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## IMPROPER INTEGRALS DEPENDING ON A PARAMETER. CONVERGENCE CONDITIONS

Let's consider *unowned* integral of a function of two variables $f(x,\alpha)$

$$\Phi(\alpha) = \int_a^{b} f(x,\alpha)\,dx \quad , \tag{1}$$

determined by $x$ in between $a \le x < b$ and at $\alpha \in \Omega$. The value of this integral, generally speaking, depends on the value $\alpha$, at which it is taken. Here the symbol $b$ can mean either a real number or $+\infty$.

An improper integral by its definition is *unilateral limit* from a definite (Riemannian) integral, when, for example, the upper limit of the latter tends to $b$ left. Let us recall that in a definite integral the integrand, as well as the integration interval, must be *limited*. This is not required in the improper integral.

In this case, the dependence of value (1) on the value $\alpha$ is *functional*, since the limit (of the Riemannian integral sum as the partition of the integration interval tends to zero is small), if it exists, must be unique. That is, the integral in formula (1) specifies *function* from variable $\alpha$.

A natural question arises: how are the properties of a function $\Phi(\alpha)$ depend on the properties of the function $f(x,\alpha)$ ?

Or, more specifically, is it possible, by performing any operation on a function $\Phi(\alpha)$ (say, calculating its limit, differentiating or integrating it with respect to $\alpha$ ) "rearrange" this operation and integration over $x$ in (1) ?

We have already seen that in general this can be done *it is forbidden*, even for definite (Riemannian) integrals (see example 1 in topic 5). But, naturally, it seems interesting to find out whether this is possible, and if possible, then under what conditions, for improper integrals of the form (1).

To answer this question, let us first give two definitions:

A) Integral (1) is called *point by point* convergent on the set $\Omega$ , If

$$\forall \alpha \in \Omega \;\; \text{и} \;\; \forall \varepsilon > 0: \; \exists \delta_{\varepsilon,\alpha} > 0: \; \forall \delta : \; \delta_{\varepsilon,\alpha} < \delta < b \quad \Rightarrow \quad \left| \int\limits_{\delta}^{b} f(x,\alpha)\, dx \right| < \varepsilon \, .$$

B) Integral (1) is called *evenly* convergent on the set $\Omega$ , If

$$\forall \varepsilon > 0 \; : \; \exists \delta_{\varepsilon} > 0 \; : \; \forall \alpha \in \Omega \;\; \text{и} \;\; \forall \delta : \; \delta_{\varepsilon} < \delta < b \quad \Rightarrow \quad \left| \int\limits_{\delta}^{b} f(x,\alpha)\, dx \right| < \varepsilon \, .$$

Note that although the definitions of A) and B) are similar in text, there are significant differences between them.

The first definition simply means the existence of an improper integral for every fixed $\alpha \in \Omega$.

According to this definition, in the case of pointwise convergence there is a selection rule $\delta_{\varepsilon,\alpha}$ (according to predetermined values $\varepsilon$ And $\alpha$ ), which ensures the fulfillment of the conditions specified in the definition. However, this rule may be *different* for different values $\alpha \in \Omega$.

The second definition requires the existence of a selection rule $\delta_{\varepsilon}$, ensuring the fulfillment of the conditions specified in the definition, at once for *everyone* values $\alpha$ from $\Omega$.

It is clear that the second definition is more "strict" than the first. That is, from *uniform* convergence of integral (1) follows *pointwise*, but not vice versa.

The following statements (theorems) hold.

Theorem 1. If

1) $f(x,\alpha)$ is continuous on the set $K:\{\, a \le x < b,\ \alpha \in [\alpha_1,\alpha_2]\,\}$ And

2) integral (1) converges uniformly on $[\alpha_1,\alpha_2]$ ,

That $\Phi(\alpha)$ continuous on $[\alpha_1,\alpha_2]$ and the formula is correct

$$\int_{\alpha_1}^{\alpha_2} d\alpha \int_a^{b} f(x,\alpha)\,dx = \int_a^{b} dx \int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha .$$

Theorem 2. If

1) functions $f(x,\alpha)$ And $\dfrac{\partial f}{\partial \alpha}(x,\alpha)$ continuous on the set $K:\{\, a \le x < b,\ \alpha \in [\alpha_1,\alpha_2]\,\}$ And

2) integral (1) converges pointwise on $[\alpha_1,\alpha_2]$ , and the integral $\displaystyle\int_a^{b} \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx$ converges uniformly on $[\alpha_1,\alpha_2]$ ,

then the formula is correct

$$\frac{\partial \Phi}{\partial \alpha}(\alpha) = \int_a^{b} \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx .$$

From these theorems follows the importance of the property of uniform convergence in problems of studying improper integrals depending on a parameter. However, directly using the definition of B) can be very labor-intensive.

