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Explanations for the introduction to HARMONIC ANALYSIS

We will use some concepts and methods discussed earlier in the courses of mathematical analysis and linear algebra to construct analogues of the coordinate description of elements in linear spaces that do not have a basis.

Main feature n -dimensional linear space is the existence in it *basis*, that is:

> orderly *final* recruitment $\{g_1, g_2, \ldots, g_n\}$ linearly independent elements such that adding any element of space to it makes this set linearly dependent.

In this case *every* linear space element x may be presented *the only way* as a linear combination of basis elements

$$x = \sum_{j=1}^{n} \xi_j g_j , \tag{1}$$

where is an ordered set of numbers ξ_j (called *coordinates* element x in the basis $\{g_1, g_2, \ldots, g_n\}$) gives a complete description of this element. The maximum possible number of linearly independent elements in a linear space with a basis is usually called *dimension* space.

It is known that the basis does not exist in any linear space where there are linearly independent elements. For example, in $LC_{[-1,\,1]}$ – linear function space $x(\tau)$, continuous on the segment $\tau \in [-1,\,1]$ – set of elements of the form $\{1, \tau, \tau^2, \ldots, \tau^k\}$ linearly independent for any non-negative integer k. But for this set there is always an element τ^{k+1}, adding which does not violate linear independence (check this yourself). Consequently, there is no basis in such a linear space, and a standard coordinate description of the elements is impossible in it.

On the other hand, the convenience of coordinate representation of linear space elements is quite obvious. Therefore, it seems appropriate to attempt to generalize the concept of coordinates in such a way that their use would be possible in linear spaces with *unlimited* number of linearly independent elements.

Let us briefly describe the ideas of two possible approaches to solving this problem.

1) Let there be a linear n-dimensional space in which the scalar product of elements is defined (x,y), that is, *Euclidean* space E^n. If $\{g_1, g_2, \ldots, g_n\}$ – basis in E^n, then for each element x due to (1) will be true

$$\sum_{j=1}^{n}(g_j, g_k)\xi_k = (x, g_k) \quad \forall k = [1,n]. \tag{2}$$

Here (2) is a system of linear equations, which (by virtue of Cramer's theorem) is always consistent and uniquely solvable with respect to coordinates, since its main matrix is a Gram matrix - non-singular for a linearly independent set of elements $\{g_1, g_2, \ldots, g_n\}$.

Note that in the case *orthonormal* basis $\{e_1, e_2, \ldots, e_n\}$ (which always exists in E^n) system (2) breaks down into n independent equalities of the form

$$\xi_k = (x, e_k) \quad \forall k = [1,n], \tag{3}$$

because $(e_j, e_k) = \delta_{jk} = \begin{cases} 1, \text{ если } j = k, \\ 0, \text{ если } j \neq k. \end{cases}$

On the other hand, according to the axiomatics of Euclidean space, the scalar product exists regardless of whether the given space is finite-dimensional or not. Therefore, formulas (3) *conditionally* can be taken to determine the coordinates of the element $x \in E$ – in Euclidean space with an unlimited number of linearly independent elements.

Conditions that must be met in order for the members of the sequence $\{\xi_k\}$, Where

$$\xi_k = (x, e_k) \quad \forall k = [1,+\infty), \tag{4}$$

could be considered as analogues of element coordinates $x \in E$, are:

- all elements in a set of linearly independent $\{e_1, e_2, \ldots, e_n, \ldots\}$ must be normalized and pairwise orthogonal, that is, $(e_j, e_k) = \delta_{jk} \ \forall j,k \in N$;
- row $\sum_{j=1}^{+\infty} \xi_j e_j$ must be convergent at some point $\Omega \subseteq E$;
- the sum of this series must coincide (or be close in some sense) to x.

Let us recall here (in connection with the use of the concept of series convergence) that as *norms* element $x \in E$ number can be accepted $\|x\| = \sqrt{(x,x)}$, and as *metrics* − (measures of proximity (or distance) for elements x And y) − number $\rho(x,y) = \|x - y\|$. This allows the concepts to be used *fundamentality* And *convergence* for a sequence of elements in E and, specifically, for sequences of partial sums of the series $\sum_{j=1}^{+\infty} \xi_j e_j$.

