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COMPLETENESS OF SYSTEMS OF FUNCTIONS.

Below we will use the following definitions and theoretical facts.

In any finite-dimensional linear space, all norms are equivalent (in terms of convergence). In infinite-dimensional spaces this may not hold.

Linear space of continuous on $[a,b]$ functions $f(x)$ with the norm $\|f\|_{CL[a,b]} = \max\limits_{x\in[a,b]}|f(x)|$ we will denote it as $CL[a,b]$ or $C[a,b]$.

Linear space of continuous on $[a,b]$ functions $f(x)$ with the norm $\|f\|_{CL_1[a,b]} = \int\limits_a^b |f(x)|\,dx$ we will denote it as $CL_1[a,b]$.

Linear space of continuous on $[a,b]$ functions $f(x)$ with the norm $\|f\|_{CL_2[a,b]} = \sqrt{\int\limits_a^b f^2(x)\,dx}$ we will denote it as $CL_2[a,b]$.

In this case, there are useful estimates:

$$\|f\|_{CL_1[a,b]} \le (b-a)\|f\|_{CL[a,b]} \quad \text{And} \quad \|f\|_{CL_2[a,b]} \le \sqrt{b-a}\,\|f\|_{CL[a,b]}.$$

It can be shown that from norm convergence $\| f \|_{CL_2[a,b]}$ (as they sometimes say, *convergence in mean square*) does not follow norm convergence $\| f \|_{CL[a,b]}$ (*uniform* convergence), and from norm convergence $\| f \|_{CL_1[a,b]}$ (convergence *on average*) does not imply convergence in the mean square.

Definition. Counting system of elements $\{ g_1(x), g_2(x), \ldots, g_n(x), \ldots \}$ in linear normed space L called *full*, If

$$\forall f(x) \in L \;\; \forall \varepsilon > 0 \quad \to \quad \exists k \in \mathbf{N} \;\; \exists\, \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbf{R} : \qquad \left\| f - \sum_{j=1}^{k} \lambda_j g_j \right\| < \varepsilon \, .$$

Note that *negation* this definition looks like:

$$\exists f_0(x) \in L \;\; \exists \varepsilon_0 > 0 \quad \to \quad \forall k \in \mathbf{N} \;\; \forall\, \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbf{R} : \qquad \left\| f_0 - \sum_{j=1}^{k} \lambda_j g_j \right\| \geq \varepsilon_0 \, .$$

Fair *Weierstrass' theorem*:

1⁰. Function system $\{1, x, x^2, \ldots, x^n, \ldots\}$ full in $C[a,b]$ on any segment $[a,b]$.

2⁰. Function system $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots\}$ is complete in the space of continuous on $[-\pi, \pi]$ functions for which $f(-\pi) = f(\pi)$.

Complete systems can be used to approximate functions of a suitable class by finite polynomials with any predetermined accuracy.

Let's look at some examples *studies of systems of functions for completeness* (that is, evidence of the presence or absence of this property).

Example 01. The system of odd Legendre polynomials, supplemented by a function equal to identically 1, is complete in the space of continuous functions on the interval $[0,1].$

Solution: I like continuous $[0,1]$ function $f(x)$ can be represented in the form $f(x) = f(0) + \varphi(x),$ Where $\varphi(0) = 0.$

Function $\varphi(x)$, and hence the function $f(x) - f(0)$ can be continued in an odd way to $[-1,1].$ Means $\exists \alpha_k$ such that

$$\left\| f(x) - f(0) \cdot 1 - \sum_{k=1}^{n} \alpha_k P_{2k-1}(x) \right\| < \varepsilon \quad \text{on } [-1,1].,$$

whence the completeness of the system follows $\{1, x, x^3, \ldots , x^{2k-1}, \ldots \}$ on $[0,1].$

Example 02. The system of sines of odd multiple arcs is incomplete in the space of continuous functions on the interval $[0,1]$.

Solution: It follows from the assessment: $\max\limits_{x\in[0,1]} \left| 1-\sum\limits_{j=1}^{n}\lambda_j \sin(2j-1)x \right| \ge 1$, because there is a continuous $f(x)=1 \quad \forall x\in[0,1]$, and any function of the form $\sum\limits_{j=1}^{n}\lambda_j \sin(2j-1)x$ equal to 0 at $x=0$.