In a number of cases, the following statements are useful for solving practical problems:

**I.** *Necessary and sufficient condition for uniform convergence.*

Integral (1) converges uniformly if and only if

$$\lim_{A\to b}\sup_{\alpha\in\Omega}\int_A^b f(x,\alpha)\,dx = 0\,.$$

**II.** *Denial of uniform convergence.*

Integral (1) does not converge uniformly if

$$\exists\varepsilon_0>0 \text{ и } \alpha_0\in\Omega\ :\ \forall\delta>0\quad \exists\xi_0 :\ \delta<\xi_0<b \quad \Rightarrow \quad \left|\int_{\xi_0}^{b} f(x,\alpha_0)\,dx\right| \ge \varepsilon_0\,.$$

**III.** *Weierstrass sign* (*sufficient condition for uniform convergence*).

If exists $A$ such that the function $\phi(x)$, defined on $[A,+\infty)$, satisfies the conditions:

1) $\forall x \in [A,+\infty)$ и $\forall \alpha \in \Omega: \quad |f(x,\alpha)| \le \phi(x)$,

2) improper integral $\int\limits_A^{b} \phi(x)\,dx$ converges

then integral (1) converges uniformly on $\Omega$.

**IV.** *Cauchy criterion* (*necessary and sufficient condition for uniform convergence*).

Integral (1) converges uniformly if and only if

$$\forall\varepsilon > 0 \ : \ \exists\delta_\varepsilon \in (a,b) \ : \forall\alpha\in\Omega \ \ \forall\delta'\in[\,\delta_\varepsilon, b) \ \text{ и } \ \forall\delta''\in[\,\delta_\varepsilon, b) \ \ \Rightarrow \ \ \left|\int\limits_{\delta'}^{\delta''} f(x,\alpha)\,dx\right| < \varepsilon\,.$$

**V.** *Negation of the Cauchy criterion*. In order for the improper integral (1) not to converge uniformly, it is necessary and sufficient that

$$\exists\varepsilon_0 > 0 \ \ \forall\delta\in[a,b): \ \exists\alpha_0\in\Omega\,, \ \ \exists\delta_0'\in[\,\delta, b) \ \text{ и } \ \exists\delta_0''\in[\,\delta, b) \ \ \Rightarrow \ \ \left|\int\limits_{\delta_0'}^{\delta_0''} f(x,\alpha_0)\,dx\right| \ge \varepsilon_0\,.$$

Please note that the integrals in **IV** And **V** *certain* (rimanovskie).

**WE.** *Dirichlet test* (*sufficient condition for uniform convergence*).

Integral of the form $\int\limits_A^{b} f(x,\alpha)g(x,\alpha)\,dx$ converges uniformly in $\alpha$ on a set $\Omega$, if on the set $x \in [A,b)$ at every $\alpha \in \Omega$ functions $f(x,\alpha),\ g(x,\alpha)$ и $g'_x\,(x,\alpha)$ *continuous* by $x$ and satisfy the conditions:

1) $\lim\limits_{x\to b} g(x,\alpha) = 0$ evenly on $\alpha \in \Omega$,

2) function $g'_x\,(x,\alpha)$ sign constant of $x \in [A,b)$ at every $\alpha \in \Omega$,

3) $\exists M > 0 \ \ \forall \alpha \in \Omega$ и $\forall x \in [a,b) \ \ \Rightarrow \ \ \left| \int\limits_a^x f(u,\alpha)\,du \right| \le M$.

Based on these theoretical statements, they carry out studies of improper integrals for convergence

Example 1. Investigate for uniform convergence $\Phi(\alpha) = \int\limits_A^{+\infty} \alpha e^{-\alpha x} dx$ at $A > 0$ in plurals: 1) $\alpha \in [\alpha_0, +\infty)$ , Where $\alpha_0 > 0$ and 2) $\alpha \in (0, +\infty)$ .

