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Topic 02 2024/25 academic year G.

Trigonometric Fourier series

Consider a set of functions of the form

$$\{\ 1, \cos\tau, \sin\tau, \dots \cos n\tau, \sin n\tau, \dots \quad \forall n \in \mathbf{N}, \quad \tau \in [-\pi, \pi]\ \}. \tag{1}$$

Any linear combination of these functions with an unlimited number of terms is usually called *trigonometric series*.

The set of all trigonometric series is *linear space* regarding standard operations of adding functions and multiplying a number by a function

Let us explore the possibility of representing a function in the form of a trigonometric series.

Firstly, sometimes such a representation in the form of a series (or in the form of its partial sum) for a periodic function can be found uniquely, through formulaic transformations.

Example 01. Find trigonometric series for a function $f(\tau)=\sin^4\tau$.

Solution: Let us give the form of the function $f(\tau)$ to a linear combination, using the formula for doubling the argument of trigonometric functions twice:

$$\sin^4\tau=\left[\frac{1-\cos 2\tau}{2}\right]^2=\frac{1}{4}-\frac{1}{2}\cos 2\tau+\frac{1}{4}\cos^2 2\tau=$$

$$=\frac{1}{4}-\frac{1}{2}\cos 2\tau+\frac{1}{4}\left[\frac{1+\cos 4\tau}{2}\right]=\frac{3}{8}-\frac{1}{2}\cos 2\tau+\frac{1}{8}\cos 4\tau\ .$$

Let's consider a more general method of constructing an approximation in the form of a trigonometric series.

Let us first introduce (by definition) the concept *absolute integrability of the function* $f(\tau)$ *in between* $[a,b]$ *in an improper sense*.

Let's say that $f(\tau)$ *absolutely integrable to* $[a,b]$ *in an improper sense*, If:

1) exists $\int\limits_a^b |f(\tau)|\,d\tau.$
2) number of singular points $f(\tau)$ on $[a,b]$ Certainly
And
Riemannian integral of $f(\tau)$ (that is, $\int\limits_\alpha^\beta f(\tau)\,d\tau$) exists on any interval belonging to $[a,b]$, which does not contain singular points.

Note that here we assume both the existence of the integrals $\int\limits_a^b f(\tau)\,d\tau$, and integrals $\int\limits_a^b |f(\tau)|\,d\tau.$ Their existence *not equivalent*, for example, for the Dirichlet function on the interval: $[a,b]$

$$f(\tau)=\begin{cases} 1, \text{если } \tau - \text{рациональное}, \\ -1, \text{если } \tau - \text{ирациональное}, \end{cases}$$

where absolute integrability does not imply Riemann integrability.

Then, let us transform the linear space of trigonometric series into *Euclidean space*, took for *dot product* on $[-\pi,\pi]$ functions $x(\tau)$ And $y(\tau)$, bilinear form

$$(x,y)=\int_{-\pi}^{\pi}x(\tau)y(\tau)\,d\tau \quad .$$

The set of functions (1) in this case turns out to be *orthogonal*.

Indeed, the equalities will be true:

$$\int_{-\pi}^{\pi}\cos n\tau\cos m\tau\,d\tau=0,\qquad \int_{-\pi}^{\pi}\sin n\tau\sin m\tau\,d\tau=0,\quad ,\quad \forall n\neq m$$
$$\int_{-\pi}^{\pi}\cos n\tau\sin m\tau\,d\tau=0,\qquad \forall n,m\in\mathbf{N}. \tag{2}$$

\
In addition, the following relations are true:

$$\int_{-\pi}^{\pi}1\,d\tau=2\pi,\qquad \int_{-\pi}^{\pi}\cos^2 n\tau\,d\tau=\pi\qquad \int_{-\pi}^{\pi}\sin^2 n\tau\,d\tau=\pi,\qquad \forall n\in\mathbf{N}.$$

It is clear that the properties of trigonometric series for system (1) depend on how the coefficients in these series are chosen. We organize this choice so that the trigonometric series is some representation of the function $f(\tau)$ on $[-\pi,\pi]$.

Let's consider a specific method for choosing the coefficients of a trigonometric series. Let us assume that a series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos n\tau+b_n\sin n\tau) \tag{3}$$

converges to a function $f(\tau)$ on $[-\pi,\pi]$ *evenly*, then its coefficients are found from the following reasoning.

This means that the equality is true $f(\tau)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{+\infty}(a_n\cos n\tau+b_n\sin n\tau)\quad\forall\tau\in[-\pi,\pi]$.

If we multiply both sides of this equality term by term $\cos m\tau$, and then integrate the product over τ , then by virtue of (2) we get

$$\int\limits_{-\pi}^{\pi}f(\tau)\cos m\tau\,d\tau=\frac{a_0}{2}\int\limits_{-\pi}^{\pi}\cos m\tau\,d\tau+\sum_{n=1}^{+\infty}\left(a_n\int\limits_{-\pi}^{\pi}\cos n\tau\cos m\tau\,d\tau+b_n\int\limits_{-\pi}^{\pi}\sin n\tau\cos m\tau\,d\tau\right)\quad\Rightarrow$$

$$\Rightarrow\qquad a_m=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(u)\cos mu\,du\qquad\forall m\in\{0,\mathbf{N}\}.\tag{4}$$

Likewise, after multiplying by $\sin m\tau$, we find that $b_m=\dfrac{1}{\pi}\int\limits_{-\pi}^{\pi}f(u)\sin mu\,du\qquad\forall m\in\mathbf{N}.$

Substitution τ on u made for ease of recording, since the value of a definite integral does not depend on the identifier (symbol) denoting the integration variable.

By definition, we accept formulas (4) as defining the coefficients of the trigonometric series in the case of any absolutely integrable function $f(\tau)$.

In this case, the resulting trigonometric series (3) is called *near Fourier* for this function on $[-\pi,\pi]$.

It turns out that the sum of the Fourier series (if it exists) does not always coincide identically with $f(\tau)$ on $[-\pi,\pi]$, therefore it is customary to use the function for comparison $f(\tau)$ and its Fourier series special designation:

$$f(\tau) \sim \frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n \cos n\tau + b_n \sin n\tau).$$

Moreover, not every convergent trigonometric series is necessarily a Fourier series of any function on $[-\pi,\pi]$.

