USING TENSORS IN FIELD THEORY

Basic definitions and notation for tensors

For arbitrary natural n ordered set consisting of n^k numbers of the form

$$\left\{ \xi_{i_1 i_2 \ldots i_k}, \text{ Where } i_r = [1, n] \quad \forall r = [1, k], \ \forall k \right\}$$

we will call *k-dimensional matrix* or *matrix the size* n^k (k − any non-negative *whole* number)

Tensor V Λ^n let's call *k*-dimensional matrix, the values of the elements of which vary according to *linear formulas* when moving from one basis to another. Moreover, the proportionality coefficients in these linear formulas are *transition matrix elements* (possibly both direct and reverse).

Comment: here, transition matrices are understood as both themselves and the result of their transposition - there is no difference, since the elements in them are the same.

In other words, the tensor is *multidimensional matrix plus special linear rule* changes in its components when the basis changes.

In practice, it turns out that tensors are a convenient tool for quantitative description of various objects and phenomena (primarily physical).

We will further assume that the original basis in Λ^n There is $\{ \vec{g}_1, \vec{g}_2, \dots, \vec{g}_n \}$, and "new" – $\{ \vec{g}'_1, \vec{g}'_2, \dots, \vec{g}'_n \}$, while the direct transition matrix (from the original to the "new") $\|S\| = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{Vmatrix}$, and the inverse matrix – $\|S\|^{-1} = \|T\| = \begin{Vmatrix} \tau_{11} & \tau_{12} & \dots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \dots & \tau_{2n} \\ \dots & \dots & \dots & \dots \\ \tau_{n1} & \tau_{n2} & \dots & \tau_{nn} \end{Vmatrix}$.

Let us also agree that all characteristics related to the "new" basis will be marked with a top stroke.

SOME EXAMPLES

Consider a three-dimensional linear space Λ^3, in which the original basis is $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$, and "new" – $\{\vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$, Then:

1°. Coordinate representation *vector* $\vec{r}$ V Λ^3 $\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\|$ there is a tensor because

$$\left\|\begin{matrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{matrix}\right\| = \|S\|^{-1}\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\| \quad\Rightarrow\quad \left\|\begin{matrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{matrix}\right\| = \|T\|\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\| \quad\Rightarrow\quad \left\|\begin{matrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{matrix}\right\| = \left\|\begin{matrix}\tau_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \tau_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \tau_{33}\end{matrix}\right\|\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\|,$$

which can be written as

$$\xi'_s = \sum_{t=1}^{3} \tau_{st}\xi_t \quad \forall s = [1,3].$$

The linearity of the formulas for transforming the components of the numerical description of an object (in this case, a vector) according to the elements of the inverse transition matrix is obvious

Tensors of this type are usually called *monovalent contravariant* (that is, changing like basis vectors when *reverse* transition).

For reference, we recall that the transition formulas in the case under consideration are as follows:

$$\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\| = \|S\|\left\|\begin{matrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{matrix}\right\|;\quad \left\|\begin{matrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{matrix}\right\| = \|T\|\left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\|;\quad \left\|\begin{matrix}\vec{g}'_1\\ \vec{g}'_2\\ \vec{g}'_3\end{matrix}\right\| = \|S\|^{\mathrm{T}}\left\|\begin{matrix}\vec{g}_1\\ \vec{g}_2\\ \vec{g}_3\end{matrix}\right\| \quad \text{и} \quad \left\|\begin{matrix}\vec{g}_1\\ \vec{g}_2\\ \vec{g}_3\end{matrix}\right\| = \|T\|^{\mathrm{T}}\left\|\begin{matrix}\vec{g}'_1\\ \vec{g}'_2\\ \vec{g}'_3\end{matrix}\right\|.$$

2°. Coordinate representation *linear functional* (*linear function*) $f(\vec{r})$ in space Λ^3 , having the form

$$f(\vec{r}) = \sum_{j=1}^{3} \varphi_j \xi_j = \left\| \varphi_1 \quad \varphi_2 \quad \varphi_3 \right\| \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| ,$$

