PROPER INTEGRALS DEPENDING ON PARAMETERS

Let's consider *definite* (Riemannian) integral of a function of two variables $f(x,\alpha)$, defined on a closed rectangle $K:\{\, a \le x \le b; c \le \alpha \le d \,\}$, which is taken by variable x ranging from a to b.

It is clear that the value of this integral will, generally speaking, depend on the value α, at which it is taken.
Since the Riemannian integral, by its definition, is *limit* so-called *Riemann sum* (and it is known that the limit, if exists), then it *the only one*. Therefore, the dependence of the value on the value α is *functional*.
Simply put, in the case under consideration this integral is *function* variable α :

$$\Phi(\alpha) = \int_a^b f(x,\alpha)\,dx \quad . \tag{1}$$

A natural question arises here: how are the properties of a function $\Phi(\alpha)$ depend on the properties of the function $f(x,\alpha)$?

Or, more specifically,

> is it possible by performing any operation with a function $\Phi(\alpha)$ (say, calculating a limit, differentiation or integration with respect to α), "rearrange" this operation and integration into (1)?

Answer: In general, this cannot be done.

Let's explain this as follows:

example 1. *Let* $\Phi(\alpha)=\int\limits_0^1 \frac{\alpha}{x^2+\alpha^2}\,dx\,,\quad \alpha>0$,

need to find $\lim\limits_{\alpha\to+0}\Phi(\alpha)$.

Solution: This integral is “taken”, and from $\Phi(\alpha)=\operatorname{arctg}\frac{x}{\alpha}\Big|_0^1$ we get that $\lim\limits_{\alpha\to+0}\Phi(\alpha)=\frac{\pi}{2}$ On the other side, $\int\limits_0^1 \lim\limits_{\alpha\to+0}\frac{\alpha}{x^2+\alpha^2}\,dx=0$.

The possibility of changing the "order of actions" for the operations of passing to the limit and calculating the Riemannian integral is determined by the following Theorem 1:

If the function $f(x,\alpha)$ continuous on $K:\{\ a \le x \le b; c \le \alpha \le d\ \}$, That $\Phi(\alpha)$ continuous, and therefore integrable on $[c,d]$.

It should be taken into account that

- continuity of function $f(x,\alpha)$, as the equality of value and limit at a point, assumes here *double* (*multiple*) vector limit $\left\| \begin{matrix} x \\ \alpha \end{matrix} \right\|$. This is, on the one hand, a rather strict requirement, as example 1 shows;
- on the other hand, the fulfillment of this strict condition allows in some cases to find the values of “not taken” integrals.

An example 2. *Find* $I = \lim\limits_{\alpha \to 0} \int\limits_1^9 \frac{e^{\alpha x}}{\sqrt{x}} dx$.

Solution: Here it will not be possible to start with the Newton-Leibniz formula, since the corresponding indefinite integral is not expressed in terms of elementary functions.
However, due to *continuity* integrand function *two* variables α And x, the equalities will be valid

$$I = \lim_{\alpha \to 0} \int_1^9 \frac{e^{\alpha x}}{\sqrt{x}} dx = \int_1^9 \lim_{\alpha \to 0} \frac{e^{\alpha x}}{\sqrt{x}} dx = \int_1^9 \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_1^9 = 4.$$

An example 3. *Find* $I=\int\limits_0^1 \frac{x^b-x^a}{\ln x}\sin\ln\frac{1}{x}\,dx$, *Where* $b>a>0$.

Solution: 1) Note that $\frac{x^b-x^a}{\ln x}=\int\limits_a^b x^u\,du$. Then

$$I=\int\limits_0^1 \frac{x^b-x^a}{\ln x}\sin\ln\frac{1}{x}\,dx=-\int\limits_0^1 dx\int\limits_a^b x^u\sin\ln x\,du \, . \qquad (2)$$

If we add zeros at points with $u\in[a,b]$ And $x=0$ integrand function of the internal integral, then we will make it continuous (*Why*?) on the set $K:\{\,a\le u\le b;\,0\le x\le 1\,\}$.

Now, (by Theorem 1) changing the order of integration, we obtain by substituting $x=e^t$ And $dx=e^t\,dt$,

$$I=-\int\limits_a^b du\int\limits_{-\infty}^0 e^{(u+1)t}\sin t\,dt \, .$$

Let us denote by J internal integral. For him (*remember the material of the first course*) by double sequential integration "by parts" we obtain an equation of the form: $J=-1-(u+1)^2 J$, where $J=-\frac{1}{1+(u+1)^2}$.

Returning to the calculation of the integral $I,$ from (2) we finally obtain

$$I=\int\limits_a^b \frac{du}{1+(u+1)^2}=\operatorname{arctg}(b+1)-\operatorname{arctg}(a+1).$$

The conditions that determine the possibility of rearranging the operations of differentiation and integration in Riemannian integrals are given by Theorem 2:

If in the integral $\Phi(\alpha) = \int\limits_{\lambda(\alpha)}^{\mu(\alpha)} f(x,\alpha)\,dx$ functions $f(x,\alpha),\ \lambda(\alpha),\ \mu(\alpha)$ *continuously differentiable* by α on $K : \{\, a \le x \le b; c \le \alpha \le d \,\}$, then the equality is true

$$\Phi'(\alpha) = f(\mu(\alpha),\alpha)\,\mu'(\alpha) - f(\lambda(\alpha),\alpha)\,\lambda'(\alpha) + \int\limits_{\lambda(\alpha)}^{\mu(\alpha)} \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx\,. \tag{3}$$

Formula (3) may be useful in finding integrals.

An example 4. *Applying the differentiation formula with respect to the parameter* α *in the integral* $I(\alpha)=\int\limits_0^b \frac{dx}{x^2+\alpha^2}$, *Where* $b>0$, *find the integral* $J(\alpha)=\int\limits_0^b \frac{dx}{(x^2+\alpha^2)^2}$.

Solution: Since Theorem 2 applies in this case, we have

$$I'_\alpha(\alpha) = -2\alpha\, J(\alpha) \,. \tag{4}$$

On the other hand, according to the rules of integration

$$I(\alpha)=\int\limits_0^b \frac{dx}{x^2+\alpha^2} = \frac{1}{\alpha}\operatorname{arctg}\frac{b}{\alpha} \,,$$

and this gives

$$I'_\alpha(\alpha) = -\frac{1}{\alpha^2}\operatorname{arctg}\frac{b}{\alpha} - \frac{1}{\alpha}\cdot\frac{1}{1+\frac{b^2}{\alpha^2}}\cdot\frac{b}{\alpha^2} \,.$$

And, finally, from equality (4) we find that

$$\int\limits_0^b \frac{dx}{(x^2+\alpha^2)^2} = J(\alpha) = -\frac{1}{2\alpha} I'_\alpha(\alpha) = \frac{1}{2\alpha^3}\operatorname{arctg}\frac{b}{\alpha} + \frac{1}{2\alpha^2}\cdot\frac{b}{\alpha^2+b^2} \,.$$