Like other classes of function series, Fourier series are used as a possible way of representing (or describing) a function $f(\tau)$. Note that this representation is an alternative to representing this function, for example, as a Taylor series. Indeed, the Taylor series expansion is performed in a small neighborhood of a certain point $\tau_0 \in [-\pi, \pi]$, while the Fourier series represents $f(\tau)$ over the entire (not small!) segment $[-\pi, \pi]$.

Therefore, of primary interest are the conditions that allow us to draw conclusions about the properties of the function itself based on the properties of the sum of the Fourier series $f(\tau)$.

Let us describe these conditions by first giving the following definitions.

> Function $f(\tau)$ we'll call *piecewise continuous on a segment* $[-\pi, \pi]$, if it is continuous at every point of this segment, except, perhaps, for a finite number of points at which it has finite one-sided limits.
>
> Function $f(\tau)$ we'll call *piecewise smooth on a segment* $[-\pi, \pi]$, if its derivative is piecewise continuous on the interval $[-\pi, \pi]$.

Firstly, an important property of Fourier series is described by *Riemann's oscillation lemma*.

The coefficients of the Fourier series for any absolutely integrable function tend to zero as $n \to \infty$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0. \tag{5}$$

Note that these formulas can be refined if additional assumptions are made about the properties of the function $f(\tau)$. See Theorems C and D.

Secondly, the following statements are true:

Theorem A. 1) Fourier series for a piecewise smooth function $f(\tau)$ converges at every point $\tau_0 \in (-\pi,\pi)$ to value $\dfrac{f(\tau_0+0)+f(\tau_0-0)}{2}$, which implies convergence at points of continuity $f(\tau)$ to value $f(\tau_0)$.

2) At points $\tau_0=-\pi$ And $\tau_0=\pi$ The Fourier series converges to the value $\dfrac{f(-\pi+0)+f(\pi-0)}{2}$.

Theorem B. If $f(\tau)$ has on $[-\pi,\pi]$ *continuous derivatives* up to order $N-1$ inclusive, for which $f^{(k)}(-\pi)=f^{(k)}(\pi)\quad \forall k=[0,N-1]$ And *piecewise continuous derivative* order N, then the Fourier series converges *absolutely and evenly* to function $f(\tau)$ on $[-\pi,\pi]$ and at the same time

$$\left| f(\tau)-S_m(\tau,f(\tau)) \right| < \frac{\beta_m}{m^{N-1/2}} \qquad \forall \tau \in [-\pi,\pi],$$

Where $\lim\limits_{m\to\infty}\beta_m=0$, A $S_m(\tau,f(\tau))=\dfrac{a_0}{2}+\sum\limits_{n=1}^{m}(a_n\cos n\tau+b_n\sin n\tau)$ - partial sum of the Fourier series (of the order m) for function $f(\tau)$.

Example 02. Find the trigonometric Fourier series expansion of the 'step function'

$$f(\tau)=\begin{cases} 0, \text{ если } -\pi \le \tau \le 0, \\ 1, \text{ если } \quad 0<\tau \le \pi \end{cases}$$

and plot the sum of this series.

Solution: 1. Note that when $n \ge 1$ All $a_n = 0$ in force $\int\limits_0^{\pi} \cos n\tau \, d\tau = 0$, while $a_0 = \frac{1}{\pi}\int\limits_0^{\pi} d\tau = 1$.

2. Let's find

$$b_n = \frac{1}{\pi}\int\limits_0^{\pi} \sin n\tau \, d\tau = -\frac{1}{\pi n}\cos n\tau \Big|_0^{\pi} = \frac{1-\cos n\pi}{\pi n} = \begin{cases} 0, \text{ если } n = 2k, \\ \frac{2}{\pi n}, \text{ если } n = 2k+1. \end{cases}$$

3. Then, moving from the index n to index k, we get that

$$f(\tau) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{+\infty} \frac{1}{2k+1}\sin(2k+1)\tau .$$