There is *tensor with components* $\varphi_j \;\; j = [1,3]$, since (from the linear algebra course) it is known that

$$\left\| \varphi'_1 \quad \varphi'_2 \quad \varphi'_3 \right\| = \left\| \varphi_1 \quad \varphi_2 \quad \varphi_3 \right\| \left\| S \right\| \qquad \text{или, что то же самое} \qquad \left\| \begin{matrix} \varphi'_1 \\ \varphi'_1 \\ \varphi'_1 \end{matrix} \right\| = \left\| S \right\|^{\mathrm{T}} \left\| \begin{matrix} \varphi_1 \\ \varphi_1 \\ \varphi_1 \end{matrix} \right\| .$$

In coordinates, this relationship is obviously linear in the components of the direct transition matrix $\|S\|$ and looks like

$$\varphi'_s = \sum_{t=1}^{3} \sigma_{ts} \varphi_t \quad \forall s = [1,3] \qquad \text{or} \qquad \begin{cases} \varphi'_1 = \sigma_{11}\varphi_1 + \sigma_{21}\varphi_2 + \sigma_{31}\varphi_3 \ , \\ \varphi'_2 = \sigma_{12}\varphi_1 + \sigma_{22}\varphi_2 + \sigma_{32}\varphi_3 \ , \\ \varphi'_3 = \sigma_{13}\varphi_1 + \sigma_{23}\varphi_2 + \sigma_{33}\varphi_3 \ . \end{cases}$$

Please note that the indices in the entry for the rule for changing tensor components of a linear form $f(\vec{r})$ arranged differently than in example 1°. The tensors from example 2° are usually called *monovalent covariant* (that is, changing like basis vectors when *direct* transition).

3°. Coordinate representation *linear transformation* $\hat{A}$ V Λ^3 (which is a third order square matrix $\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix}$) is a tensor, since from the well-known formula $\|\hat{A}\|_{g'} = \|S\|^{-1}\|\hat{A}\|_{g}\|S\|$ it follows that

$$\begin{Vmatrix} \alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\ \alpha'_{21} & \alpha'_{22} & \alpha'_{23} \\ \alpha'_{31} & \alpha'_{32} & \alpha'_{33} \end{Vmatrix} = \|T\| \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix} \|S\|$$

or

$$\alpha'_{st} = \sum_{j=1}^{n}\sum_{i=1}^{n} \tau_{sj}\sigma_{it}\alpha_{ji} \quad \forall s = [1,3], \quad \forall t = [1,3]$$

Tensors of this type are called *bivalent, once covariant and once contravariant*.

4°. Coordinate representation *bilinear functional* (*bilinear form*) $B(\vec{x},\vec{y})$ V Λ^3 will also be a tensor, since it is a third-order square matrix $\left\|\begin{matrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{matrix}\right\|$, for which, when changing the basis, the formula is valid $\|B\|_{g'} = \|S\|^{\mathrm{T}}\|B\|_g\|S\|$. This equality can also be written in the form

$$\left\|\begin{matrix}\beta'_{11} & \beta'_{12} & \beta'_{13}\\ \beta'_{21} & \beta'_{22} & \beta'_{23}\\ \beta'_{31} & \beta'_{32} & \beta'_{33}\end{matrix}\right\| = \|S\|^{\mathrm{T}}\left\|\begin{matrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{matrix}\right\|\|S\|$$

or

$$\beta'_{st} = \sum_{j=1}^{n}\sum_{i=1}^{n}\sigma^{\mathrm{T}}_{js}\sigma_{it}\beta_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{sj}\sigma_{it}\beta_{ji} \quad \forall s=[1,3], \quad \forall t=[1,3].$$

Note that the last formula was written using the obvious equality $\sigma^{\mathrm{T}}_{sj} = \sigma_{js}$. Such tensors are called *bivalent, doubly covariant*

DEFINITION OF TENSOR IN THE GENERAL CASE

From the above examples we can conclude that the valence of a tensor is the dimension of its matrix − k, A p − "number of covariances" (q – "number of contravariances") is determined by the number of uses of the matrix $\|S\|$ (resp. matrices $\|T\|$) in the formula for recalculating the values of tensor components when changing the basis.

In other words,

in the example 1° $1 = k = p + q = 0 + 1$,
in the example 2° $1 = k = p + q = 1 + 0$,
in the example 1° $2 = k = p + q = 1 + 1$,
in the example 1° $2 = k = p + q = 2 + 0$.