Solution. 1) Let's apply the criterion **I**. We have (check these equalities yourself)

$$\lim_{A\to+\infty} \sup_{\alpha\in[\alpha_0,+\infty)} \int\limits_A^{+\infty} \alpha e^{-\alpha x} dx = \lim_{A\to+\infty} \sup_{\alpha\in[\alpha_0,+\infty)} e^{-\alpha A} = \lim_{A\to+\infty} e^{-\alpha_0 A} = 0,$$

that is, the integral converges uniformly.

2) Likewise,

$$\lim_{A\to+\infty} \sup_{\alpha>0} \int\limits_A^{+\infty} \alpha e^{-\alpha x} dx = \lim_{A\to+\infty} \sup_{\alpha>0} e^{-\alpha A} = \lim_{A\to+\infty} e^{-1} = e^{-1} > 0,$$

because among the positive $\alpha$ for anyone $A > 0$ there will be $\alpha_0 = \frac{1}{A}$ .
Consequently, uniform convergence on the second set with respect to $\alpha$ No.

Example 2. Find $\lim\limits_{a\to 0}\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2+\alpha^2}$ .

Solution. From the assessment $\frac{1}{1+x^2+\alpha^2}\le\phi(x)=\frac{1}{1+x^2}\quad\forall\alpha\in\mathbf{R}$ based on **III** (Weierstrass), due to the convergence of the improper integral $\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2}=\operatorname{arctg}x\Big|_{-\infty}^{+\infty}=\pi$, we conclude that the integral specified in the condition converges uniformly in $\alpha$ on $\mathbf{R}$ .

Therefore, by Theorem 1 (due to continuity in $\alpha$ ) the equality will be true

$$\lim_{a\to 0}\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2+\alpha^2}=\int\limits_{-\infty}^{+\infty}\lim_{a\to 0}\frac{dx}{1+x^2+\alpha^2}=\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2}=\operatorname{arctg}x\Big|_{-\infty}^{+\infty}=\pi\ .$$

Example 3. Investigate for uniform convergence $\Phi(\alpha)=\int\limits_0^{+\infty}\frac{dx}{1+\alpha^2x^2}$ in plurals: 1) $\alpha\in[\alpha_0,+\infty)$, Where $\alpha_0>0$ And 2) $\alpha\in(0,+\infty)$.

Solution. 1) Let us apply definition B) on page 3, which in this problem has the form

$$\forall\varepsilon>0\ :\ \exists\delta_\varepsilon>0\ :\ \forall\alpha\in[\alpha_0,+\infty)\ \text{ и }\ \forall\delta:\ \delta_\varepsilon<\delta<+\infty\ \Rightarrow\ \left|\int\limits_\delta^{+\infty}\frac{dx}{1+\alpha^2x^2}\right|<\varepsilon\,.$$

We need to find a rule according to which for each predetermined positive $\varepsilon$ can be specified $\delta_\varepsilon$, ensuring the fulfillment of this inequality.

Let us take advantage of the fact that the corresponding indefinite integral can be taken, i.e. equality $\int\frac{dx}{1+\alpha^2x^2}=\frac{1}{\alpha}\operatorname{arctg}\alpha x+C$.

According to the Newton-Leibniz formula (for the improper integral) and the properties of the function $\operatorname{arctg} x$, for any positive $\varepsilon, \delta_\varepsilon$ And $\alpha$ the relations will be valid

$$0 < \frac{\pi}{2\alpha} - \frac{1}{\alpha}\operatorname{arctg}\alpha\delta_\varepsilon = \frac{\varepsilon}{2} < \varepsilon \quad\Rightarrow\quad \alpha\delta_\varepsilon = \operatorname{tg}\left(\frac{\pi}{2} - \frac{\alpha\varepsilon}{2} + \pi k\right) \quad \forall k \in \mathbf{Z}.$$

Due to the periodicity of the function $\operatorname{tg} x$, view $\delta_\varepsilon$ is the same for any whole $k$, so let's put $k = 0$. Then for $\delta_\varepsilon$ we have

$$\alpha\delta_\varepsilon = \operatorname{tg}\left(\frac{\pi}{2} - \frac{\alpha\varepsilon}{2}\right) \quad\Rightarrow\quad \delta_\varepsilon = \frac{1}{\alpha}\operatorname{ctg}\frac{\alpha\varepsilon}{2}.$$

Finally (check it out for yourself) since $\frac{1}{\alpha}\operatorname{ctg}\frac{\alpha\varepsilon}{2}$ at $\alpha\varepsilon < \pi$ monotonically decreases in $\alpha$, then as the desired dependence $\delta_\varepsilon$ from $\varepsilon$ you can take it $\delta_\varepsilon = \frac{1}{\alpha_0}\operatorname{ctg}\frac{\alpha_0\varepsilon}{2}$. Which proves the uniform convergence of the integral.