4. Schedule *series sums* will look like

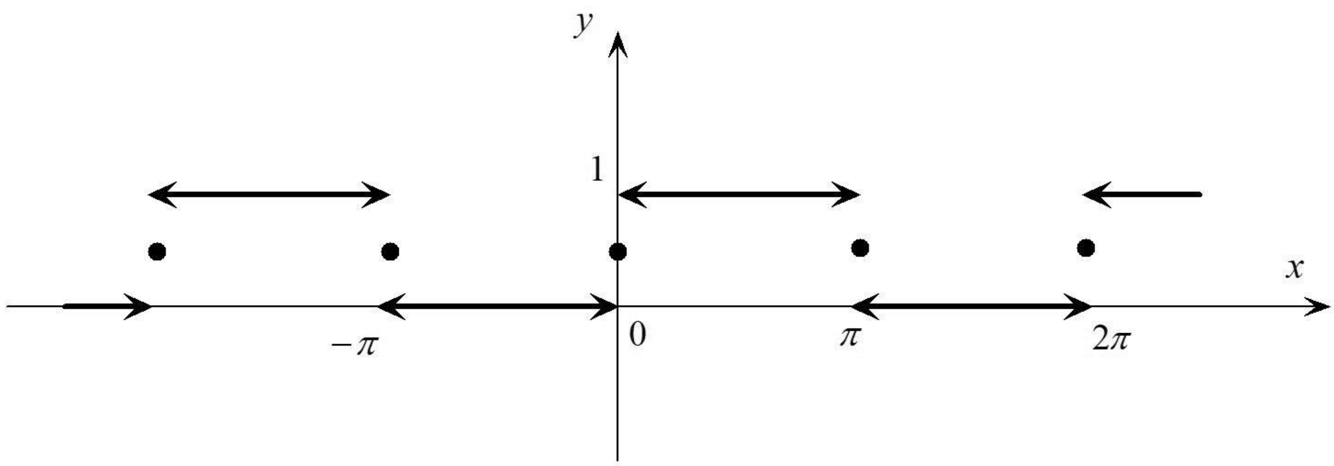


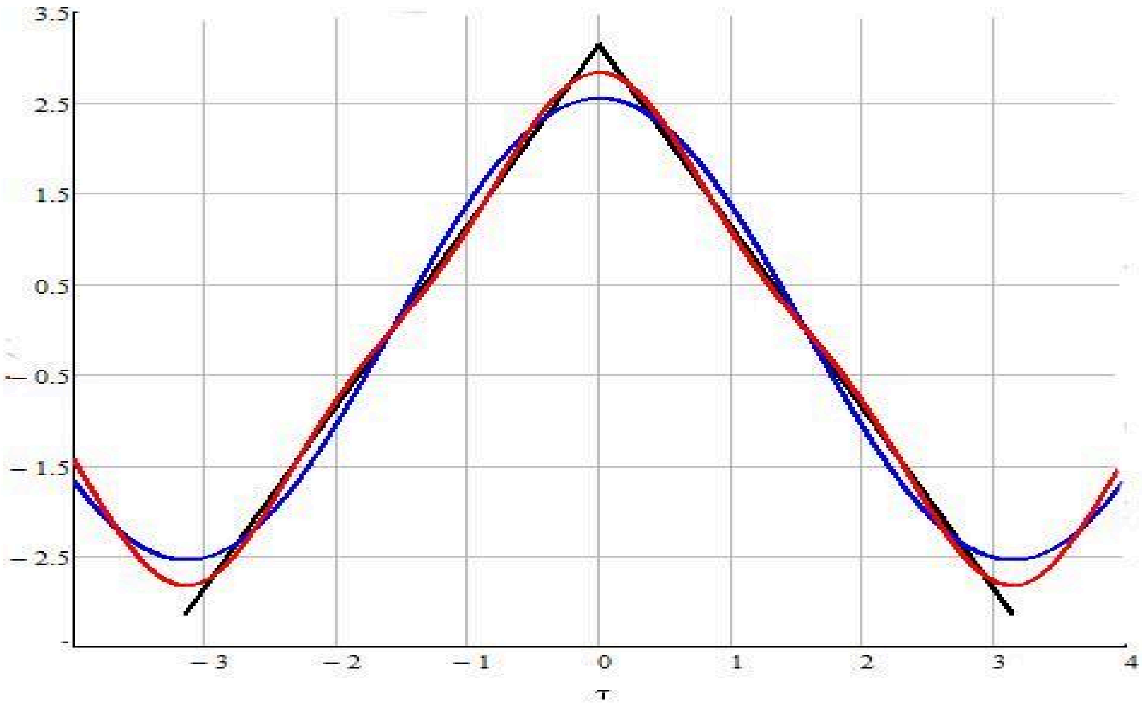
Fig. 1.

The following figures illustrate the statements of Theorems A and B.

In Fig. Figure 2 shows two partial sums of the Fourier series for the function $f(\tau) = \pi - 2|\tau|$, as well as the graph of the function itself (highlighted in black) on the segment $[-\pi, \pi]$. When solving example 04 it will be shown that $f(\tau) \sim \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\tau}{(2m+1)^2}$.

In this case $N = 1$. Partial amounts for $m = 0$ (blue line) and $m = 1$ (red line) respectively have the form

$$S_1(\tau, f(\tau)) = \frac{8}{\pi}\cos\tau \qquad\qquad S_3(\tau, f(\tau)) = \frac{8}{\pi}\cos\tau + \frac{8}{9\pi}\cos 3\tau$$.

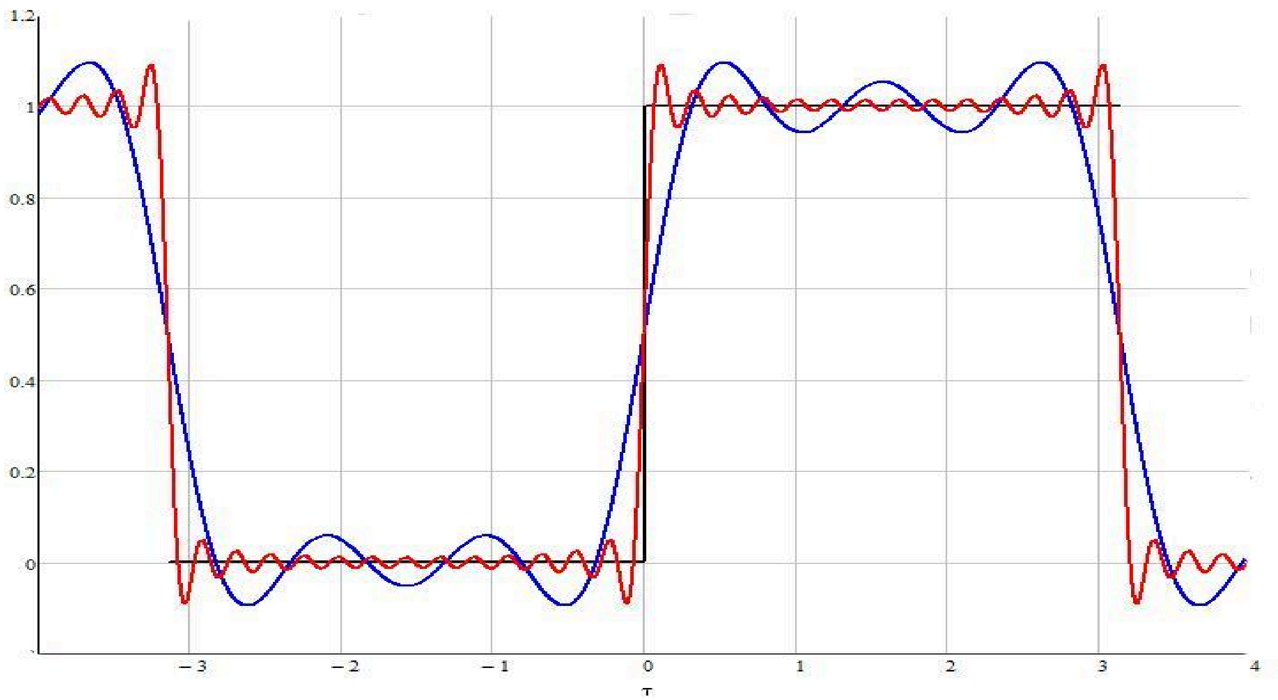


Rice. 2.

In Fig. Figure 3 shows two partial sums of the Fourier series for the 'step function' $f(\tau)=\begin{cases} 0, \text{при } -1 \le \tau \le 0, \\ 1, \text{при } \quad 0 < \tau \le 1, \end{cases}$ as well as the graph of this function (highlighted in black) on the segment $[-\pi,\pi]$.

From example 02 we have $f(\tau) \;\sim\; \frac{1}{2}+\frac{2}{\pi}\sum_{k=0}^{+\infty}\frac{1}{2k+1}\sin(2k+1)\tau$.

In this case $N=0$. Partial sums of order 5 (blue line) and order 29 (red line) are of the form $S_{2p+1}(\tau, f(\tau)) = \frac{1}{2}+\frac{2}{\pi}\sum_{k=0}^{p}\frac{\sin(2k+1)\tau}{2k+1}$, Where $p = 2 \text{ и } 14$ respectively.



