

Metric sets and normed linear spaces

Let us first recall the definitions *convergence* And *fundamentality* for number sequences.

- 1) Number A is *limit* number sequence $\{x_n\}$ (symbolically this is written as $\lim_{n \rightarrow \infty} x_n = A$), If
- $$\forall \varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \rightarrow |x_n - A| < \varepsilon.$$

- 2) Number sequence $\{x_n\}$ called *fundamental*, If:

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \text{ и } \forall m \geq N_\varepsilon \rightarrow |x_n - x_m| < \varepsilon$$

or, what is the same,

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \text{ и } \forall p \in \mathbf{N} \rightarrow |x_{n+p} - x_n| < \varepsilon.$$

Takes place **theorem**:

In order for a number sequence to have a limit (was convergent), it is necessary and sufficient for it to be fundamental.

Definition 01 A multitude $X := \{x, y, z, \dots\}$ called *metric*, if each ordered pair of its elements x and y is associated with a unique non-negative number $\rho(x, y)$ such that $\forall x, y, z \in X$ axioms are satisfied:

$$1) \rho(x, y) = 0 \Leftrightarrow x = y,$$

$$2) \rho(x, y) = \rho(y, x),$$

$$3) \rho(x, y) \leq \rho(x, z) + \rho(z, y) \cdot (\text{triangle inequality})$$

In a metric set, you can use the definitions *convergent* and *fundamental* sequences.

Fundamentality always follows from convergence, but not vice versa!

The set is called *complete*, if every fundamental sequence in it converges to an element of this set.

Definition 02 *Linear space* $X := \{x, y, z, \dots\}$ called *normalized*, If to everyone its element x compliant *the only thing* non-negative number $\|x\|$ such that $\forall x, y \in X$ And $\forall \lambda \in \mathbf{R}$ axioms are satisfied:

- 1) $\|\lambda x\| = |\lambda| \|x\|$, (*uniformity of norm*),
- 2) $\|x + y\| \leq \|x\| + \|y\|$, (*triangle inequality*),
- 3) $\|x\| = 0 \Leftrightarrow x = o$.

In a normalized linear space, we can take as a metric

$$\rho(x, y) = \|x - y\|.$$

A complete normed space is usually called *Banachov* space.

Example 02. Let $\{x_n\}$ And $\{y_n\}$ are fundamental sequences in some metric set X . Show that the sequence $\{\rho(x_n, y_n)\}$ converges.

Solution: 1) By the triangle inequality we have

$$\rho(x_n, y_n) \leq \rho(x_n, x_m) + \rho(x_m, y_m) + \rho(y_m, y_n)$$

or

$$\rho(x_n, y_n) - \rho(x_m, y_m) \leq \rho(x_n, x_m) + \rho(y_m, y_n).$$

If you swap places n And m , then we get

$$-\rho(x_n, y_n) + \rho(x_m, y_m) \leq \rho(x_m, x_n) + \rho(y_m, y_n)$$

or

$$\rho(x_n, y_n) - \rho(x_m, y_m) \geq -(\rho(x_n, x_m) + \rho(y_m, y_n)).$$

Means,

$$|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_n, x_m) + \rho(y_m, y_n).$$

From fundamentality $\{x_n\}$ And $\{y_n\}$ it follows that

$$\forall \varepsilon > 0 \exists N_\varepsilon : \forall n, m \geq N \rightarrow \rho(x_n, x_m) < \frac{\varepsilon}{2} ; \rho(y_m, y_n) < \frac{\varepsilon}{2} .$$

Then

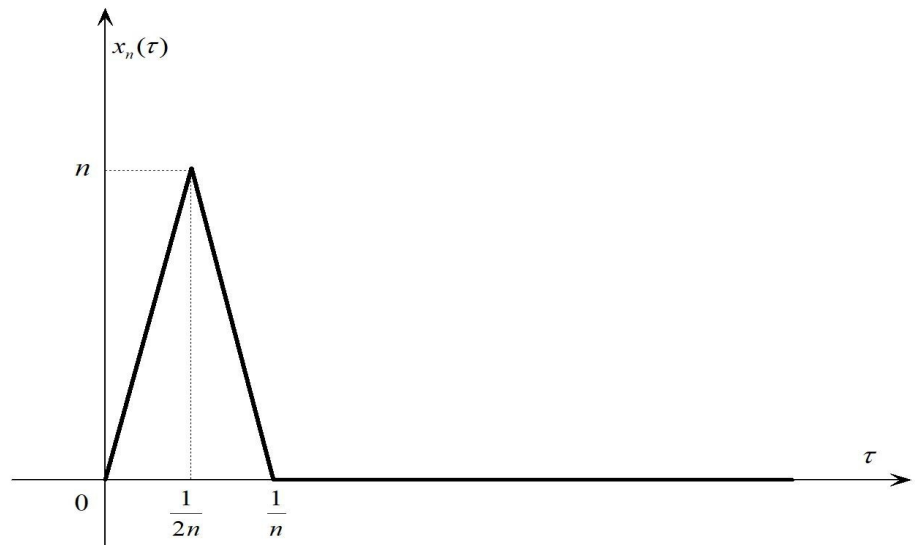
$$|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_n, x_m) + \rho(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

So the sequence $\{\rho(x_n, y_n)\}$ fundamental, and since it *numeric*, then it is convergent, according to the Cauchy criterion.

