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IMPROPER INTEGRALS DEPENDING ON A PARAMETER.
CALCULATION OF INTEGRALS DEPENDING ON PARAMETERS

When solving problems, it is assumed that the following (tabular) integrals can be used, as known from the theoretical part of the course "Harmonic Analysis":

1^0. *Dirichlet integral*: $\int\limits_0^{+\infty} \frac{\sin \alpha x}{x} dx = \frac{\pi}{2} \operatorname{sign} \alpha , \qquad \alpha \in \mathbf{R}$.

2^0. *Integral Laplace*: $\int\limits_0^{+\infty} \frac{x \sin \alpha x}{1+x^2} dx = \frac{\pi}{2} e^{-|\alpha|} \operatorname{sign} \alpha , \quad \alpha \in \mathbf{R}$.

$$\int\limits_0^{+\infty} \frac{\cos \alpha x}{1+x^2} dx = \frac{\pi}{2} e^{-|\alpha|} , \qquad \alpha \in \mathbf{R}.$$

3^0. *Euler–Poisson integral*: $\int\limits_0^{+\infty} e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} . \qquad \alpha > 0$.

It may be useful in a number of tasks *Frullani formula*:

If the function $f(x)$ continuous on $[0,+\infty)$ and for everyone $A>0$ integral $\int\limits_{A}^{+\infty}\frac{f(x)}{x}dx$ converges

That $\forall a>0$ And $\forall b>0$ equality is true $\int\limits_{0}^{+\infty}\frac{f(ax)-f(bx)}{x}dx=f(0)\ln\frac{b}{a}$.

For illustration, consider the following problems.

Example 1. Calculate $\int\limits_0^{+\infty} \frac{\sin x^3}{x} dx$.

Solution. Let's perform a variable change $x^3 = u$ and use the value of the Dirichlet integral at $\alpha = 1$

$$\int\limits_0^{+\infty} \frac{\sin x^3}{x} dx = \frac{1}{3}\int\limits_0^{+\infty} \frac{\sin x^3}{x^3} 3x^2 dx = \frac{1}{3}\int\limits_0^{+\infty} \frac{\sin x^3}{x^3} dx^3 =$$

$$= \frac{1}{3}\int\limits_0^{+\infty} \frac{\sin u}{u} du = \frac{1}{3}\frac{\pi}{2} = \frac{\pi}{6}.$$

Example 2. Calculate $\int\limits_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx$, If $\alpha > 0$ And $\beta > 0$.

Solution. Note that from the convergence of the Euler-Poisson integral it follows that

$$0 < \int\limits_A^{+\infty} \frac{e^{-x^2}}{x} dx \le \frac{1}{A} \int\limits_A^{+\infty} e^{-x^2} dx < \frac{1}{A} \int\limits_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2A}$$

for any fixed $A > 0$.

Therefore, we can apply the Frullani formula. We have

$$\int\limits_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx = \int\limits_0^{+\infty} \frac{e^{-\left(\sqrt{\alpha}\, x\right)^2} - e^{-\left(\sqrt{\beta}\, x\right)^2}}{x} dx = e^{-0^2} \ln \frac{\sqrt{\alpha}}{\sqrt{\beta}} = \frac{1}{2} \ln \frac{\beta}{\alpha}$$

.

Example 2a. Calculate $\int\limits_0^{+\infty} \frac{\sin(\alpha x + \delta) - \sin(\beta x + \delta)}{x} dx$, If $\alpha > 0$ And $\beta > 0$.

Solution. Note that from the convergence of the Dirichlet integral it follows that the integral

$$\int\limits_A^{+\infty} \frac{\sin(\alpha x + \delta)}{x} dx$$

converges for any fixed $A > 0$.

Therefore, applying the Frullani formula gives

$$\int\limits_0^{+\infty} \frac{\sin(\alpha x + \delta) - \sin(\beta x + \delta)}{x} dx = \sin\delta \cdot \ln\frac{\alpha}{\beta}\,.$$

Example 3. Calculate $$\Phi(\alpha)=\int\limits_0^{+\infty}\frac{1-e^{-\alpha x}}{x}\cos x\,dx\,,\ \ \alpha>0$$.

Solution. Function $\frac{1-e^{-\alpha x}}{x}$ decreases monotonically by $x\in(0,+\infty)$, a function $\cos x$ has a limited antiderivative. That's why $\Phi(\alpha)$ converges *point by point* according to the Dirichlet criterion.

Integral of $$\frac{\partial}{\partial\alpha}\left(\frac{1-e^{-\alpha x}}{x}\cos x\right)=e^{-\alpha x}\cos x$$ converges *evenly*, so we can use the equality (this integral is taken)

$$\Phi'(\alpha)=\int\limits_0^{+\infty}e^{-\alpha x}\cos x\,dx=\frac{\alpha}{1+\alpha^2}.$$

Where do we get it from?

$$\Phi(\alpha)=\frac{1}{2}\ln\left(1+\alpha^2\right)+C$$.

And, because $\Phi(0)=0$, That $C=0$ and finally $\Phi(\alpha)=\frac{1}{2}\ln\left(1+\alpha^2\right)$.

Example 4. Calculate $\int\limits_0^{+\infty} \frac{x - \sin x}{x^3}\, dx$.

Solution. Consider the parametric integral

$$\Phi(\alpha) = \int\limits_0^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3}\, dx\,, \quad \alpha > 0\,,$$

which coincides with the original one at $\alpha = 1$.

Note that $\Phi(\alpha)$ converges uniformly, since it can be represented in the form

$$\int\limits_0^{1} \frac{\alpha x - \sin \alpha x}{x^3}\, dx + \int\limits_1^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3}\, dx\,,$$

where the first integral is definite, taken from a continuous function of two variables,
and for the second (improper) the following estimate is valid:

$$\int\limits_1^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3}\, dx \quad \le \quad 2\alpha \int\limits_1^{+\infty} \frac{dx}{x^2}\,.$$

Next, $\Phi'(\alpha)$ also converges *evenly*, since the estimate is valid

$$\frac{\partial}{\partial\alpha}\frac{\alpha x-\sin\alpha x}{x^3} = \frac{1-\cos\alpha x}{x^2} \le \frac{2}{x^2}.$$

Finally, using the Dirichlet integral, we find

$$\Phi''(\alpha)=\int\limits_0^{+\infty}\frac{\sin\alpha x}{x}dx=\frac{\pi}{2}$$, because $\alpha>0$.

Now. integrating over twice α function $\Phi''(\alpha)$, we get that

$$\Phi(\alpha)=\frac{\pi}{4}\alpha^2+\frac{C\pi}{2}\alpha+D.$$

Because $\lim\limits_{\alpha\to+0}\Phi(\alpha)=0$, then by continuity we can put $\Phi(0)=0$. Then $\Phi(\alpha)=\frac{\pi}{4}\alpha^2$ and the value of the original integral is $\Phi(1)=\frac{\pi}{4}$.

In Example 4, the question may naturally arise: wouldn't it be easier to parametrize the integral using the formula $\Phi(\alpha)=\int\limits_0^{+\infty}\frac{x-\sin\alpha x}{x^3}dx,\ \alpha>0$? What is your opinion on this matter?