Rice. 3.

Formulas (5) clarify:

Theorem C: If $f(\tau)$ has a period 2π And $f^{(k-1)}(\tau)$ piecewise smooth on $[-\pi,\pi]$, then the coefficients of the Fourier series $a_n, b_n = o\left(\frac{1}{n^k}\right)$ при $n \to \infty$.

Here $k \ge 1$.

Theorem D: If $f(\tau)$ has a period 2π and derivative $f^{(k-2)}(\tau)$ at $k > 2$ continuous on $[-\pi,\pi]$, A $f^{(k-1)}(\tau)$ – piecewise continuously differentiable on $[-\pi,\pi]$, then the coefficients of the Fourier series $a_n, b_n = O\left(\frac{1}{n^k}\right)$ при $n \to \infty$.

Note that in Theorem D $f^{(k-1)}(\tau)$ is not continuous, but $k \ge 2$.

It follows from Theorem B that not every convergent trigonometric series is a Fourier series of some piecewise continuous function.

For example, a series $\sum\limits_{k=2}^{+\infty}\frac{\sin kx}{\ln k}$ by the Dirichlet criterion converges pointwise on $[-\pi,\pi]$, but for its coefficients $b_k=\frac{1}{\ln k}\neq O\left(\frac{1}{k}\right)$.

Example 03. Estimate the order in which the coefficients of the Fourier series tend to zero for the function $f(\tau)=(\tau^2-\pi^2)\sin^2\tau$.

Solution: 1. This function is infinitely differentiable on the interval $(-\pi,\pi)$. Therefore, to make it smooth $2\pi-$ periodic continuation to the entire real axis is sufficient to perform for any non-negative integer k equality $f^{(k)}(-\pi+0)=f^{(k)}(\pi-0)$.

2. We need to find out to what extent *the maximum* k this equality will be satisfied. To do this, we will sequentially calculate the derivatives of the function $f(\tau)$, found. for example, using the Leibniz formula, and compare their limits at $\tau\to\pi-0$ And $\tau\to-\pi+0$.

3. We have for $k=0$ $f(-\pi+0)=0=f(\pi-0)$.

At $k=1$

$$f'(\tau)=\tau(1-\cos 2\tau)+(\tau^2-\pi^2)\sin 2\tau \ .$$

Means, $f'(-\pi+0)=0=f'(\pi-0)$.

At $k=2$

$$f''(\tau)=(1-\cos 2\tau)+4\tau\sin 2\tau+2(\tau^2-\pi^2)\cos 2\tau \ .$$

Then $f''(-\pi+0)=0=f''(\pi-0)$.

If $k=3$, That

$$f^{(3)}(\tau)=6\sin 2\tau+12\tau\cos 2\tau-4(\tau^2-\pi^2)\sin 2\tau \ .$$

Then it will be $f^{(3)}(-\pi+0)=-12\pi\neq 12\pi=f^{(3)}(\pi-0)$.

4. Since derivatives from $f(\tau)$ are continuous up to order 2, and the derivative of order 3 is piecewise continuous, then by virtue of Theorem D, we obtain the required estimate $a_n, b_n=O\left(\dfrac{1}{n^4}\right)$.

In the practical use of Fourier series, there is often a need to obtain an expansion of a specific function not over the entire orthogonal system, but over some subset of it.

Example 04. Find the Fourier series expansion of the function $f(\tau) = \pi - 2\tau \quad \tau \in (0, \pi]$ according to the system

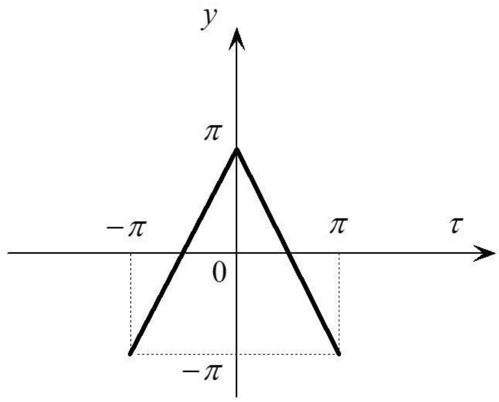
1) even when $\tau \in [-\pi, \pi]$ functions $\{1, \cos\tau, \ldots \cos n\tau, \ldots \quad \forall n \in \mathbf{N}\}$,

2) odd when $\tau \in [-\pi, \pi]$ functions $\{\sin\tau, \ldots \sin n\tau, \ldots \quad \forall n \in \mathbf{N}\}$.

Construct sum graphs for each of these series.

Solution: The basis for solving this problem can be the statement following from formulas (2): for *even* on $[-\pi, \pi]$ functions $f(\tau)$ *All* odds $b_n \quad \forall n \in \mathbf{N}$ in the Fourier series (3) *are equal to zero*. Similarly, for an odd function all coefficients $a_n = 0$ for all non-negative integers n.

1. In the first case, let us define $f(\tau)$ for the entire segment $[-\pi, 0]$ so that it turns out to be even on the segment $[-\pi, \pi]$ (see Fig. 4a) In this case, everything $b_n = 0$.



Rice. 4a

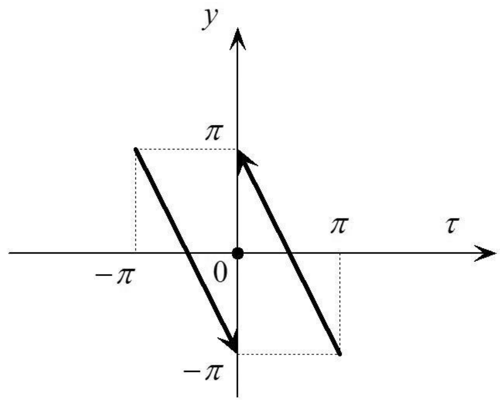


Fig.4c

2. Find the coefficients a_n .

From Fig. 4a, according to the geometric meaning of the definite integral, it is obvious that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,d\tau = \frac{1}{\pi}\int_{0}^{\pi}(\pi - 2\tau)\,d\tau = 0 .$$

For natural n we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\cos n\tau\,d\tau = \frac{2}{\pi}\int_{0}^{\pi}(\pi - 2\tau)\cos n\tau\,d\tau =$$

$$= 2\int_{0}^{\pi}\cos n\tau\,d\tau - \frac{4}{\pi}\int_{0}^{\pi}\tau\cos n\tau\,d\tau =$$

$$= \frac{2\sin n\tau}{n}\Bigg|_0^{\pi} - \left[\frac{4\tau\sin n\tau}{\pi n}\Bigg|_0^{\pi} - \frac{4}{\pi n}\int_{0}^{\pi}\sin n\tau\,d\tau\right] =$$

$$= -\frac{4\cos n\tau}{\pi n^2}\Bigg|_0^{\pi} = \frac{(-4)(\cos n\pi - 1)}{\pi n^2} = \begin{cases} 0, & \text{если } n = 2m, \\ \dfrac{8}{\pi n^2}, & \text{если } n = 2m+1. \end{cases}$$

Finally, for the required series we obtain

$$f(\tau) \sim \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\tau}{(2m+1)^2} .$$

The graph of the sum of this series is shown in Fig. 5.