If this is the case, then the following version of the tensor definition can be used.

We will say that in a real linear space Λ^n determined *tensor type* (q, p) q *times contravariant* And p *times covariant* (*or* $(p+q)$ *-valence*), if in Λ^n it is characterized by an ordered set n^{p+q} numbers $\xi_{j_1 j_2 \ldots j_q i_1 i_2 \ldots i_p}$ (Where $j_m = [1, n]$; $m = [1, q]$ – contravariant indices and $i_s = [1, n]$; $s = [1, p]$ – covariant), transforming upon transition from the basis $\{g_1, g_2, \ldots, g_n\}$ to the base $\{g'_1, g'_2, \ldots, g'_n\}$ by law

$$\xi'_{j'_1 j'_2 \ldots j'_q i'_1 i'_2 \ldots i'_p} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_p=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_q=1}^{n} \sigma_{i_1 i'_1} \sigma_{i_2 i'_2} \ldots \sigma_{i_p i'_p} \quad \tau_{j'_1 j_1} \tau_{j'_2 j_2} \ldots \tau_{j'_q j_q} \; \xi_{j_1 j_2 \ldots j_q i_1 i_2 \ldots i_p} \, ,$$

Where $i'_s = [1, n]$; $s = [1, p]$ And $j'_m = [1, n]$; $m = [1, q]$, A σ_{ij} And τ_{ij} are respectively the components of the transition matrix and its inverse.

The cumbersomeness of notation and the unreadability of tensors when using the standard notation scheme are obvious already from the example of this definition. Therefore, tensor calculus uses a special, more compact form of description of tensor objects and operations with them, which is based on the following rules.

1°. If any of the indexes takes all values from 1 to n, then this list of values is not indicated in the tensor record. For example, record $\alpha_i = \beta_i$ means that

$$\alpha_i = \beta_i \quad \forall i = [1, n].$$

2°. *The order of the indices in the notation of tensors is important*. In order to avoid possible ambiguity, the following rule is applied: if it is necessary to write out all the components of the tensor sequentially (for example, in the form of one line), then first of all (if possible!) the indices located closer to the right end of the index list are increased.

For example, tensor ξ_{ijs} v Λ^2 has the following order of components:

$$\xi_{111}, \xi_{112}, \xi_{121}, \xi_{122}, \xi_{211}, \xi_{212}, \xi_{221}, \xi_{222} \; .$$

3°. Covariant and contravariant indices generally differ as follows: covariant indices are written as *lower*, and contravariant ones - as *upper* indexes.

In a number of important special cases (which include the issues we consider below), the difference between covariant and contravariant indices is not significant or is completely absent. In our formulas we will use only subscripts. Relevant explanations will be given later.

4°. Let there be an expression that is the product of factors having repeating indices, and each such index appears in the expression *smooth twice*. Then, by definition, it is considered that this expression is *the sum of such products written out for all values of the repeating index*. This amount is usually called *convolution of the tensor by index*.
If there are several pairs of matching indices in an expression, multiple summation occurs.
Note that using the same index in a tensor notation more than twice is prohibited by definition.

Example: value of a quadratic functional (quadratic form)

$$\Phi(\vec{r}) = \sum_{j=1}^{n}\sum_{i=1}^{n}\varphi_{ji}\,\xi_j\,\xi_i$$

can now be written as double convolution $\varphi_{ji}\,\xi_j\,\xi_i$.

An example of the simultaneous use of the rules formulated above is the system *n* linear equations with *n* unknown:

$$\begin{cases} \alpha_{11}\xi_1 + \alpha_{12}\xi_2 + \ldots + \alpha_{1n}\xi_n = \beta_1 \\ \alpha_{21}\xi_1 + \alpha_{22}\xi_2 + \ldots + \alpha_{2n}\xi_n = \beta_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{n1}\xi_1 + \alpha_{n2}\xi_2 + \ldots + \alpha_{nn}\xi_n = \beta_n \end{cases} \quad \text{or} \quad \sum_{i=1}^{n} \alpha_{ki}\xi_i = \beta_k \quad \forall k = [1,n],$$

whose tensor notation has the form $\alpha_{ki}\xi_i = \beta_k$.