As an example, consider in $EC_{[-1,\,1]}$ – Euclidean function space $x(\tau)$, continuous on the interval $[-1,1]$ and with scalar product $(x,y) = \int_{-1}^{1} x(\tau) y(\tau)\, d\tau$ − sequence of linearly independent elements $\{\, g_k(\tau) = \tau^k \;\; k = 0,1,2,3,\ldots \,\}$.

The elements of such a sequence are not pairwise orthogonal. For example,

$$(g_1, g_3) = \int_{-1}^{1} g_1(\tau) g_3(\tau)\, d\tau = \int_{-1}^{1} \tau^4 d\tau = \frac{2}{5} \neq 0 .$$

However, in this case, using the Gram–Schmidt orthogonalization procedure (applicable to sets with an unlimited number of elements), it is possible to construct an orthogonal set, and after normalization, an orthonormal one. As a result (this is proven in the course of harmonic analysis), we obtain a sequence of power polynomials of the form:

$$e_0(\tau)=1,\quad e_1(\tau)=\tau,\quad e_2(\tau)=\frac{1}{2}\left(3\tau^2-1\right),\quad \text{□}\quad e_k(\tau)=\frac{1}{2^k k!}\frac{d^k}{d\tau^k}\left(\tau^2-1\right)^k,\quad \text{□}\quad ,$$

called *Legendre polynomials*.

And this new set of polynomials can be used to represent on the segment $[-1,1]$ any continuous function $x(\tau)$ using a series

$$x(\tau)=\sum_{k=0}^{+\infty}\xi_k e_k(\tau),\quad \text{где}\quad \xi_k=\int_{-1}^{1}x(\tau)g_k(\tau)\,d\tau\,. \tag{5}$$

Series of the form (5) are usually called *Fourier series* and their partial sums are approximations of the function $x(\tau)$ *throughout the entire period* $[-1,1]$ while power series (such as the Taylor series) give only *local* approximation for fixed points.

2) Questions about convergence conditions and other properties of Fourier series are discussed in the course of harmonic analysis. Here we will draw attention to the fact that orthogonal systems of elements in infinite-dimensional Euclidean spaces can also be constructed according to schemes that are fundamentally different from the Gram–Schmidt method.

Let's analyze one of these schemes, based on the use of properties *self-adjoint linear transformations* (operators) operating in E .

Let us first recall some necessary information from the linear algebra course, using the equality $y = \hat{A}x$ to indicate a fact: element y is the result of the transformation action $\hat{A}$ per element x . Element y in this case it is customary to call *way* element x , and the element x – *prototype* element y .

1⁰. Linear transformation $\hat{A}^*$ called *conjugate* linear transformation $\hat{A}$, If $\forall x, y \in E: \ (\hat{A}x, y) = (x, \hat{A}^* y)$. That is, the scalar product under the action of the operator $\hat{A}$ by its first factor coincides with the scalar product under the action $\hat{A}^*$ to the second factor.

2⁰. Linear transformation $\hat{R}$ called *self-adjoint* linear transformation if $\forall x, y \in E: \ (\hat{R}x, y) = (x, \hat{R}y)$.

3⁰. Non-zero element f called *eigenvector* linear transformation $\hat{A}$ who answers *eigenvalue* λ , If $\hat{A}f = \lambda f$.

4⁰. Eigenvectors of a self-adjoint transformation corresponding to *various* eigenvalues, *pairwise orthogonal*. Moreover, from the eigenvectors of any self-adjoint transformation to E^n can be formed *orthonormal basis*.

5⁰. For any linear transformation $\hat{A}$ linear transformations $\hat{A}\hat{A}^*$ And $\hat{A}^*\hat{A}$ – *self-adjoint* and have *non-negative* eigenvalues.

Let us recall the proofs of the statements given in 4⁰ and 5⁰.

Paragraph 4⁰. Let f_1 And f_2 – eigenvectors $\hat{R}$, then from $\begin{cases}\hat{R}f_1=\lambda_1 f_1,\\ \hat{R}f_2=\lambda_2 f_2\end{cases}$ should

$$\begin{cases}(\hat{R}f_1,f_2)=\lambda_1(f_1,f_2),\\ (f_1,\hat{R}f_2)=\lambda_2(f_1,f_2).\end{cases}$$

Due to self-adjointness $\hat{R}$ (that is, equality of the left sides) we will have $\lambda_1(f_1,f_2)=\lambda_2(f_1,f_2)$ and, since $\lambda_1\neq\lambda_2$, That $(f_1,f_2)=0$.