3. In the second case, we further define $f(\tau)$ for the entire segment $[-\pi,0]$ so that it turns out to be odd on the segment $[-\pi,\pi]$ (see Fig. 4c) In this case, everything $a_n = 0$.

We'll find b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\sin n\tau\, d\tau = \frac{2}{\pi}\int_{0}^{\pi}(\pi - 2\tau)\sin n\tau\, d\tau =$$

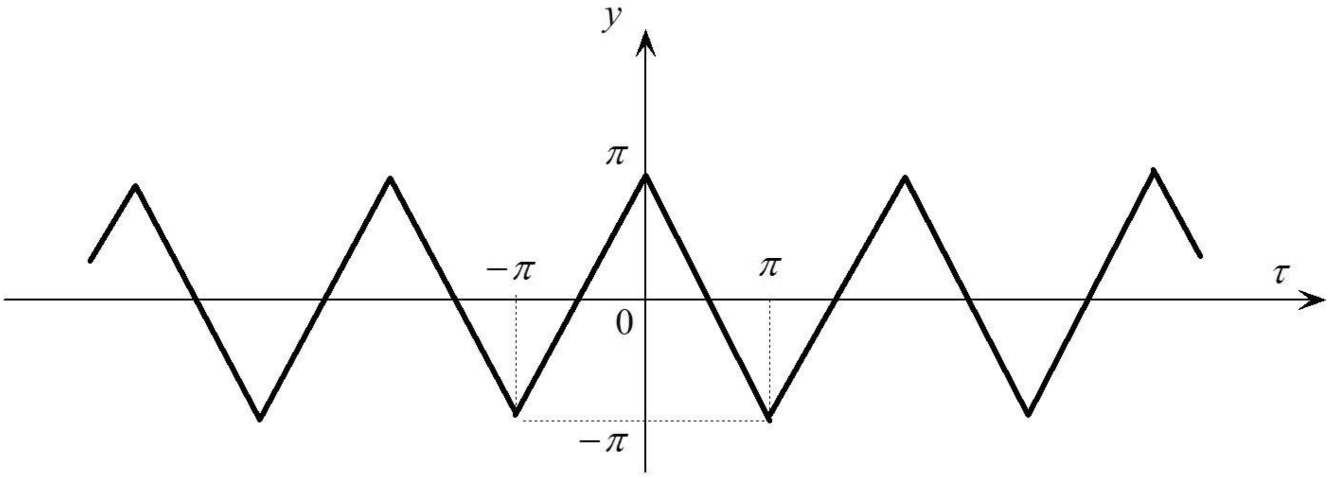
$$= 2\int_0^{\pi}\sin n\tau\, d\tau - \frac{4}{\pi}\int_0^{\pi}\tau\sin n\tau\, d\tau =$$

$$= -\frac{2\cos n\tau}{n}\Bigg|_0^{\pi} - \frac{4}{\pi}\left[-\frac{\tau\cos n\tau}{n}\Bigg|_0^{\pi} + \int_0^{\pi}\cos n\tau\, d\tau\right] =$$

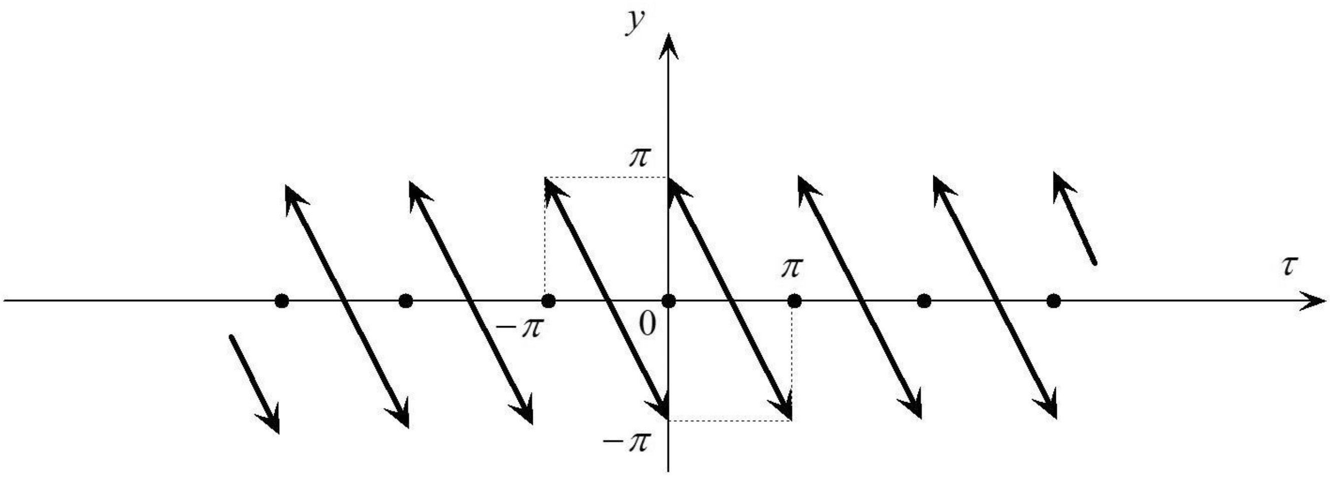
$$= -\frac{2\cos n\pi}{n} + \frac{2}{n} - \frac{4}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\sin n\tau}{n}\Bigg|_0^{\pi}\right] =$$

$$= \frac{2\cos n\pi}{n} + \frac{2}{n} = \begin{cases} 0, & \text{если } n = 2m-1, \\ \dfrac{4}{n}, & \text{если } n = 2m. \end{cases}$$

Finally, for the required series we obtain $f(\tau) \sim 2\sum_{m=1}^{\infty}\frac{\sin 2m\tau}{m}$.
The graph of the sum of this series is shown in Fig. 6.



