

COMPLETENESS OF SYSTEMS OF FUNCTIONS.

Below we will use the following definitions and theoretical facts.

In any finite-dimensional linear space, all norms are equivalent (in terms of convergence). In infinite-dimensional spaces this may not hold.

Linear space of continuous on $[a, b]$ functions $f(x)$ with the norm $\|f\|_{CL[a,b]} = \max_{x \in [a,b]} |f(x)|$ we will denote it as $CL[a, b]$ or $C[a, b]$.

Linear space of continuous on $[a, b]$ functions $f(x)$ with the norm $\|f\|_{CL_1[a,b]} = \int_a^b |f(x)| dx$ we will denote it as $CL_1[a, b]$.

Linear space of continuous on $[a, b]$ functions $f(x)$ with the norm $\|f\|_{CL_2[a,b]} = \sqrt{\int_a^b f^2(x) dx}$ we will denote it as $CL_2[a, b]$.

In this case, there are useful estimates:

$$\|f\|_{CL_1[a,b]} \leq (b-a) \|f\|_{CL[a,b]} \quad \text{And} \quad \|f\|_{CL_2[a,b]} \leq \sqrt{b-a} \|f\|_{CL[a,b]}.$$

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It can be shown that from norm convergence $\|f\|_{C_{L_2}[a,b]}$ (as they sometimes say, *convergence in mean square*) does not follow norm convergence $\|f\|_{C_{L[a,b]}}$ (*uniform convergence*), and from norm convergence $\|f\|_{C_{L_1}[a,b]}$ (*convergence on average*) does not imply convergence in the mean square.

Definition. Counting system of elements $\{g_1(x), g_2(x), \dots, g_n(x), \dots\}$ in linear normed space L called *full*, If

$$\forall f(x) \in L \quad \forall \varepsilon > 0 \quad \rightarrow \quad \exists k \in \mathbf{N} \quad \exists \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbf{R} : \quad \left\| f - \sum_{j=1}^k \lambda_j g_j \right\| < \varepsilon .$$

Note that *negation* this definition looks like:

$$\exists f_0(x) \in L \quad \exists \varepsilon_0 > 0 \quad \rightarrow \quad \forall k \in \mathbf{N} \quad \forall \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbf{R} : \quad \left\| f_0 - \sum_{j=1}^k \lambda_j g_j \right\| \geq \varepsilon_0 .$$

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Fair *Weierstrass' theorem*:

- 1°. Function system $\{1, x, x^2, \dots, x^n, \dots\}$ full in $C[a, b]$ on any segment $[a, b]$.
- 2°. Function system $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots\}$ is complete in the space of continuous on $[-\pi, \pi]$ functions for which $f(-\pi) = f(\pi)$.

Complete systems can be used to approximate functions of a suitable class by finite polynomials with any predetermined accuracy.

Let's look at some examples *studies of systems of functions for completeness* (that is, evidence of the presence or absence of this property).

Example 01. The system of odd Legendre polynomials, supplemented by a function equal to identically 1, is complete in the space of continuous functions on the interval $[0,1]$.

Solution: Any continuous $[0,1]$ function $f(x)$ can be represented in the form $f(x) = f(0) + \varphi(x)$, where $\varphi(0) = 0$.

Function $\varphi(x)$, and hence the function $f(x) - f(0)$ can be continued in an odd way to $[-1,1]$. Means $\exists \alpha_k$ such that

$$\left\| f(x) - f(0) \cdot 1 - \sum_{k=1}^n \alpha_k P_{2k-1}(x) \right\| < \varepsilon \quad \text{on } [-1,1],$$

whence the completeness of the system follows $\{1, x, x^3, \dots, x^{2k-1}, \dots\}$ on $[0,1]$.

Example 02. The system of sines of odd multiple arcs is incomplete in the space of continuous functions on the interval $[0,1]$.

Solution: It follows from the assessment: $\max_{x \in [0,1]} \left| 1 - \sum_{j=1}^n \lambda_j \sin(2j-1)x \right| \geq 1$, because there is a continuous $f(x) = 1 \quad \forall x \in [0,1]$, and any function of the form $\sum_{j=1}^n \lambda_j \sin(2j-1)x$ equal to 0 at $x = 0$.

Problem 03. Show that the system of functions $\{x^2, x^4, x^6, \dots, x^{2n}, \dots\}$

- 1) not complete $C[-1,2]$,
- 2) full in $C[1,2]$.

