Metric sets and normed linear spaces

Let us first recall the definitions *convergence* And *fundamentality* for number sequences.

1) Number A is *limit* number sequence $\{x_n\}$ (symbolically this is written as $\lim\limits_{n\to\infty} x_n = A$), If

$$\forall\varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \quad \to \quad |x_n - A| < \varepsilon .$$

2) Number sequence $\{x_n\}$ called *fundamental*, If:

$$\forall\varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \text{ и } \forall m \geq N_\varepsilon \quad \to \quad |x_n - x_m| < \varepsilon$$

or, what is the same,

$$\forall\varepsilon > 0 \quad \exists N_\varepsilon : \forall n \geq N_\varepsilon \text{ и } \forall p \in \mathbf{N} \quad \to \quad |x_{n+p} - x_n| < \varepsilon .$$

Takes place **theorem**:

> *In order for a number sequence to have a limit* (*was convergent*)*, it is necessary and sufficient for it to be fundamental.*

Definition 01 A multitude $X := \{x, y, z, \ldots\}$ called *metric*, If *each* ordered pair of its elements x And *and* compliant *the only thing* non-negative number $\rho(x, y)$ such that $\forall x, y, z \in X$ axioms are satisfied:

1) $\rho(x, y) = 0 \quad \Leftrightarrow \quad x = y$,

2) $\rho(x, y) = \rho(y, x)$,

3) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$. (*triangle inequality*)

In a metric set, you can use the definitions *convergent* And *fundamental* sequences.

Fundamentality always follows from convergence, but not vice versa!

The set is called *complete*, if every fundamental sequence in it converges to an element of this set.

Definition 02 *Linear space* $X := \{x, y, z, \ldots\}$ called *normalized*, If *to everyone* its element x compliant *the only thing* non-negative number $\|x\|$ such that $\forall x, y \in X$ And $\forall \lambda \in \mathbf{R}$ axioms are satisfied:

1) $\|\lambda x\| = |\lambda| \, \|x\|$, (*uniformity of norm*),
2) $\|x + y\| \le \|x\| + \|y\|$, (*triangle inequality*),
3) $\|x\| = 0 \quad \Leftrightarrow \quad x = o$.

In a normalized linear space, we can take as a metric

$$\rho(x, y) = \|x - y\|.$$

A complete normed space is usually called *Banakhov* space.

Example 02. Let $\{x_n\}$ And $\{y_n\}$ are fundamental sequences in some metric set *X*. Show that the sequence $\{\rho(x_n, y_n)\}$ converges.

Solution: 1) By the triangle inequality we have

$$\rho(x_n, y_n) \le \rho(x_n, x_m) + \rho(x_m, y_m) + \rho(y_m, y_n)$$

or

$$\rho(x_n, y_n) - \rho(x_m, y_m) \le \rho(x_n, x_m) + \rho(y_m, y_n) .$$

If you swap places *n* And *m*, then we get

$$-\rho(x_n, y_n) + \rho(x_m, y_m) \le \rho(x_m, x_n) + \rho(y_m, y_n)$$

or

$$\rho(x_n, y_n) - \rho(x_m, y_m) \ge -\left(\rho(x_n, x_m) + \rho(y_m, y_n)\right) .$$

Means,

$$\left| \rho(x_n, y_n) - \rho(x_m, y_m) \right| \le \rho(x_n, x_m) + \rho(y_m, y_n) .$$

From fundamentality $\{x_n\}$ And $\{y_n\}$ it follows that

$$\forall\varepsilon > 0\ \exists N_\varepsilon :\quad \forall n,m \ge N \quad \to \qquad \rho(x_n, x_m) < \frac{\varepsilon}{2} \quad ; \quad \rho(y_m, y_n) < \frac{\varepsilon}{2} \ .$$