Example 03. In a set of continuous $[0,1]$ functions $x(\tau)$ functional sequence

$$x_n(\tau) = \begin{cases} 2n^2\tau, & \text{если } \tau \in \left[0, \frac{1}{2n}\right], \\ 2n(1-n\tau), & \text{если } \tau \in \left[\frac{1}{2n}, \frac{1}{n}\right], \\ 0, & \text{если } \tau \in \left[\frac{1}{n}, 1\right] \end{cases}$$

converges pointwise to a function identically equal to zero (see figure).



But it will not be convergent if the metric is given by the formula:

$$\rho(x, y) = \int_0^1 |x(\tau) - y(\tau)| d\tau$$

because $\rho(x_n(\tau), 0) = \frac{1}{2} n \frac{1}{n} = \frac{1}{2} \forall n$.

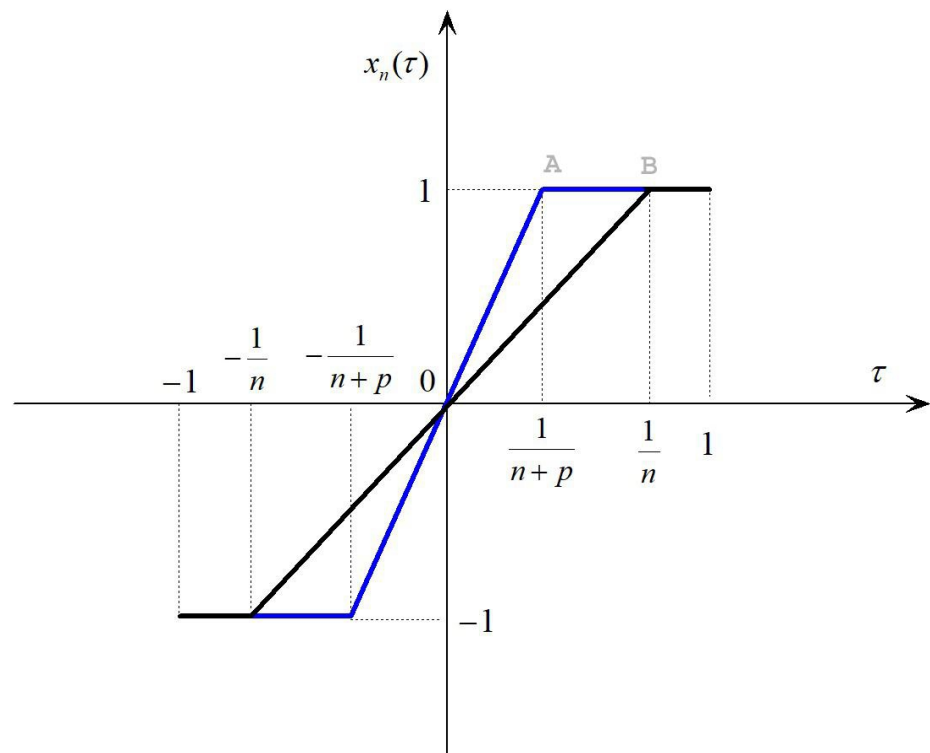
Example 04. Show that a linear space of continuous $[-1, 1]$ functions is not complete if the norm is defined by the formula

$$\|x(\tau)\| = \sqrt{\int_{-1}^1 x^2(\tau) d\tau}.$$

Solution: 1) It is enough to show that in a linear space of continuous $[-1, 1]$ functions $x(\tau)$ functional sequence

$$x_n(\tau) = \begin{cases} -1, & \text{если } \tau \in \left[-1, -\frac{1}{n}\right], \\ n\tau, & \text{если } \tau \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ 1, & \text{если } \tau \in \left[\frac{1}{n}, 1\right] \end{cases}$$

is fundamental, but does not converge to a continuous function (see Fig. 2).



Really, $\lim_{n \rightarrow \infty} x_n(\tau) = \text{sgn } \tau$, which is not continuous.

2) Let us show that the functional sequence $\{x_n(\tau)\}$ fundamental.

Let us obtain an estimate for the squared norm of the difference of functions

$x_{n+p}(\tau)$ And $x_n(\tau)$. We have

$$\begin{aligned} \frac{1}{2} \|x_{n+p}(\tau) - x_n(\tau)\|^2 &= \int_0^{\frac{1}{n+p}} [(n+p)\tau - n\tau]^2 d\tau + \int_{\frac{1}{n+p}}^{\frac{1}{n}} [1 - n\tau]^2 d\tau = \\ &= \int_0^{\frac{1}{n+p}} p^2 \tau^2 d\tau + \int_{\frac{1}{n+p}}^{\frac{1}{n}} (1 - n\tau)^2 d\tau = \frac{p^2 \tau^3}{3} \Big|_0^{\frac{1}{n+p}} + \frac{(n\tau - 1)^3}{3n} \Big|_{\frac{1}{n+p}}^{\frac{1}{n}} = \\ &= \frac{p^2}{3n(n+p)^2} \leq \frac{1}{3n}. \end{aligned}$$

3) Other methods of assessment.

A) Let $n < m$, Then

$$\begin{aligned}\|x_n(\tau) - x_m(\tau)\|^2 &= \int_{-1}^1 (x_n(\tau) - x_m(\tau))^2 d\tau = \int_{-\frac{1}{n}}^{\frac{1}{n}} (x_n(\tau) - x_m(\tau))^2 d\tau \leq \\ &\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} (|x_n(\tau)| + |x_m(\tau)|)^2 d\tau \leq 4 \int_{-\frac{1}{n}}^{\frac{1}{n}} d\tau = \frac{8}{n}.\end{aligned}$$

IN) Use the fact that the area of a triangle OAB equal

$$S_{OAB} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+p} \right) \cdot 1 = \frac{1}{2n} \cdot \frac{p}{n+p} \leq \frac{1}{2n}.$$