

## PROPER INTEGRALS DEPENDING ON PARAMETERS

Let's consider *definite* (Riemannian) integral of a function of two variables  $f(x, \alpha)$ , defined on a closed rectangle  $K : \{ a \leq x \leq b; c \leq \alpha \leq d \}$ , which is taken by variable  $x$  ranging from  $a$  to  $b$ .

It is clear that the value of this integral will, generally speaking, depend on the value  $\alpha$ , at which it is taken.

Since the Riemannian integral, by its definition, is *limit* so-called *Riemann sum* (and it is known that the limit, if exists), then it *the only one*. Therefore, the dependence of the value on the value  $\alpha$  is *functional*.

Simply put, in the case under consideration this integral is *function* variable  $\alpha$  :

$$\Phi(\alpha) = \int_a^b f(x, \alpha) dx \quad (1)$$

A natural question arises here: how are the properties of a function  $\Phi(\alpha)$  depend on the properties of the function  $f(x, \alpha)$  ?

Or, more specifically,

is it possible by performing any operation with a function  $\Phi(\alpha)$  (say, calculating a limit, differentiation or integration with respect to  $\alpha$ ), “rearrange” this operation and integration into (1)?

**Answer:** In general, this cannot be done.

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Let's explain this as follows:

example 1.  $\Phi(\alpha) = \int_0^1 \frac{\alpha}{x^2 + \alpha^2} dx, \quad \alpha > 0$   
Let  $\lim_{\alpha \rightarrow +0} \Phi(\alpha)$   
need to find  $\lim_{\alpha \rightarrow +0} \Phi(\alpha)$ .

Solution: This integral is "taken", and from  $\Phi(\alpha) = \arctg \frac{x}{\alpha} \Big|_0^1$  we get that  
 $\lim_{\alpha \rightarrow +0} \Phi(\alpha) = \frac{\pi}{2}$  On the other side,  $\int_0^1 \lim_{\alpha \rightarrow +0} \frac{\alpha}{x^2 + \alpha^2} dx = 0$ .

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The possibility of changing the “order of actions” for the operations of passing to the limit and calculating the Riemannian integral is determined by the following Theorem 1:

If the function  $f(x, \alpha)$  continuous on  $K : \{ a \leq x \leq b; c \leq \alpha \leq d \}$ , That  $\Phi(\alpha)$  continuous, and therefore integrable on  $[c, d]$ .

It should be taken into account that

- continuity of function  $f(x, \alpha)$ , as the equality of value and limit at a point, assumes here *double (multiple)* vector limit  $\begin{pmatrix} x \\ \alpha \end{pmatrix}$ . This is, on the one hand, a rather strict requirement, as example 1 shows;
- on the other hand, the fulfillment of this strict condition allows in some cases to find the values of “not taken” integrals.

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An example 2. Find 
$$I = \lim_{\alpha \rightarrow 0} \int_1^9 \frac{e^{\alpha x}}{\sqrt{x}} dx$$
.

Solution: Here it will not be possible to start with the Newton-Leibniz formula, since the corresponding indefinite integral is not expressed in terms of elementary functions.

However, due to *continuity* integrand function *two* variables  $\alpha$  And  $x$ , the equalities will be valid

$$I = \lim_{\alpha \rightarrow 0} \int_1^9 \frac{e^{\alpha x}}{\sqrt{x}} dx = \int_1^9 \lim_{\alpha \rightarrow 0} \frac{e^{\alpha x}}{\sqrt{x}} dx = \int_1^9 \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_1^9 = 4.$$

An example 3. Find 
$$I = \int_0^1 \frac{x^b - x^a}{\ln x} \sin \ln \frac{1}{x} dx,$$
 Where  $b > a > 0$ .

Solution: 1) Note that 
$$\frac{x^b - x^a}{\ln x} = \int_a^b x^u du.$$
 Then

$$I = \int_0^1 \frac{x^b - x^a}{\ln x} \sin \ln \frac{1}{x} dx = - \int_0^1 dx \int_a^b x^u \sin \ln x du \quad (2)$$

If we add zeros at points with  $u \in [a, b]$  And  $x = 0$  integrand function of the internal integral, then we will make it continuous (Why?) on the set  $K : \{ a \leq u \leq b; 0 \leq x \leq 1 \}$ .

Now, (by Theorem 1) changing the order of integration, we obtain by substituting  $x = e^t$  And  $dx = e^t dt$ ,

$$I = - \int_a^b du \int_{-\infty}^0 e^{(u+1)t} \sin t dt$$

Let us denote by  $J$  internal integral. For him (*remember the material of the first course*) by double sequential integration “by parts” we obtain an equation of the

form:  $J = -1 - (u+1)^2 J$ , where  $J = -\frac{1}{1+(u+1)^2}$ .

Returning to the calculation of the integral  $I$ , from (2) we finally obtain

$$I = \int_a^b \frac{du}{1+(u+1)^2} = \operatorname{arctg}(b+1) - \operatorname{arctg}(a+1).$$

The conditions that determine the possibility of rearranging the operations of differentiation and integration in Riemannian integrals are given by Theorem 2:

$$\Phi(\alpha) = \int_{\lambda(\alpha)}^{\mu(\alpha)} f(x, \alpha) dx$$

If in the integral functions  $f(x, \alpha), \lambda(\alpha), \mu(\alpha)$  continuously differentiable by  $\alpha$  on  $K : \{ a \leq x \leq b; c \leq \alpha \leq d \}$ , then the equality is true

$$\Phi'(\alpha) = f(\mu(\alpha), \alpha) \mu'(\alpha) - f(\lambda(\alpha), \alpha) \lambda'(\alpha) + \int_{\lambda(\alpha)}^{\mu(\alpha)} \frac{\partial f}{\partial \alpha}(x, \alpha) dx. \quad (3)$$

Formula (3) may be useful in finding integrals.

An example 4. Applying the differentiation formula with respect to the parameter  $\alpha$  in

the integral  $I(\alpha) = \int_0^b \frac{dx}{x^2 + \alpha^2}$ , Where  $b > 0$ , find the integral

$$J(\alpha) = \int_0^b \frac{dx}{(x^2 + \alpha^2)^2}.$$

Solution: Since Theorem 2 applies in this case, we have

$$I'_\alpha(\alpha) = -2\alpha J(\alpha). \quad (4)$$

On the other hand, according to the rules of integration

$$I(\alpha) = \int_0^b \frac{dx}{x^2 + \alpha^2} = \frac{1}{\alpha} \operatorname{arctg} \frac{b}{\alpha},$$

and this gives

$$I'_\alpha(\alpha) = -\frac{1}{\alpha^2} \operatorname{arctg} \frac{b}{\alpha} - \frac{1}{\alpha} \cdot \frac{1}{1 + \frac{b^2}{\alpha^2}} \cdot \frac{b}{\alpha^2}.$$

And, finally, from equality (4) we find that

$$\int_0^b \frac{dx}{(x^2 + \alpha^2)^2} = J(\alpha) = -\frac{1}{2\alpha} I'_\alpha(\alpha) = \frac{1}{2\alpha^3} \operatorname{arctg} \frac{b}{\alpha} + \frac{1}{2\alpha^2} \cdot \frac{b}{\alpha^2 + b^2}.$$