Now, finally, the definition of a tensor can be given in the following formulation.

We will say that in a real linear space Λ^n determined *tensor type* (q,p) q *times contravariant* And p *times covariant*, if in Λ^n it is characterized by an ordered set n^{p+q} numbers $\xi_{j_1 j_2 \ldots j_q i_1 i_2 \ldots i_p}$ (Where $j_m = [1,n]$; $m = [1,q]$ – contravariant indices and $i_s = [1,n]$; $s = [1,p]$ – covariant), transforming upon transition from the basis $\{g_1, g_2, \ldots, g_n\}$ to the base $\{g_1', g_2', \ldots, g_n'\}$ by law

$$\xi'_{j_1' j_2' \ldots j_q' i_1' i_2' \ldots i_p'} = \sigma_{i_1 i_1'} \sigma_{i_2 i_2'} \ldots \sigma_{i_p i_p'} \; \tau_{j_1' j_1} \tau_{j_2' j_2} \ldots \tau_{j_q' j_q} \; \xi_{j_1 j_2 \ldots j_q i_1 i_2 \ldots i_p} ,$$

Where σ_{ij} And τ_{ij} are respectively the components of the transition matrix and its inverse.

CASE OF EUCLIDEAN SPACES

We will consider *n*-dimensional Euclidean space E^n with right orthonormal bases $\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ And $\{ \vec{e}'_1, \vec{e}'_2, \vec{e}'_3 \}$.

From the course of linear algebra it is known that orthogonal matrices (and only they!) serve as transition matrices from one ONB to another. Moreover, for any orthogonal matrix $\|S\|$ equality is true $\|S\|^{-1} = \|S\|^{\mathrm{T}}$, so the difference between covariant and contravariant cases in Euclidean space with ONB *absent*, that is, all tensor formulas look the same when written both through upper and lower indices. And in further calculations we will use only subscripts.

Slightly deviating from the main direction of presentation of the issue under consideration, we note that in an arbitrary basis of Euclidean space there is a difference in the covariant and contravariant forms of writing the same tensor, but it is not fundamental. Indeed, the presence of the scalar product operation and the Gram matrix (which is a bivalent, doubly covariant tensor) allows us to unambiguously transfer these records from one form to another and back. It is clear that in an arbitrary finite-dimensional linear space Λ^n such manipulations are impossible.
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If you are interested in the details, you can look at them in any resource on tensor calculus on the topic “The operation of raising and lowering tensor indices in Euclidean space.”

Now we present some useful notations and relationships, written (if possible) both in traditional vector-coordinate form and in tensor form.

1°. Vector decomposition $\vec{x}$ on the basis $\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ V E^3 :

$$\vec{x} = \xi_1 \vec{e}_1 + \xi_2 \vec{e}_2 + \xi_3 \vec{e}_3 \quad \text{or} \quad \vec{x} = \xi_i \vec{e}_i$$

2°. *Kronecker symbol* is a bivalent tensor $\delta_{ji} = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$ It is easy to show that in any basis its components coincide with the elements of the identity matrix. Moreover, its convolution $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$.

3°. *Levi-Cevita symbol* is a trivalent tensor

$$\varepsilon_{jik} = \begin{cases} 0, & \textit{если среди } i, j, k \textit{ есть равные}, \\ 1, & \textit{если } \{i, j, k\} - \textit{четная перестановка}, \\ -1, & \textit{если } \{i, j, k\} - \textit{нечетная перестановка}. \end{cases}$$

Its components do not change when indexes are cyclically rearranged. The convolutions for this tensor have the form:

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} , \quad \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \quad \text{And} \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6 .$$

It is easy to check that the value of the Levi-Cevita symbol does not change during a cyclic permutation of indices and changes sign to the opposite during an anticyclic permutation.