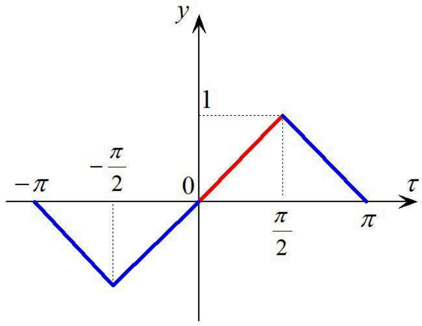
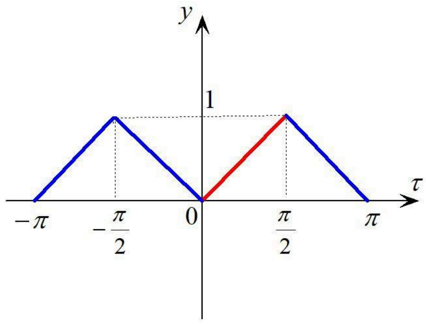
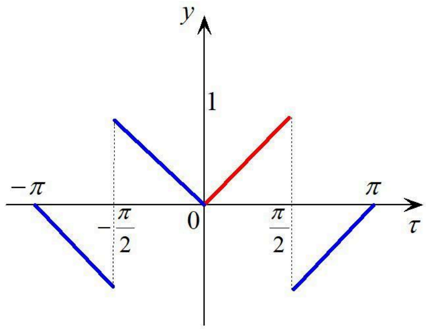
Rice. 5.

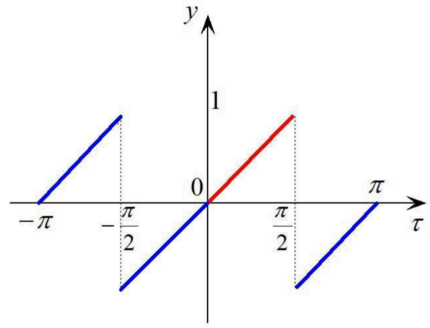


Rice. 6.

Make sure that Theorem D holds for both cases.

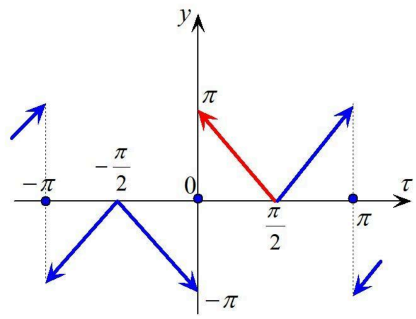
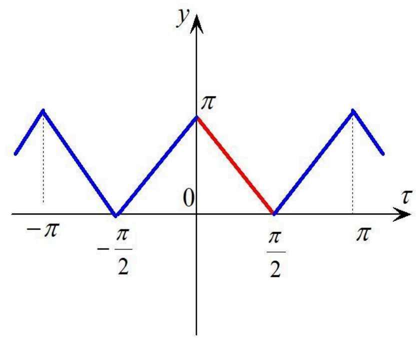
A more complex example is the problem of Fourier series expansion of a function $f(\tau)$, defined on the interval $\left[0,\frac{\pi}{2}\right]$, by system *multiple even or odd arcs*. Methods for extending this function to a segment $[-\pi,\pi]$, providing the required type of decomposition, are shown in the following table.

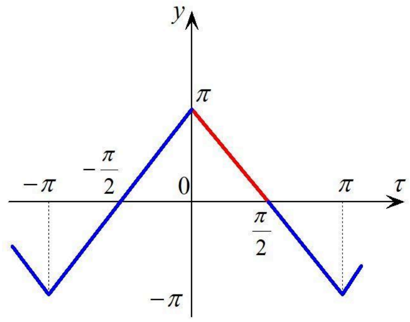
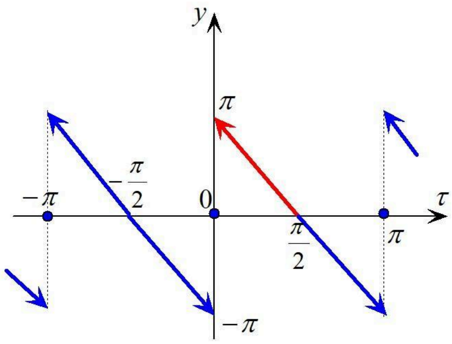
System type	**Additional determination method**
Sines of odd multiple arcs $\sin(2n+1)\tau$	
Cosines of even multiple arcs $\cos 2n\tau$	
Cosines of odd multiple arcs $\cos(2n+1)\tau$	

Sines of even multiple arcs $\sin 2n\tau$	

Example 04a. Construct a similar table of options for the Fourier series expansion of the function $f(\tau)=\pi-2\tau \quad \tau\in(0,\frac{\pi}{2}]$ by systems of multiple even or odd arcs. Compare the solutions obtained in Example 4 with this table.

Solution:

System type	**Additional determination method**
Sines of odd multiple arcs $\sin(2n+1)\tau$	
Cosines of even multiple arcs $\cos 2n\tau$	

Cosines of odd multiple arcs $\cos(2n+1)\tau$	
Sines of even multiple arcs $\sin 2n\tau$	

An important question for practice is: under what conditions is formal differentiation or integration of the Fourier series of a function $f(\tau)$ there will be a Fourier series corresponding to the function $f'(\tau)$ or $\int\limits_{\tau_0}^{\tau} f(u)\,du$?
The answer to this question is given by:

Theorem E: If $f(\tau)$ has a period 2π and is piecewise smooth on $[-\pi,\pi]$, then its Fourier series (3) converges uniformly to $f(\tau)$ on the entire real axis.

Theorem F: If $f(\tau)$ has a period 2π and is piecewise smooth on $[-\pi,\pi]$, then the Fourier series of the function $f'(\tau)$ looks like

$$f'(\tau) \quad \sim \quad \sum_{n=1}^{+\infty} n(-a_n \sin n\tau + b_n \cos n\tau).$$

Theorem G: If $f(\tau)$ has a period 2π and piecewise continuous on $[-\pi,\pi]$, then the Fourier series for the function $\Phi(\tau) = \int\limits_{\tau_0}^{\tau} f(u)\,du$ is obtained by formal integration (3) over the variable τ .