Then

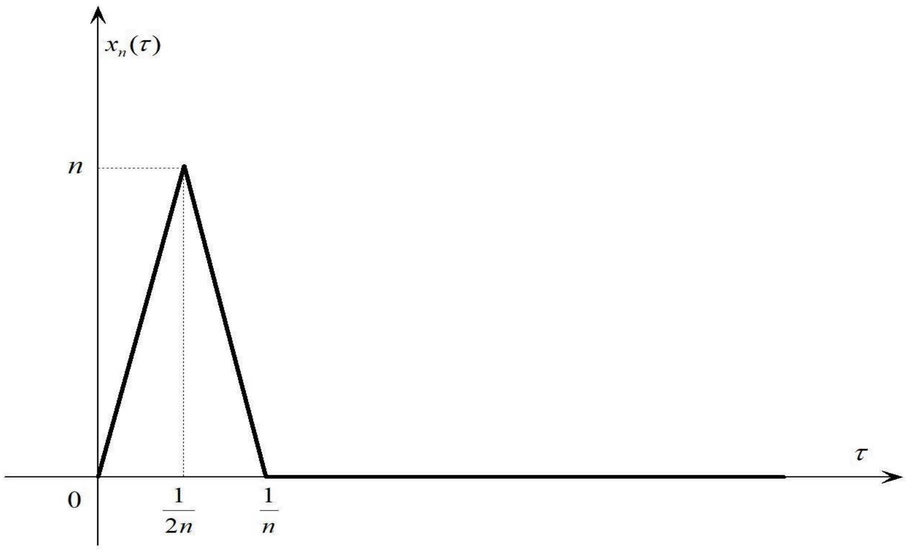
$$\left|\, \rho(x_n, y_n) - \rho(x_m, y_m) \,\right| \le \rho(x_n, x_m) + \rho(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

So the sequence $\{\rho(x_n, y_n)\}$ fundamental, and since it *numeric*, then it is convergent, according to the Cauchy criterion.

Example 03. In a set of continuous $[0,1]$ functions $x(\tau)$ functional sequence

$$x_n(\tau)=\begin{cases} 2n^2\tau, & \text{если } \tau\in\left[0,\dfrac{1}{2n}\right],\\ 2n(1-n\tau), & \text{если } \tau\in\left[\dfrac{1}{2n},\dfrac{1}{n}\right],\\ 0, & \text{если } \tau\in\left[\dfrac{1}{n},1\right]\end{cases}$$

converges pointwise to a function identically equal to zero (see figure).



But it will not be convergent if the metric is given by the formula:

$$\rho(x,y)=\int_0^1 |\,x(\tau)-y(\tau)\,|\,d\tau\,.$$

because $\rho(x_n(\tau),0)=\dfrac{1}{2}n\dfrac{1}{n}=\dfrac{1}{2}\;\forall n$.