4°. Let in E^3 vectors are given $\vec{a} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \alpha_3 \vec{e}_3$, $\vec{b} = \beta_1 \vec{e}_1 + \beta_2 \vec{e}_2 + \xi_3 \vec{e}_3$ And $\vec{c} = \kappa_1 \vec{e}_1 + \kappa_2 \vec{e}_2 + \xi_3 \vec{e}_3$ with tensor representations, respectively $\vec{a} = \alpha_i$, $\vec{b} = \beta_i$ And $\vec{c} = \kappa_i$. Then in tensor form *dot product* written simply as a convolution $(\vec{a}, \vec{b}) = \alpha_i \beta_i$, and for *vector product* $\vec{c} = [\vec{a}, \vec{b}]$ we have the formula $\kappa_i = \varepsilon_{ijk} \alpha_j \beta_k$.

Please note that the relationships indicated in paragraphs 3° and 4° require justification. Perform this rationale yourself.

FORMULAS OF FIELD THEORY IN TENSOR FORM

Let us introduce into consideration the vector-differential operator *nabla,* written symbolically as $\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial \xi_1} + \vec{e}_2 \frac{\partial}{\partial \xi_2} + \vec{e}_3 \frac{\partial}{\partial \xi_3}$ or, in tensor form $\vec{\nabla} = \vec{e}_i \frac{\partial}{\partial \xi_i}$, which is equivalent $\vec{\nabla} = \frac{\partial}{\partial \xi_i}$. In this case, quite often it is necessary to use some combinations of this operator, determined by the rules of mathematical analysis and linear algebra. Let's write down the main ones.

1°. *Scalar field gradient*. Let in some area $\Omega \subseteq E^3$ a continuously differentiable function is given $u(\vec{r}) = u(\xi_1, \xi_2, \xi_3)$. Her *gradient* called the vector grad *in* = $\vec{e}_1 \frac{\partial u}{\partial \xi_1} + \vec{e}_2 \frac{\partial u}{\partial \xi_2} + \vec{e}_3 \frac{\partial u}{\partial \xi_3}$. Symbolically, this can be written as the action of the nabla operator on the scalar function grad *in*= $\vec{\nabla} u$, and in tensor format in the form of convolution - decomposition over the basis - as grad *in*= $\vec{e}_i \frac{\partial u}{\partial \xi_i}$.

2°. *Divergence of a vector field.* Let in some area $\Omega \subseteq E^3$ a continuously differentiable vector function is given $\vec{A}(\vec{r}) = \vec{A}(\xi_1, \xi_2, \xi_3)$. Her *divergence* called a scalar function $\operatorname{div}\vec{A} = \frac{\partial A_1}{\partial \xi_1} + \frac{\partial A_2}{\partial \xi_2} + \frac{\partial A_3}{\partial \xi_3}$. In symbolic form, through the scalar product, this function can be written as $\operatorname{div}\vec{A} = (\vec{\nabla}, \vec{A})$, and in tensor form the definition of divergence is $\operatorname{div}\vec{A} = \frac{\partial A_i}{\partial \xi_i}$.

3°. *Vector field rotor.* Let in some area $\Omega \subseteq E^3$ a continuously differentiable vector function is given $\vec{A}(\vec{r}) = \vec{A}(\xi_1, \xi_2, \xi_3)$. Her *rotor* called a vector function $\operatorname{rot}\vec{A} = \det \begin{Vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\ A_1 & A_2 & A_3 \end{Vmatrix}$. In symbolic form, through the vector product, this function can be written as $\operatorname{rot}\vec{A} = [\vec{\nabla}, \vec{A}]$, and in tensor form the definition of the rotor will be $\operatorname{rot}\vec{A} = \varepsilon_{ijk} \frac{\partial A_j}{\partial \xi_i} \vec{e}_k$.

4°. *Laplace operator* (*laplacian*). Let in some area $\Omega \subseteq E^3$ a twice continuously differentiable vector function is given $\vec{A}(\vec{r}) = \vec{A}(\xi_1, \xi_2, \xi_3)$. Her *laplacian* called a vector function $\Delta \vec{A} = \frac{\partial^2 \vec{A}}{\partial \xi_1^2} + \frac{\partial^2 \vec{A}}{\partial \xi_2^2} + \frac{\partial^2 \vec{A}}{\partial \xi_3^2}$. In symbolic form, through the scalar product, this function can be written as $\Delta = (\vec{\nabla}, \vec{\nabla})$, and the tensor definition of the Laplacian is: $\Delta = \frac{\partial^2}{\partial \xi_i \, \partial \xi_i}$.