2) We prove the absence of uniform convergence of the integral by negating the criterion **V** (Cauchy), which in this case has the form

$$\exists \varepsilon_0 > 0 \ : \ \forall \delta > 0 : \ \exists \alpha_0 > 0 \, , \ \exists \delta_0^{'} \ge \delta \ \text{ и } \ \exists \delta_0^{"} \ge \delta \quad \left| \int\limits_{\delta_0^{'}}^{\delta_0^{"}} \frac{dx}{1+\alpha_0^2 x^2} \right| \ge \varepsilon_0 \, .$$

The integrand function is positive and decreases monotonically in $x$. Therefore, the assessment is fair

$$\left| \int\limits_{\delta_0^{'}}^{\delta_0^{"}} \frac{dx}{1+\alpha_0^2 x^2} \right| \ge \frac{\delta_0^{"} - \delta_0^{'}}{1+\alpha_0^2 \delta_0^{"2}} = \varepsilon_0 \, .$$

Moreover, the last equality is true for the following *acceptable* (check it out!) meanings

$$\forall \delta > 0 : \qquad \delta_0^{'} = \delta , \quad \delta_0^{"} = \delta_0^{'} + 1 , \quad \alpha_0 = \frac{1}{\delta_0^{"}} , \quad \varepsilon_0 = \frac{1}{2} \, .$$

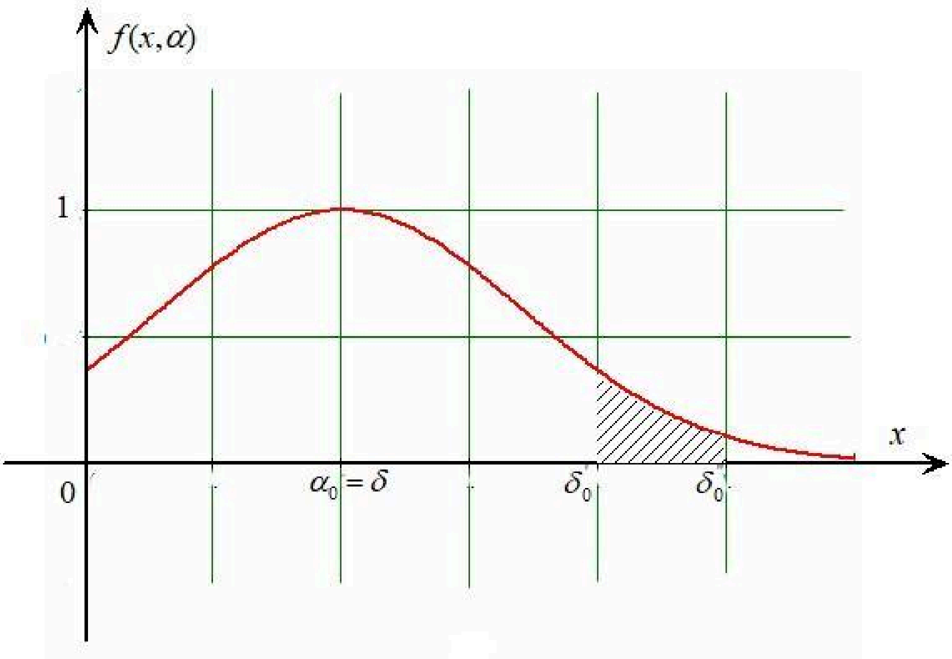
Indeed, for these parameter values the equalities are satisfied

$$\frac{\delta_0^{"} - \delta_0^{'}}{1+\alpha_0^2 \delta_0^{"2}} = \frac{\left(\delta_0^{'} + 1\right) - \delta_0^{'}}{1 + \left(\dfrac{1}{\delta_0^{"}}\right)^2 \delta_0^{"2}} = \frac{1}{2} = \varepsilon_0 \, ,$$

those. There is no non-uniform convergence by negating the Cauchy criterion.