Problem 03. Show that the system of functions $\{x^2, x^4, x^6, \dots, x^{2n}, \dots\}$
1) not complete $C[-1,2]$,
2) full in $C[1,2]$.

Solution. 1) Let $P_n(x) = \sum_{k=0}^{n-1} \alpha_k x^{2k+2}$.

In abundance $C[-1,2]$ *available* function $f_0(x) \equiv 1,$ for which *exists* point $x_0 = 0$ such that

$$\max_{x\in[-1,2]} |f_0(x) - P_n(x)| \ge |f_0(x_0) - P_n(x_0)| = |1 - 0| = 1 = \varepsilon_0$$

at *any* $P_n(x)$.
Then from *denial* determining the completeness of a system of functions should be
that the system of functions $\{x^2, x^4, x^6, \dots, x^{2n}, \dots\}$ not full in $C[-1,2]$.

2) For an arbitrary function $f(x) \in C[1,2]$ rate the size

$$\max_{x\in[1,2]} \left| f(x) - \sum_{k=0}^{n-1} \alpha_k x^{2k+2} \right| = \max_{x\in[1,2]} x^2 \cdot \left| \frac{f(x)}{x^2} - \sum_{k=0}^{n-1} \alpha_k x^{2k} \right| =$$

when replacing $t = x^2 \quad \Rightarrow \quad x = \sqrt{t}$

$$= \max_{t\in[1,4]} t \cdot \left| \frac{f(\sqrt{t})}{t} - \sum_{k=0}^{n-1} \alpha_k t^k \right| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon,$$

because $\max_{t\in[1,4]} t \le 4$ And $\max_{t\in[1,4]} \left| \frac{f(\sqrt{t})}{t} - \sum_{k=0}^{n-1} \alpha_k t^k \right| < \frac{\varepsilon}{4}$ according to the theorems of Weierstrass

due to continuity on $[1,4]$ functions $\frac{f(\sqrt{t})}{t}$.
Therefore, by defining the completeness of a system of functions, the system $\{x^2, x^4, x^6, \dots, x^{2n}, \dots\}$ full in $C[1,2]$.

Problem 04. Show that the system of functions $\{x, x^3, x^5, \ldots, x^{2n+1}, \ldots\}$

1) not complete $C[-4,\pi]$,

2) full in $C[\pi,4]$.

Solution. 1) Let $P_n(x)=\sum_{k=0}^{n-1}\alpha_k x^{2k+1}$.

In abundance $C[-4,\pi]$ *available* function $f_0(x)\equiv 1,$ for which *exists* point $x_0=0\in C[-4,\pi]$ such that

$$\max_{x\in[-4,\pi]}\left|f_0(x)-P_n(x)\right|\ge\left|f_0(x_0)-P_n(x_0)\right|=\left|1-0\right|=1=\varepsilon_0$$

at *any* $P_n(x).$

Then from *denial* determining the completeness of a system of functions should be

that the system of functions $\{x, x^3, x^5, \ldots, x^{2n+1}, \ldots\}$ not full in $C[-4,\pi].$

2) Arbitrary function $f(x)\in C[\pi,4]$ continue continuously in an odd way on $[-4,4].$ We denote the resulting function $g(x)\in C[-4,4].$ True for her equality $g(x)=-g(-x)\quad\forall x\in[-4,4].$

For $g(x)$ Weierstrass's theorem is true:

$$\forall\varepsilon>0\;\exists R_n(x)=\sum_{k=0}^{2n+1}\alpha_k x^k:\quad\forall x\in[-4,4]\quad\Rightarrow\quad\left|g(x)-R_n(x)\right|<\varepsilon.$$

But it will also be true $\forall x\in[-4,4]\quad\Rightarrow\quad\left|g(-x)-R_n(-x)\right|<\varepsilon.$

Note also that $P_n(x)=\dfrac{R_n(x)-R_n(-x)}{2}\quad\forall x\in[-4,4]$ and what of the oddness functions $g(x)$ equality follows $g(x)=\dfrac{g(x)-g(-x)}{2}\quad\forall x\in[-4,4].$

Let's evaluate now

$$\left|g(x)-P_n(x)\right|=\left|\frac{g(x)-g(-x)}{2}-\frac{R_n(x)-R_n(-x)}{2}\right|\le$$

$$\le\frac{1}{2}\left|g(x)-R_n(x)\right|+\frac{1}{2}\left|g(-x)-R_n(-x)\right|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\quad\forall x\in[-4,4].$$

This means that the system of functions $\{x, x^3, x^5, \ldots, x^{2n+1}, \ldots\}$ full in $C[-4,4],$ and, therefore, in $C[\pi,4].$ Because, by construction, $g(x)=f(x)\quad\forall x\in[\pi,4]$.