When calculating various combinations in symbolic form, it should be remembered that the nabla operator in these combinations, on the one hand, behaves like a vector (that is, it satisfies the known relations from the vector algebra course), and on the other hand, it is a differentiation operator acting on scalar or vector functions, appearing in the entry to the right of it according to the rules for calculating derivatives.

Let's explain this with examples.

1°. Find $\operatorname{div}\operatorname{rot}\vec{A}$.

Let's solve the problem first *symbolic method*. According to the properties of scalar, mixed and vector products of vectors, we have

$$\operatorname{div}\operatorname{rot}\vec{A} = (\vec{\nabla},[\vec{\nabla},\vec{A}]) = (\vec{A},[\vec{\nabla},\vec{\nabla}]) = ([\vec{\nabla},\vec{\nabla}],\vec{A}) = (\vec{o},\vec{A}) = 0 .$$

Tensor method. The Levi-Cevita symbol is equal to zero if at least one of the pairs of its indices has the same values, therefore, in tensor form, a threefold convolution of 27 terms $\operatorname{div}\operatorname{rot}\vec{A} = \frac{\partial}{\partial \xi_k}\varepsilon_{ijk}\frac{\partial A_j}{\partial \xi_i}\vec{e}_k$ there will be only six non-zeros.

With a plus sign for index values $\{i,j,k\}$ – $\{1,2,3\}$, $\{3,1,2\}$ And $\{2,3,1\}$ – this is:

$$\frac{\partial^2 A_2}{\partial \xi_3\,\partial \xi_1},\ \frac{\partial^2 A_1}{\partial \xi_2\,\partial \xi_3}\ \text{и}\ \frac{\partial^2 A_3}{\partial \xi_1\,\partial \xi_2},$$

and with a minus sign for index values $\{i,j,k\}$ – $\{2,1,3\}$, $\{3,2,1\}$ And $\{1,3,2\}$ – this is:

$$\frac{\partial^2 A_1}{\partial \xi_3\,\partial \xi_2},\ \frac{\partial^2 A_2}{\partial \xi_1\,\partial \xi_3}\ \text{и}\ \frac{\partial^2 A_3}{\partial \xi_2\,\partial \xi_1},$$

which ultimately gives the convolution a value of zero.

2°. For vector fields $\vec{A}$ And $\vec{B}$ find $\operatorname{div}[\vec{A},\vec{B}]$.

Symbolic method. According to the formula for differentiating a product of functions, the properties of a mixed product and the rule for writing the action of an operator, we have

$$\operatorname{div}[\vec{A},\vec{B}] = (\vec{\nabla},[\vec{A},\vec{B}]) = (\vec{\nabla},[\overset{\downarrow}{\vec{A}},\vec{B}]) + (\vec{\nabla},[\vec{A},\overset{\downarrow}{\vec{B}}]) = (\vec{B},[\vec{\nabla},\overset{\downarrow}{\vec{A}}]) + (\vec{A},[\overset{\downarrow}{\vec{B}},\vec{\nabla}]) =$$

$$= (\vec{B},[\vec{\nabla},\vec{A}]) - (\vec{A},[\vec{\nabla},\vec{B}]) = (\vec{B},\operatorname{rot}\vec{A}) - (\vec{A},\operatorname{rot}\vec{B})\ .$$

Upper vertical arrow $\downarrow$ We showed which of the factors in the vector product of fields is affected by the nabla operator.

Tensor method. Here we take into account that during a cyclic permutation of indices, the Levi-Cevita symbol does not change its values, but during an anticyclic permutation (that is, with a one-time change in the order of two adjacent indices), it changes the signs of the values to the opposite. We have

$$\operatorname{div}[\vec{A},\vec{B}] = \frac{\partial}{\partial\xi_k}[\vec{A},\vec{B}]_k = \frac{\partial}{\partial\xi_k}(\varepsilon_{ijk}A_iB_j) = \varepsilon_{ijk}\frac{\partial A_i}{\partial\xi_k}B_j + \varepsilon_{ijk}A_i\frac{\partial B_j}{\partial\xi_k} =$$

$$= B_j\varepsilon_{ijk}\frac{\partial A_i}{\partial\xi_k} - A_i\,\varepsilon_{kji}\frac{\partial B_j}{\partial\xi_k} = (\vec{B},\operatorname{rot}\vec{A}) - (\vec{A},\operatorname{rot}\vec{B})\ .$$

