{\partial \vec{E}}{\partial t} & \operatorname{div} \vec{B} = 0. \end{cases}$$

If we take the rotor from both sides of the first equation, we get the equality
\

$$\operatorname{rot}\operatorname{rot}\vec{E} = -\frac{\partial}{\partial t}\operatorname{rot}\vec{B}\,,$$

the right side of which is, by virtue of the third equation $-\dfrac{\partial^2 \vec{E}}{\partial t^2}$, and the left one is calculated using the nabla operator and the formula for the double vector product

$$\operatorname{rot}\operatorname{rot}\vec{E} = [\nabla,[\nabla,\vec{E}]] = \nabla(\nabla,\vec{E}) - (\nabla,\nabla)\vec{E} = \operatorname{grad}\ \operatorname{div}\vec{E} - \Delta\vec{E}\,.$$

Since, due to the second equation of the system $\operatorname{div}\vec{E} = 0$, then we arrive at the equation $\Delta\vec{E} = \dfrac{\partial^2 \vec{E}}{\partial t^2}$.

For field $\vec{B}$ the reasoning is similar.

Example 8. Let $r=\sqrt{x^2+y^2+z^2}$. For which differentiable function $\Phi(r)$ view field $\vec{F}=\Phi(r)\,\vec{r}$ will it be solenoidal?

Solution: `1) Necessary and sufficient condition for the field to be solenoidal $\vec{F}$ looks like $\operatorname{div}\vec{F}=0$ or $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$. Therefore, first we find the partial derivatives of the components $\vec{F}$.

Since $\left\|\vec{F}\right\|=\left\|x\Phi(r)\quad y\Phi(r)\quad z\Phi(r)\right\|^{\mathrm{T}}$, then we have

$$\frac{\partial P}{\partial x}=\Phi'_r\frac{x^2}{r}+\Phi,$$
$$\frac{\partial Q}{\partial y}=\Phi'_r\frac{y^2}{r}+\Phi,$$
$$\frac{\partial R}{\partial z}=\Phi'_r\frac{z^2}{r}+\Phi.$$

2) Then the condition $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$ takes the form $\frac{d\Phi}{dr}r+3\Phi=0$.

Having solved the resulting differential equation (for example, using the method of separation of variables), we obtain that $\Phi(r)=\frac{C}{r^3}$.