Solution. 1) Let $P_n(x) = \sum_{k=0}^{n-1} \alpha_k x^{2k+2}$.

In abundance $C[-1,2]$ available function $f_0(x) \equiv 1$, for which exists point $x_0 = 0$ such that

$$\max_{x \in [-1,2]} |f_0(x) - P_n(x)| \geq |f_0(x_0) - P_n(x_0)| = |1 - 0| = 1 = \varepsilon_0$$

at any $P_n(x)$.

Then from *denial* determining the completeness of a system of functions should be

that the system of functions $\{x^2, x^4, x^6, \dots, x^{2n}, \dots\}$ not full in $C[-1,2]$.

2) For an arbitrary function $f(x) \in C[1,2]$ rate the size

$$\max_{x \in [1,2]} \left| f(x) - \sum_{k=0}^{n-1} \alpha_k x^{2k+2} \right| = \max_{x \in [1,2]} x^2 \cdot \left| \frac{f(x)}{x^2} - \sum_{k=0}^{n-1} \alpha_k x^{2k} \right| =$$

when replacing $t = x^2 \Rightarrow x = \sqrt{t}$

$$= \max_{t \in [1,4]} t \cdot \left| \frac{f(\sqrt{t})}{t} - \sum_{k=0}^{n-1} \alpha_k t^k \right| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon,$$

because $\max_{t \in [1,4]} t \leq 4$ And $\max_{t \in [1,4]} \left| \frac{f(\sqrt{t})}{t} - \sum_{k=0}^{n-1} \alpha_k t^k \right| < \frac{\varepsilon}{4}$ according to the theorems of Weierstrass

due to continuity on $[1,4]$ functions $\frac{f(\sqrt{t})}{t}$.

Therefore, by defining the completeness of a system of functions, the system $\{x^2, x^4, x^6, \dots, x^{2n}, \dots\}$ full in $C[1,2]$.

Problem 04. Show that the system of functions $\{x, x^3, x^5, \dots, x^{2n+1}, \dots\}$
 1) not complete $C[-4, \pi]$,
 2) full in $C[\pi, 4]$.

Solution. 1) Let $P_n(x) = \sum_{k=0}^{n-1} \alpha_k x^{2k+1}$.

In abundance $C[-4, \pi]$ available function $f_0(x) \equiv 1$, for which exists point $x_0 = 0 \in C[-4, \pi]$ such that

$$\max_{x \in [-4, \pi]} |f_0(x) - P_n(x)| \geq |f_0(x_0) - P_n(x_0)| = |1 - 0| = 1 = \varepsilon_0$$

at any $P_n(x)$.

Then from *denial* determining the completeness of a system of functions should be

that the system of functions $\{x, x^3, x^5, \dots, x^{2n+1}, \dots\}$ not full in $C[-4, \pi]$.

2) Arbitrary function $f(x) \in C[\pi, 4]$ continue continuously in an odd way on $[-4, 4]$. We denote the resulting function $g(x) \in C[-4, 4]$. True for her equality $g(x) = -g(-x) \quad \forall x \in [-4, 4]$.

For $g(x)$ Weierstrass's theorem is true:

$$\forall \varepsilon > 0 \exists R_n(x) = \sum_{k=0}^{2n+1} \alpha_k x^k : \forall x \in [-4, 4] \quad |g(x) - R_n(x)| < \varepsilon.$$

But it will also be true $\forall x \in [-4, 4] \quad |g(-x) - R_n(-x)| < \varepsilon$.

Note also that $P_n(x) = \frac{R_n(x) - R_n(-x)}{2} \quad \forall x \in [-4, 4]$ and what of the oddness

functions $g(x)$ equality follows $g(x) = \frac{g(x) - g(-x)}{2} \quad \forall x \in [-4, 4]$.

Let's evaluate now

$$\begin{aligned} |g(x) - P_n(x)| &= \left| \frac{g(x) - g(-x)}{2} - \frac{R_n(x) - R_n(-x)}{2} \right| \leq \\ &\leq \frac{1}{2} |g(x) - R_n(x)| + \frac{1}{2} |g(-x) - R_n(-x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall x \in [-4, 4]. \end{aligned}$$

This means that the system of functions $\{x, x^3, x^5, \dots, x^{2n+1}, \dots\}$ full in $C[-4, 4]$, and, therefore, in $C[\pi, 4]$. Because, by construction, $g(x) = f(x) \quad \forall x \in [\pi, 4]$.