Example 4. Investigate for uniform convergence $\Phi(\alpha)=\int\limits_0^{+\infty} e^{-(x-\alpha)^2}\,dx$ in plurals: 1) $\alpha\in(-\infty,0)$ and 2) $\alpha\in(0,+\infty)$ .

Solution. 1) Let us apply the Weierstrass test. For any $\alpha<0$ inequality is true $\int\limits_0^{+\infty} e^{-(x-\alpha)^2}\,dx<\int\limits_0^{+\infty} e^{-x^2}\,dx$ . Since the last integral converges, the original integral on the set $\alpha\in(-\infty,0)$ converges uniformly.




2) For $\alpha \in (0,+\infty)$ on a set $x > \alpha$ the integrand function is positive and monotonically decreasing in $x$ . Then by negation of the Cauchy criterion

$$\exists \varepsilon_0 = \frac{1}{e^4}: \quad \forall \delta > 0: \quad \exists \alpha_0 = \delta \quad u \quad \exists \delta'_0 = \delta + 1, \quad \exists \delta''_0 = \delta + 2 \quad \Rightarrow$$

$$\Rightarrow \quad \left| \int\limits_{\delta'_0}^{\delta''_0} e^{-(x-\alpha_0)^2} dx \right| \geq e^{-(\delta''_0 - \alpha_0)^2} (\delta''_0 - \delta'_0) =$$

$$= e^{-(\delta+2-\delta)^2} ((\delta+2) - (\delta+1)) = \frac{1}{e^4} = \varepsilon_0 \; .$$

What does it mean that the integral does not converge uniformly on the set? $\alpha \in (0,+\infty)$ .

Example 5. Investigate integrals for uniform convergence $\Phi_1(\alpha)=\int\limits_1^{+\infty}\frac{\sin x}{x^\alpha}dx$ And $\Phi_2(\alpha)=\int\limits_1^{+\infty}\frac{\sin x}{x^\alpha+\sin x}dx$ on set: $\alpha\ge\alpha_0>0$ .

Solution. 1) To study the integral $\Phi_1(\alpha)$ Let's apply the Dirichlet criterion. Integral of the form $\int\limits_1^{+\infty}f(x,\alpha)g(x,\alpha)\,dx$ converges uniformly in $\alpha$ on a set $\Omega$, since on the set $x\in[1,+\infty)$ at every $\alpha\ge\alpha_0>0$ functions $f(x,\alpha),\ g(x,\alpha)$ и $g'_x\ (x,\alpha)$ continuous in $x$ and satisfy the conditions:

In this case $f(x,\alpha)=\sin x$ is continuous and $g(x,\alpha)=\frac{1}{x^\alpha}$ continuously differentiable with respect to $x$. In addition, we have

$1^0$. $\lim\limits_{x\to+\infty}\frac{1}{x^\alpha}=0$ at $\forall\alpha>0$, and the uniformity of this passage to the limit follows from the obvious inequality

$$\frac{1}{x^{\alpha_0}}\ge\frac{1}{x^\alpha}\qquad\forall x\in[1,+\infty),\ \forall\alpha\in[\alpha_0,+\infty)\ .$$

$2^0$. function $g'_x\ (x,\alpha)=-\frac{\alpha}{x^{\alpha+1}}$ sign constant of $x\in[1,+\infty)$ at every $\alpha\ge\alpha_0>0$ ,

$3^0$. Finally, $\left|\int\limits_1^x\sin u\,du\right|=\left|-\cos x+\cos 1\right|\le M=2$ $x\in[1,+\infty)\forall$ at every $\alpha\ge\alpha_0>0$

This means (by the Dirichlet criterion) the integral $\Phi_1(\alpha)$ converges uniformly.

2). Let us now consider the case of the integral $\Phi_2(\alpha)$. Condition $2^0$ will not be satisfied here. Indeed, the function $g'_x(x,\alpha) = -\dfrac{\alpha x^{\alpha-1} + \cos x}{\left(x^\alpha + \sin x\right)^2}$ is not constant in sign $x \in [1,+\infty)$ at $\alpha \in (0,1)$. In other words, the passage to the limit of $1^0$ takes place, but it *non-monotonic*. Therefore, the Dirichlet test is useless here.