3°. Show that vectors $\vec{E}$ And $\vec{B}$, satisfying the system of Maxwell's equations for vacuum

$$\begin{cases} \operatorname{rot}\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}; \quad \operatorname{div}\vec{E} = 0; \\ \operatorname{rot}\vec{B} = \dfrac{\partial \vec{E}}{\partial t}; \quad \operatorname{div}\vec{B} = 0, \end{cases}$$

will be a solution to wave equations of the form:

$$\Delta\vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{And} \quad \Delta\vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}.$$

Symbolic method. Let's take the rotor from both sides of the first of the system equations. For the left side according to the formula "*bac-cab*» get

$$\operatorname{rot}\operatorname{rot}\vec{E} = [\vec{\nabla},[\vec{\nabla},\vec{E}]] = \vec{\nabla}(\vec{\nabla},\vec{E}) - \overset{\downarrow}{\vec{E}}(\vec{\nabla},\vec{\nabla}) = \vec{\nabla}(\vec{\nabla},\vec{E}) - (\vec{\nabla},\vec{\nabla})\overset{\downarrow}{\vec{E}} = \vec{\nabla}(\vec{\nabla},\vec{E}) - \Delta\vec{E} =$$

$$= \operatorname{grad}\operatorname{div}\vec{E} - \Delta\vec{E}.$$

But since, by virtue of Maxwell’s second equation $\operatorname{div}\vec{E} = 0$, That $\operatorname{rot}\operatorname{rot}\vec{E} = -\Delta\vec{E}$. For the right side (taking into account Maxwell’s third equation) we find that

$$-\operatorname{rot}\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\operatorname{rot}\vec{B} = -\frac{\partial}{\partial t}\frac{\partial \vec{E}}{\partial t} = -\frac{\partial^2 \vec{E}}{\partial t^2}.$$

So, $-\Delta\vec{E} = -\dfrac{\partial^2 \vec{E}}{\partial t^2} \quad \Leftrightarrow \quad \Delta\vec{E} = \dfrac{\partial^2 \vec{E}}{\partial t^2}$. The second wave equation is derived similarly. Check it out for yourself.

Tensor method. Let's check the validity of the formula $\operatorname{rot}\operatorname{rot}\vec{E} = \operatorname{grad}\operatorname{div}\vec{E} - \Delta\vec{E}$.
We have

$$\operatorname{rot}\operatorname{rot}\vec{E} = \varepsilon_{ijk}\frac{\partial \operatorname{rot} E_j}{\partial \xi_i}\vec{e}_k = \varepsilon_{ijk}\frac{\partial}{\partial \xi_i}\left(\varepsilon_{stj}\frac{\partial E_t}{\partial \xi_s}\right)\vec{e}_k =$$

By performing a cyclic permutation of indices in the first factor and using the formula for convolution of the Levi-Cevita symbol by the last index (see page 6), we obtain

$$= \varepsilon_{kij}\frac{\partial}{\partial \xi_i}\left(\varepsilon_{stj}\frac{\partial E_t}{\partial \xi_s}\right)\vec{e}_k = \delta_{ks}\delta_{it}\frac{\partial^2 E_t}{\partial \xi_i\,\partial \xi_s}\vec{e}_k - \delta_{kt}\delta_{is}\frac{\partial^2 E_t}{\partial \xi_i\,\partial \xi_s}\vec{e}_k =$$

$$= \frac{\partial^2 E_i}{\partial \xi_i\,\partial \xi_k}\vec{e}_k - \frac{\partial^2 E_k}{\partial \xi_i^2}\vec{e}_k = \operatorname{grad}\operatorname{div}\vec{E} - \Delta\vec{E}\,.$$

The convolution with the Kronecker symbol used here is trivial: for example, $\delta_{lj}\Omega_i = \Omega_j$ or $\delta_{lj}\delta_{st}\Omega_{isk} = \Omega_{jtk}$.

Perform the reduction to the wave equation in tensor form as a final test.