Problem 05. Find out whether the system of functions will be complete

$$\{1, \cos x, \cos 2x, \ldots, \cos nx, \ldots\}$$

1) on $C[-2,4]$,
2) on $C[2,4]$.

Solution. 1) Recall: the system $\{1, \cos x, \cos 2x, \ldots, \cos nx, \ldots\}$ *is full* on $[a,b]$, If

$$\forall f(x) \quad \forall \varepsilon > 0 \quad \exists P_n(x) = \sum_{k=1}^{n} \alpha_n \cos kx$$

such that

$$|f(x) - P_n(x)| < \varepsilon \qquad \forall x \in [a,b].$$

Negation this definition is:

function system $\{1, \cos x, \cos 2x, \ldots, \cos nx, \ldots\}$ *is not full* on $[a,b]$, If $\exists \varepsilon_0 > 0, \quad \exists x_0 \in [a,b]$ such that $|f(x_0) - P_n(x_0)| \geq \varepsilon_0 \qquad \forall P_n(x).$

2^0. Let us assume the opposite: this system is complete $[-2,4]$.

Then, for not equal to identically 0, *odd* continuous function $f(x)$ there is a point $x_0 \in (0,2)$, such that

$f(x_0) = A > 0$ And $\forall \varepsilon > 0$ $|f_0(x_0) - P_n(x_0)| < \varepsilon$, Where $P_n(x) = \sum\limits_{k=0}^{n} \alpha_k \cos kx$.
Note that $P_n(x)$ There is *even* construction function.

At the same time $\exists x' = -x_0 \in (-2,0)$, where due to oddness $f(x)$ and parity $P_n(x)$, the following estimate would be fair:

$$|f_0(x') - P_n(x')| = |f_0(x_0) + P_n(x_0)| > 2A - \varepsilon,$$

from which follows the incompleteness of the system $\{1, \cos x, \cos 2x, \dots \cos kx, \dots\}$ on $[-2,2]$, and, therefore, on $[-2,4]$. But this contradicts the original assumption.

Finally, if $f(x) < 0 \;\; \forall x \in (0,2)$, then we carry out similar reasoning for the continuous odd function $g(x) = -f(x)$.

3°. Рассмотрим случай $C[2,4]$.

Пусть $S_n = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\frac{\pi k x}{l} + b_k \sin\frac{\pi k x}{l}\right)$ частичные суммы ряда Фурье для функции $f(x)$ $x \in [-l, l]$, а $\sigma_n = \frac{S_0 + S_1 + \ldots + S_n}{n+1}$ – соответствующие им суммы Фейера.

Доопределим $f(x)$ на $[-4,4]$ четным образом, получим четную, непрерывную функцию $g(x)$ такую, что $g(x) \equiv f(x)$ $\forall x \in [2,4]$. Частичные суммы ряда Фурье (равно как и суммы Фейера) для функции $g(x)$ будут некоторыми линейными комбинациями функций из системы $\{1; \cos x; \cos 2x; \ldots \cos nx; \ldots\}$.

Воспользуемся теперь теоремой Фейера о том, что, если функция $g(x)$ непрерывна на $[-l, l]$ и $g(-l) = g(l)$, то функциональная последовательность $\{\sigma_n\}$ сходится равномерно к сумме ряда Фурье для функции $g(x)$. Из этой теоремы следует, что $\forall \varepsilon > 0$ $\exists N_0$ такое, что $\forall m \geq N_0$ $\sup\limits_{x \in R} |g(x) - \sigma_m| < \varepsilon$, но тогда будет верно и $\forall \varepsilon > 0$ $\exists N_0$ такое, что $\forall m \geq N_0$ $\sup\limits_{x \in [2,4]} |f(x) - \sigma_m| < \varepsilon$. Что доказывает полноту системы функций $\{1; \cos x; \cos 2x; \ldots \cos nx; \ldots\}$ на $C[2,4]$.