*Point 5*⁰. Equalities are valid $\forall x,y\in E:\ (\hat{A}^*\hat{A}x,y)=(\hat{A}x,\hat{A}y)=(x,\hat{A}^*\hat{A}y)$. But this is what self-conjugation means. $\hat{A}^*\hat{A}$.

From $\hat{A}^*\hat{A}f=\lambda f$ we have equality $(\hat{A}^*\hat{A}f,f)=\lambda(f,f)$. Where $(\hat{A}f,\hat{A}f)=\lambda(f,f)$ and therefore, by virtue of the axiomatics of Euclidean space, it is true that $\lambda\geq 0$.

Let us now consider many different functions $x(\tau)$, having a derivative *any* order on the segment $[-1,1]$, and having, together with their derivatives, equal values at the ends of this segment. It is easy to verify that this set is a linear space.

Let's turn it into Euclidean space by introducing the scalar product of elements x And y according to the formula

$$(x, y) = \int_{-1}^{1} x(\tau) y(\tau)\, d\tau.$$

Differentiation operator $\hat{D} = \frac{d}{d\tau}$ will be a linear transformation acting in this space.

Let us find the conjugate transformation for it. Integration by parts gives equalities

$$(\hat{D}x, y) = \int_{-1}^{1} \frac{dx}{d\tau}(\tau) y(\tau)\, d\tau = x(\tau) y(\tau)\Big|_{-1}^{1} - \int_{-1}^{1} x(\tau) \frac{dy}{d\tau}(\tau)\, d\tau = \int_{-1}^{1} x(\tau)\Big(-\frac{dy}{d\tau}(\tau)\Big)\, d\tau = (x, \hat{D}^* y).$$

Where $\hat{D}^* = -\frac{d}{d\tau}$. Then the operator $\hat{D}^*\hat{D} = -\frac{d}{d\tau}\Big(\frac{d}{d\tau}\Big) = -\frac{d^2}{d\tau^2}$ will be self-adjoint and with non-negative eigenvalues.

Let's find eigenvectors for it. Condition $\hat{D}^* \hat{D} f = \lambda f$ in this case (if we put $\lambda = \omega^2 \geq 0$) is a differential equation of the form

$$-\frac{d^2 f}{d\tau^2} = \omega^2 f,$$

the general real solution of which is described by the formula $f(\tau) = A\cos\omega\tau + B\sin\omega\tau$, and $A^2 + B^2 \neq 0$, since the eigenvector is non-zero by definition.

Note that in this case the characteristic equation of the transformation $\hat{D}^* \hat{D}$ No. However, the eigenvalues can be found using the conditions

$$f^{(k)}(-1) = f^{(k)}(1) \quad \forall k = 0,1,2,\ldots \quad . \tag{6}$$

Specifically, here we get that

$$A\cos\omega 1 + B\sin\omega 1 = A\cos\omega(-1) + B\sin\omega(-1) \qquad \Rightarrow \qquad \sin\omega = 0,$$

that is, $\omega = \pi k$, Where k – any integer. Thus, the system *linearly independent* pairwise orthogonal functions that are eigenvectors of the self-adjoint operator $\hat{D}^*\hat{D}$, has the form

$$f_k(\tau) = A\cos\pi k\tau + B\sin\pi k\tau, \quad k = 0,1,2,3,\ldots.$$

This system, usually called *trigonometric*, (as well as the Legendre polynomial system) can be used for approximation on $[-1,1]$ functions $x(\tau)$, satisfying, for example, boundary conditions (6).

Such an approximation will have the form of a series $\sum_{k=0}^{\infty}(A_k\cos\pi k\tau + B_k\sin\pi k\tau)$, where are the coefficients A_k And B_k are determined by formulas similar to (5)

$$A_0 = \frac{1}{2}\int_{-1}^{1} x(\tau)\,d\tau, \quad A_k = \int_{-1}^{1} x(\tau)\cos\pi k\tau\,d\tau \qquad \text{And} \qquad B_k = \int_{-1}^{1} x(\tau)\sin\pi k\tau\,d\tau \quad k = 1,2,3,\ldots.$$