Example 05.

Задача: выяснит ли будет ли полной система функций $\{1; \cos x; \cos 2x; \dots \cos nx; \dots\}$
на $C[-2, 4]$,
на $C[2, 4]$.

Решение:

1°. Напомним: система функций $\{1; \cos x; \cos 2x; \dots \cos nx; \dots\}$ является *полной* на

$[a, b]$, если $\forall f(x) \forall \varepsilon > 0 \exists P_n(x) = \sum_{k=0}^n \alpha_k \cos kx$ такое, что

$$\left| f(x) - \sum_{k=0}^n \alpha_k \cos kx \right| < \varepsilon \quad \forall x \in [a, b].$$

Отрицание этого определения звучит так:

система функций $\{1; \cos x; \cos 2x; \dots \cos nx; \dots\}$ не является *полной* на $[a, b]$, если

$\exists f_0(x) \exists \varepsilon_0 > 0 \exists x_0 \in [a, b]$ такие, что $\forall P_n(x) = \sum_{k=0}^n \alpha_k \cos kx$

$$\left| f_0(x_0) - \sum_{k=0}^n \alpha_k \cos kx_0 \right| \geq \varepsilon_0.$$

2°. Let us assume the opposite: this system is complete $[-2,4]$.

Then, for not equal to identically 0, odd continuous function $f(x)$ there is a point $x_0 \in (0,2)$, such that

$$f(x_0) = A > 0 \quad \text{And} \quad \forall \varepsilon > 0 \quad |f_0(x_0) - P_n(x_0)| < \varepsilon, \quad \text{Where} \quad P_n(x) = \sum_{k=0}^n \alpha_k \cos kx.$$

Note that $P_n(x)$ There is even construction function.

At the same time $\exists x' = -x_0 \in (-2,0)$, where due to oddness $f(x)$ and parity $P_n(x)$, the following estimate would be fair:

$$|f_0(x') - P_n(x')| = |f_0(x_0) + P_n(x_0)| > 2A - \varepsilon,$$

from which follows the incompleteness of the system $\{1, \cos x, \cos 2x, \dots, \cos kx, \dots\}$ on $[-2,2]$, and, therefore, on $[-2,4]$. But this contradicts the original assumption.

Finally, if $f(x) < 0 \quad \forall x \in (0,2)$, then we carry out similar reasoning for the continuous odd function $g(x) = -f(x)$.

3°. Рассмотрим случай $C[2, 4]$.

Пусть $S_n = \frac{a_0}{2} + \sum_{k=1}^n \left(a_k \cos \frac{\pi kx}{l} + b_k \sin \frac{\pi kx}{l} \right)$ — частичные суммы ряда Фурье для

функции $f(x)$ $x \in [-l, l]$, а $\sigma_n = \frac{S_0 + S_1 + \dots + S_n}{n+1}$ — соответствующие им суммы

Фейера.

Доопределим $f(x)$ на $[-4, 4]$ четным образом, получим четную, непрерывную функцию $g(x)$ такую, что $g(x) = f(x) \quad \forall x \in [2, 4]$. Частичные суммы ряда Фурье (равно как и суммы Фейера) для функции $g(x)$ будут некоторыми линейными комбинациями функций из системы $\{1; \cos x; \cos 2x; \dots \cos nx; \dots\}$.

Воспользуемся теперь теоремой Фейера о том, что, если функция $g(x)$ непрерывна на $[-l, l]$ и $g(-l) = g(l)$, то функциональная последовательность $\{\sigma_n\}$ сходится равномерно к сумме ряда Фурье для функции $g(x)$. Из этой теоремы следует, что

$\forall \varepsilon > 0 \quad \exists N_0$ такое, что $\forall m \geq N_0 \quad \sup_{x \in \mathbb{R}} |g(x) - \sigma_m| < \varepsilon$, но тогда будет верно и

$\forall \varepsilon > 0 \quad \exists N_0$ такое, что $\forall m \geq N_0 \quad \sup_{x \in [2, 4]} |f(x) - \sigma_m| < \varepsilon$. Что доказывает полноту

системы функций $\{1; \cos x; \cos 2x; \dots \cos nx; \dots\}$ на $C[2, 4]$.