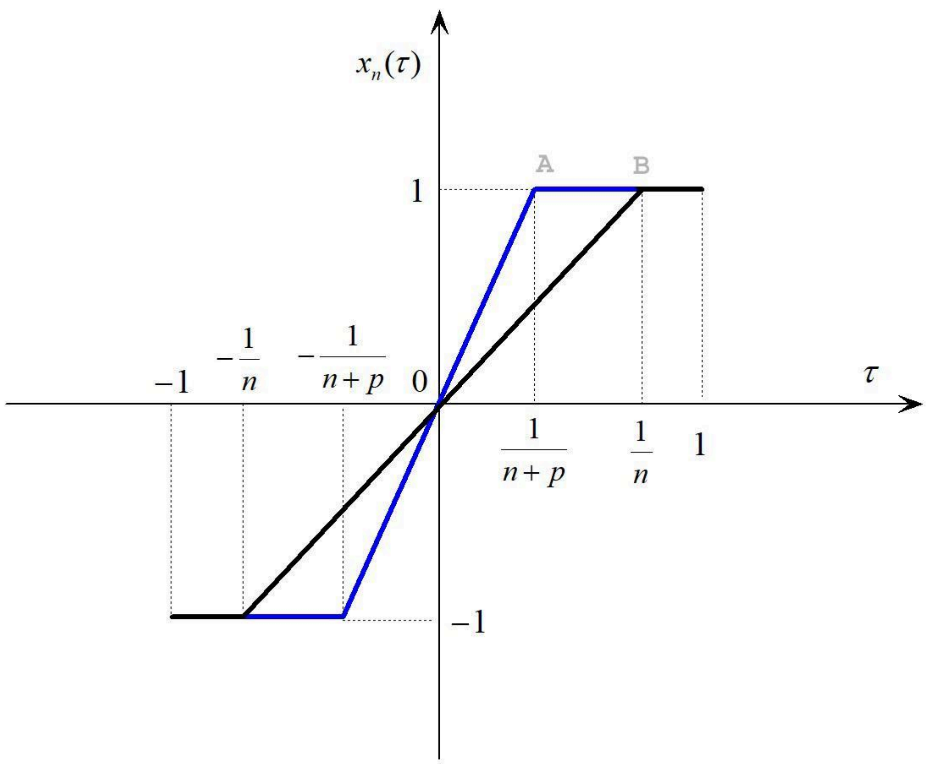
Example 04. Show that a linear space is continuous on $[-1,1]$ functions is not complete if the norm is defined by the formula

$$\| x(\tau) \| = \sqrt{\int_{-1}^{1} x^2(\tau)\,d\tau}\ .$$

Solution: 1) It is enough to show that in a linear space of continuous $[-1,1]$ functions $x(\tau)$ functional sequence

$$x_n(\tau) = \begin{cases} -1, & \text{если } \tau \in \left[-1, -\frac{1}{n}\right], \\ n\tau, & \text{если } \tau \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ 1, & \text{если } \tau \in \left[\frac{1}{n}, 1\right] \end{cases}$$

is fundamental, but does not converge to a continuous function (see Fig. 2).



Really, $\lim\limits_{n\to\infty} x_n(\tau) = \operatorname{sgn}\tau,$ which is not continuous.

2) Let us show that the functional sequence $\{x_n(\tau)\}$ fundamental.

Let us obtain an estimate for the squared norm of the difference of functions $x_{n+p}(\tau)$ And $x_n(\tau)$. We have

$$\frac{1}{2}\left\| x_{n+p}(\tau) - x_n(\tau) \right\|^2 = \int\limits_0^{\frac{1}{n+p}} [(n+p)\tau - n\tau]^2 d\tau + \int\limits_{\frac{1}{n+p}}^{\frac{1}{n}} [1 - n\tau]^2 d\tau =$$

$$= \int\limits_0^{\frac{1}{n+p}} p^2 \tau^2 d\tau + \int\limits_{\frac{1}{n+p}}^{\frac{1}{n}} (1 - n\tau)^2 d\tau = \left. \frac{p^2 \tau^3}{3} \right|_0^{\frac{1}{n+p}} + \left. \frac{(n\tau - 1)^3}{3n} \right|_{\frac{1}{n+p}}^{\frac{1}{n}} =$$

$$= \frac{p^2}{3n(n+p)^2} \le \frac{1}{3n} .$$

3) Other methods of assessment.

A) Let $n < m$, Then

$$\| x_n(\tau) - x_m(\tau) \|^2 = \int\limits_{-1}^{1} \left(x_n(\tau) - x_m(\tau) \right)^2 d\tau = \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \left(x_n(\tau) - x_m(\tau) \right)^2 d\tau \leq$$

$$\leq \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \left(| x_n(\tau) | + | x_m(\tau) | \right)^2 d\tau \leq 4 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} d\tau = \frac{8}{n}.$$

IN) Use the fact that the area of a triangle $0AB$ equal

$$S_{0AB} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+p} \right) \cdot 1 = \frac{1}{2n} \cdot \frac{p}{n+p} \leq \frac{1}{2n} .$$