To study convergence $\Phi_2(\alpha)$ Let us apply another tool: the Taylor formula in combination with the theorem proven earlier in the course of mathematical analysis that,

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|---|
| if integral $\Phi(\alpha) = P(\alpha) + Q(\alpha)$, where the integral $Q(\alpha)$ converges uniformly and absolutely, then the integrals $\Phi(\alpha)$ And $P(\alpha)$ have the same type of convergence (or divergence). |

Using the Taylor formula at $x \to +\infty$ we get equalities

$$\frac{\sin x}{x^\alpha + \sin x} = \frac{\dfrac{\sin x}{x^\alpha}}{1 + \dfrac{\sin x}{x^\alpha}} = \frac{\sin x}{x^\alpha} \cdot \left(1 - \frac{\sin x}{x^\alpha} + O\left(\frac{1}{x^{2\alpha}}\right)\right) =$$

$$= \frac{\sin x}{x^\alpha} - \frac{\sin^2 x}{x^{2\alpha}} + O\left(\frac{1}{x^{3\alpha}}\right).$$

Where does the integral come from? $\int\limits_1^{+\infty} \frac{\sin x}{x^\alpha + \sin x}\,dx$ is equivalent in the nature of convergence to the integral $\int\limits_1^{+\infty} \frac{\sin^2 x}{x^{2\alpha}}\,dx$, since the integral $\int\limits_1^{+\infty} \frac{\sin x}{x^\alpha}\,dx$ converges (as shown above) uniformly according to the Dirichlet criterion, and the integral $\int\limits_1^{+\infty} O\left(\frac{1}{x^{3\alpha}}\right)dx$ converges uniformly using the Weierstrass test (prove it yourself).

It remains to figure out the type of convergence of the integral $\int\limits_1^{+\infty} \frac{\sin^2 x}{x^{2\alpha}}\,dx$. If you remember the material from semester 2, the answer will be:

integral $\int\limits_1^{+\infty} \frac{\sin^2 x}{x^{2\alpha}}\,dx$, (*and, therefore, the integral* $\int\limits_1^{+\infty} \frac{\sin x}{x^\alpha + \sin x}\,dx$ )

diverges at $\alpha \le \frac{1}{2}$ and converges uniformly at $\alpha > \frac{1}{2}$ .

Indeed, when $\forall\delta \in (1,+\infty)$ And $0 < \alpha \le \frac{1}{2}$, picking up *n* such that $\pi n > \delta$, and integration limits $\delta_{01} = \pi n$ And $\delta_{02} = 2\pi n$, we have an estimate

$$\left| \int_{\delta_{01}}^{\delta_{02}} \frac{\sin^2 x}{x^{2\alpha}} dx \right| \ge \int_{\pi n}^{2\pi n} \frac{\sin^2 x}{x} dx \ge \frac{1}{2\pi n} \int_{\pi n}^{2\pi n} \sin^2 x \, dx = \frac{1}{2\pi n} \int_{\pi n}^{2\pi n} \frac{1 - \cos 2x}{2} dx = \frac{1}{2\pi n} \frac{\pi n}{2} = \frac{1}{4}.$$

So there is $\varepsilon_0 = \frac{1}{4}$ such that $\forall\delta \in (1,+\infty)$ there will be $\delta_{01} = \pi n$ And $\delta_{02} = 2\pi n$ belonging to $(\delta,+\infty)$, at which $\left| \int_{\delta_{01}}^{\delta_{02}} \frac{\sin^2 x}{x^{2\alpha}} dx \right| \ge \varepsilon_0$. Then the integral $\int_1^{+\infty} \frac{\sin^2 x}{x^{2\alpha}} dx$ *diverges* for $0 < \alpha \le \frac{1}{2}$ according to *negation of the Cauchy criterion*.

Note that the equality $\int_1^{+\infty} \frac{\sin^2 x}{x^{2\alpha}} dx = \frac{1}{2} \int_1^{+\infty} \frac{dx}{x^{2\alpha}} - \frac{1}{2} \int_1^{+\infty} \frac{\cos 2x}{x^{2\alpha}} dx$ does not allow the use of the comparison criterion, since the second integral on the right side is not taken from a function of constant sign.

On the other hand, due to inequality $|\sin x| \ge \sin^2 x$ from the divergence of the integral $\int_1^{+\infty} \frac{\sin^2 x}{x^{2\alpha}} dx$ at $0 < \alpha \le \frac{1}{2}$ follows the divergence of the integral $\int_1^{+\infty} \frac{|\sin x|}{x^{2\alpha}} dx$, since in this case both integrands are of constant sign.