

**IMPROPER INTEGRALS DEPENDING ON A PARAMETER.
CALCULATION OF INTEGRALS DEPENDING ON PARAMETERS**

When solving problems, it is assumed that the following (tabular) integrals can be used, as known from the theoretical part of the course “Harmonic Analysis”:

$$1^{\circ} \quad \textit{Dirichlet integral:} \quad \int_0^{+\infty} \frac{\sin \alpha x}{x} dx = \frac{\pi}{2} \operatorname{sign} \alpha, \quad \alpha \in \mathbf{R} .$$

$$2^{\circ} \quad \textit{Integral Laplace:} \quad \int_0^{+\infty} \frac{x \sin \alpha x}{1+x^2} dx = \frac{\pi}{2} e^{-|\alpha|} \operatorname{sign} \alpha, \quad \alpha \in \mathbf{R} .$$

$$\int_0^{+\infty} \frac{\cos \alpha x}{1+x^2} dx = \frac{\pi}{2} e^{-|\alpha|}, \quad \alpha \in \mathbf{R} .$$

$$3^{\circ} \quad \textit{Euler–Poisson integral:} \quad \int_0^{+\infty} e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}. \quad \alpha > 0 .$$

It may be useful in a number of tasks *Frullani formula*:

If the function $f(x)$ continuous on $[0, +\infty)$ and for everyone $A > 0$ integral $\int_A^{+\infty} \frac{f(x)}{x} dx$ converges

That $\forall a > 0$ And $\forall b > 0$ equality is true $\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a}$.

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For illustration, consider the following problems.

Example 1. Calculate $\int_0^{+\infty} \frac{\sin x^3}{x} dx$.

Solution. Let's perform a variable change $x^3 = u$ and use the value of the Dirichlet integral at $\alpha = 1$

$$\begin{aligned} \int_0^{+\infty} \frac{\sin x^3}{x} dx &= \frac{1}{3} \int_0^{+\infty} \frac{\sin x^3}{x^3} 3x^2 dx = \frac{1}{3} \int_0^{+\infty} \frac{\sin x^3}{x^3} dx^3 = \\ &= \frac{1}{3} \int_0^{+\infty} \frac{\sin u}{u} du = \frac{1}{3} \frac{\pi}{2} = \frac{\pi}{6}. \end{aligned}$$

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Example 2. Calculate $\int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx$, If $\alpha > 0$ And $\beta > 0$.

Solution. Note that from the convergence of the Euler-Poisson integral it follows that

$$0 < \int_A^{+\infty} \frac{e^{-x^2}}{x} dx \leq \frac{1}{A} \int_A^{+\infty} e^{-x^2} dx < \frac{1}{A} \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2A}$$

for any fixed $A > 0$.

Therefore, we can apply the Frullani formula. We have

$$\int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx = \int_0^{+\infty} \frac{e^{-(\sqrt{\alpha} x)^2} - e^{-(\sqrt{\beta} x)^2}}{x} dx = e^{-0^2} \ln \frac{\sqrt{\alpha}}{\sqrt{\beta}} = \frac{1}{2} \ln \frac{\beta}{\alpha}$$

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Example 2a. Calculate $\int_0^{+\infty} \frac{\sin(\alpha x + \delta) - \sin(\beta x + \delta)}{x} dx$, If $\alpha > 0$ And $\beta > 0$.

Solution. Note that from the convergence of the Dirichlet integral it follows that the integral

$$\int_A^{+\infty} \frac{\sin(\alpha x + \delta)}{x} dx$$

converges for any fixed $A > 0$.

Therefore, applying the Frullani formula gives

$$\int_0^{+\infty} \frac{\sin(\alpha x + \delta) - \sin(\beta x + \delta)}{x} dx = \sin \delta \cdot \ln \frac{\alpha}{\beta}.$$

$$\Phi(\alpha) = \int_0^{+\infty} \frac{1 - e^{-\alpha x}}{x} \cos x \, dx, \quad \alpha > 0$$

Example 3. Calculate

Solution. Function $\frac{1 - e^{-\alpha x}}{x}$ decreases monotonically by $x \in (0, +\infty)$, a function $\cos x$ has a limited antiderivative. That's why $\Phi(\alpha)$ converges *point by point* according to the Dirichlet criterion.

Integral of $\frac{\partial}{\partial \alpha} \left(\frac{1 - e^{-\alpha x}}{x} \cos x \right) = e^{-\alpha x} \cos x$ converges *evenly*, so we can use the equality (this integral is taken)

$$\Phi'(\alpha) = \int_0^{+\infty} e^{-\alpha x} \cos x \, dx = \frac{\alpha}{1 + \alpha^2}.$$

Where do we get it from?

$$\Phi(\alpha) = \frac{1}{2} \ln(1 + \alpha^2) + C$$

And, because $\Phi(0) = 0$, That $C = 0$ and finally $\Phi(\alpha) = \frac{1}{2} \ln(1 + \alpha^2)$.

Example 4. Calculate $\int_0^{+\infty} \frac{x - \sin x}{x^3} dx$.

Solution. Consider the parametric integral

$$\Phi(\alpha) = \int_0^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3} dx, \quad \alpha > 0,$$

which coincides with the original one at $\alpha = 1$.

Note that $\Phi(\alpha)$ converges uniformly, since it can be represented in the form

$$\int_0^1 \frac{\alpha x - \sin \alpha x}{x^3} dx + \int_1^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3} dx,$$

where the first integral is definite, taken from a continuous function of two variables,

and for the second (improper) the following estimate is valid:

$$\int_1^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3} dx \leq 2\alpha \int_1^{+\infty} \frac{dx}{x^2}.$$

Next, $\Phi'(\alpha)$ also converges *evenly*, since the estimate is valid

$$\frac{\partial}{\partial \alpha} \frac{\alpha x - \sin \alpha x}{x^3} = \frac{1 - \cos \alpha x}{x^2} \leq \frac{2}{x^2}.$$

Finally, using the Dirichlet integral, we find

$$\Phi''(\alpha) = \int_0^{+\infty} \frac{\sin \alpha x}{x} dx = \frac{\pi}{2}, \text{ because } \alpha > 0.$$

Now, integrating over twice α function $\Phi''(\alpha)$, we get that

$$\Phi(\alpha) = \frac{\pi}{4} \alpha^2 + \frac{C\pi}{2} \alpha + D.$$

Because $\lim_{\alpha \rightarrow +0} \Phi(\alpha) = 0$, then by continuity we can put $\Phi(0) = 0$. Then

$$\Phi(\alpha) = \frac{\pi}{4} \alpha^2 \quad \text{and the value of the original integral is} \quad \Phi(1) = \frac{\pi}{4}.$$

In Example 4, the question may naturally arise: wouldn't it be easier to parametrize the integral

$$\Phi(\alpha) = \int_0^{+\infty} \frac{x - \sin \alpha x}{x^3} dx, \quad \alpha > 0$$

using the formula

? What is your opinion on this matter?