Additional important theoretical facts

Fourier series expansion can also be performed for functions with period $2L$, specified for $\tau \in [-L, L]$ How $f(\tau)$.

In this case, the Fourier series is defined as follows

$$f(\tau) \quad \sim \quad \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos\frac{\pi n \tau}{L} + b_n \sin\frac{\pi n \tau}{L}) \,, \tag{6}$$

Where $$a_m = \frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{\pi m u}{L}\,du \qquad \forall m \in \{0, \mathbf{N}\} \qquad \text{и} \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(u)\sin\frac{\pi m u}{L}\,du \qquad \forall m \in \mathbf{N} \,. \tag{7}$$

Fair assessments:

1) If the square of the function $f(\tau)$ integrate (perhaps in an improper sense) on the interval $[-\pi,\pi]$, then for partial sums of the Fourier series the equality called *minimal property of partial sums of a Fourier series*

$$\int_{-\pi}^{\pi} \left| f(t) - S_m(\tau, f(\tau)) \right|^2 d\tau \quad = \quad \min_{\forall T_m(\tau)} \int_{-\pi}^{\pi} \left| f(t) - T_m(\tau, f(\tau)) \right|^2 d\tau \ . \tag{8}$$

The minimum on the right side of equality (8) is taken over the set of trigonometric polynomials of the following form:

$$T_m(\tau, f(\tau)) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos n\tau + b_n \sin n\tau) \ .$$

2) For the coefficients of the Fourier series, it is true *equality Parseval*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(\tau)\, d\tau \ . \tag{9}$$

Finally, we note the possibility of representing the Fourier series in complex form.

Indeed, according to Euler's formula

$$\cos n\tau = \frac{e^{in\tau} + e^{-in\tau}}{2} \quad \text{And} \quad \sin n\tau = \frac{e^{in\tau} - e^{-in\tau}}{2i},$$

which allows series (3) to be written as $\sum\limits_{n=-\infty}^{+\infty} c_n e^{in\tau}$, Where

$$c_0 = a_0, \qquad c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2} \qquad n \in \mathbf{N}.$$

If series (3) is a Fourier series, with coefficients calculated using formulas (4), then

$$c_n = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(u)e^{-inu}du \quad n \in \mathbf{Z} \qquad \Rightarrow \qquad f(\tau) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\tau}.$$

Partial sum of Fourier series and Dirichlet kernel.

Function type $D_n(\tau) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\tau$ called *Dirichlet kernel*.

Using the formula $\cos\alpha \cdot \sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$ gives

$$D_n(\tau) = \frac{1}{2\sin\frac{\tau}{2}} \left(\sin\frac{\tau}{2} + \sum_{k=1}^{n} 2\cos k\tau \cdot \sin\frac{\tau}{2} \right) =$$

$$= \frac{1}{2\sin\frac{\tau}{2}} \left(\sin\frac{\tau}{2} + \left(\sin\frac{3\tau}{2} - \sin\frac{\tau}{2} \right) + \left(\sin\frac{5\tau}{2} - \sin\frac{3\tau}{2} \right) + \dots + \left(\sin\frac{(2n+1)\tau}{2} - \sin\frac{(2n-1)\tau}{2} \right) \right) =$$

$$= \frac{\sin\frac{(2n+1)\tau}{2}}{2\sin\frac{\tau}{2}} \qquad \tau \neq 2\pi m,\ m \in \mathbf{Z}.$$

For 2π -periodic, absolutely integrable function $f(\tau)$ the partial sums of its Fourier series can be represented in the form $S_n(\tau) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(u) f(\tau+u)\, du$.

Summation of Fourier series by the method of arithmetic means.

Definition Arithmetic mean of partial sums of the Fourier series of the form $\sigma_n(\tau)=\frac{1}{n+1}\sum_{k=0}^{n}S_k(\tau)$ called *Fejer sum* for function $f(\tau)$.

Arithmetic mean of Dirichlet kernels of the form $\Phi_n(\tau)=\frac{1}{n+1}\sum_{k=0}^{n}D_k(\tau)$ called *Fejer core* for function $f(\tau)$.

We will call the Fourier series summable at the point τ by the method of arithmetic averages, if there is a finite limit at this point $\lim_{n\to\infty}\sigma_n(\tau)$.

Theorem H: If $f(\tau)$ continuous on $[-\pi,\pi]$ And $f(-\pi)=f(\pi)$, That $\sigma_n(\tau)$ converges uniformly to $f(\tau)$ on $[-\pi,\pi]$.

Bessel's inequality and Parseval's equality.

1) Let for *piecewise continuous* on the segment $[-\pi,\pi]$ functions $f(\tau)$ the trigonometric Fourier series has the form

$$f(\tau)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\tau+b_n\sin n\tau\right).$$

Then Bessel's inequality is true

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\le\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(\tau)\,d\tau,$$

from which the convergence of the series can follow $\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$.

2) If the function $f(\tau)$ *quadratically integrable* on the segment $[-\pi,\pi]$, that is, there is $\int_{-\pi}^{\pi}f^2(\tau)\,d\tau<+\infty$, then Parseval's equality holds

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(\tau)\,d\tau.$$

3) Using Parseval's equality, you can find the sums of some number series.

Example 01. Find the sum of a number series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: Fourier series for a function $f(\tau)=\tau$ looks like $\tau \sim 2\sum_{n=1}^{\infty}\frac{(-1)}{n}\sin n\tau$. Everything is here $a_n=0$, A $b_n=2\frac{(-1)^n}{n}$.

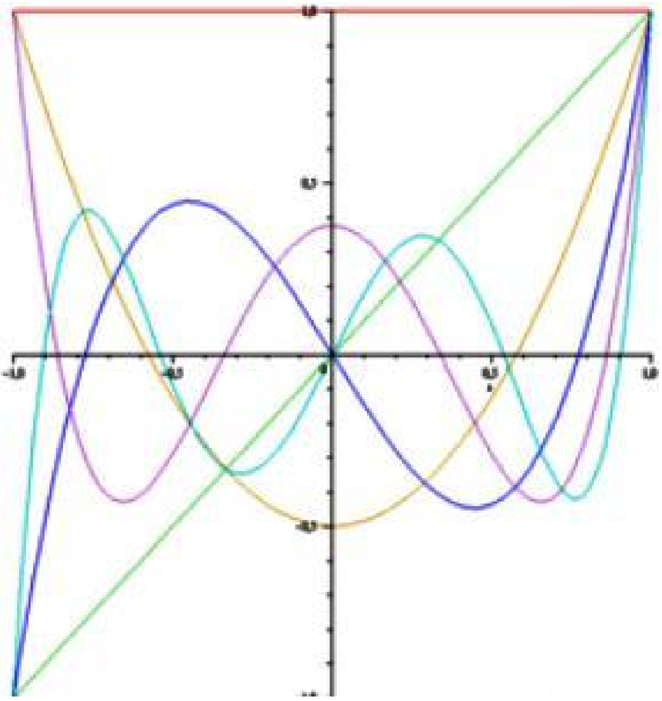
Square $f(\tau)=\tau$ integrate on $[-\pi,\pi]$, therefore Parseval's equality is true

$$\sum_{n=1}^{\infty}\frac{4}{n^2}=\frac{1}{\pi}\int_{-\pi}^{\pi}\tau^2 d\tau \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Functions can also be represented as Fourier series using other orthogonal systems. It can be shown that the system *Legendre polynomial*

$$P_n(\tau) = \frac{1}{2^n n!} \cdot \frac{d^n(\tau^2-1)^n}{d\tau^n} \quad n \in \mathbf{N}, \tau \in [-1,1] \quad \text{And} \quad P_0(\tau) = 1$$

is orthogonal if the dot product is defined as $(x,y) = \int_{-1}^{1} x(\tau)y(\tau)\,d\tau$. In this case $(P_m(\tau), P_n(\tau)) = \frac{2\delta_{nm}}{2n+1}$ and normalizing factor η_n for a polynomial $P_n(\tau)$ will be equal $\sqrt{\frac{2n+1}{2}}$.



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Example 05. Find the Fourier series expansion of the Legendre polynomial system of the function $f(\tau)=\begin{cases}1, \text{если } \tau\in[0,1],\\ 0, \text{если } \tau\in[-1,0).\end{cases}$

Solution: 1. Let the required expansion have the form $f(\tau)\sim\sum\limits_{n=0}^{+\infty} r_n P_n(\tau)$. It's easy to see that $r_0=\frac{1}{\sqrt{2}}\int\limits_{-1}^{1} f(\tau)P_0(\tau)\,d\tau=\frac{1}{\sqrt{2}}\int\limits_{0}^{1} d\tau=\frac{1}{\sqrt{2}}$.

2. Let's find the coefficients r_n for $n\geq 1$. We have

$$r_n=\sqrt{\frac{2n+1}{2}}\cdot\int\limits_{-1}^{1} f(\tau)P_n(\tau)\,d\tau=\frac{1}{2^n n!}\sqrt{\frac{2n+1}{2}}\int\limits_{0}^{1}\frac{d^n(\tau^2-1)^n}{d\tau^n}\,d\tau=$$

$$=\frac{1}{2^n n!}\sqrt{\frac{2n+1}{2}}\,\frac{d^{n-1}(\tau^2-1)^n}{d\tau^{n-1}}\Bigg|_0^1$$

Let's calculate the value of the expression $\left.\dfrac{d^{n-1}(\tau^2-1)^n}{d\tau^{n-1}}\right|_0^1$.

(10)

Due to the rules of differentiation, when $\tau = 1$ this expression obviously equals 0.

Let $\tau = 0$. According to Newton's binomial formula

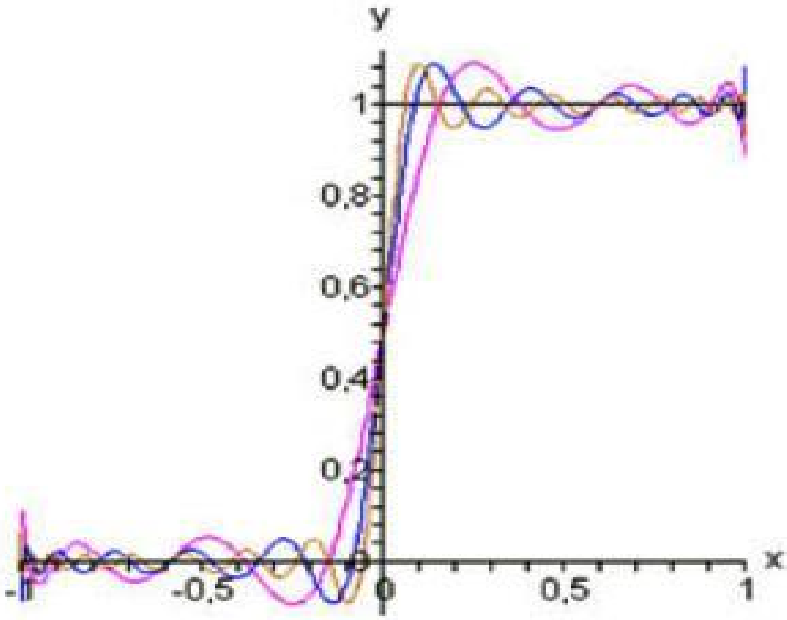
$$\begin{aligned}\frac{d^{n-1}(\tau^2-1)^n}{d\tau^{n-1}} &= \frac{d^{n-1}}{d\tau^{n-1}}\sum_{k=0}^{n} C_n^{n-k}\tau^{2n-2k}(-1)^k = \\ &= \sum_{k=0}^{n}(-1)^k C_n^{n-k}\left(\frac{d^{n-1}}{d\tau^{n-1}}\tau^{2n-2k}\right) = \\ &= \sum_{k=0}^{n} C_n^{n-k}(-1)^k(2n-2k)(2n-2k-1)\cdots(n-2k+2)\tau^{n-2k+1}.\end{aligned}$$

It should be here $n-2k+1 \geq 0$. For all such *even* n in force $\tau = 0$ we find that (10) is equal to 0. When *odd* n, and those that $n-2k+1>0$, again this expression will obviously equal 0.

There remains the case of odd n, for which $n-2k+1=0$. Note that for each $n=2m+1$ there is always one number $k=m+1$ such that this equality is true.

Substituting these values, we obtain for (10) the value $C_{2m+1}^{m+1}(-1)^{m+1}(2m)!$. Where do we find it from?

$$r_m = \frac{1}{2^{2m+1}(2m+1)!}\sqrt{\frac{4m+3}{2}}C_{2m+1}^{m+1}(-1)^{m+1}(2m)! = \frac{(-1)^{m+1}\sqrt{4m+3}\,(2m)!}{2^{2m+1}\sqrt{2}\,(m+1)!}.$$



n = 5, *n* = 10, *n* = 15