SPECIAL FUNCTIONS

The use of improper integrals depending on a parameter often arises in various applied mathematical and physical problems.

Euler integrals

A large number of practically important tasks lead to the need to use a special class of functions called *Euler integrals*.

Definition 1. *Euler integral of the first kind* (or *beta function*) is called a function of two variables of the form

$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx,\quad \alpha>0,\quad \beta>0\,.$$

. *Euler integral of the second kind* (or *gamma function*) is called a function of the form

$$\Gamma(p)=\int_0^{+\infty} x^{p-1}e^{-x}\,dx,\quad p>0\,.$$

.

One of the first to study these functions was twenty-two-year-old Leonhard Euler, which led Adrien Legendre to later call them Euler integrals.

Over time, it turned out that the number of applications that are in one way or another connected with Euler integrals is so large that it is advisable to separate them into a special class of functions.

Let us list (without justification) the main properties of Euler integrals.

1) Scope: $B(\alpha,\beta)$ exists $\forall\alpha,\beta\in(0,+\infty)$, $\Gamma(p)$ – $\forall p>0$,

2) $B(\alpha,\beta)=B(\beta,\alpha)$ "*symmetry*",

3) $B(\alpha,\beta)=\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, expression of the beta function through the gamma function,

4) $\Gamma(p+1)=p\Gamma(p)$ $p>0$ property "*reductions*",

5) $\Gamma(p)\Gamma(1-p)=\dfrac{\pi}{\sin\pi p}$ $p\in(0,1)$ property "*additions*".

Let's check, for example, formula 4).

We have

$$\Gamma(p+1)=\int\limits_0^{+\infty} x^p e^{-x}dx = -\,x^p e^{-x}\Big|_0^{+\infty} + p\int\limits_0^{+\infty} x^{p-1}e^{-x}dx = p\Gamma(p)\,.$$

Gamma and beta functions are a convenient tool for calculating some integrals, in particular many integrals that are not "taken" in elementary functions.

For Eulerian integrals, there are computer procedures for finding values, and therefore they can be used in calculations along with elementary functions.

Let's look at some examples.

Example 1. By virtue of formula 4) (the “lowering” property), the equalities turn out to be true:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \\ = n(n-1)(n-2)\Gamma(n-2) = \ldots = n(n-1)(n-2)\ldots 2\cdot 1\cdot\Gamma(1) = n!\cdot\Gamma(1)$$

or $\Gamma(n+1) = n!$, since it is obvious ☺ that $\Gamma(1) = 1$.

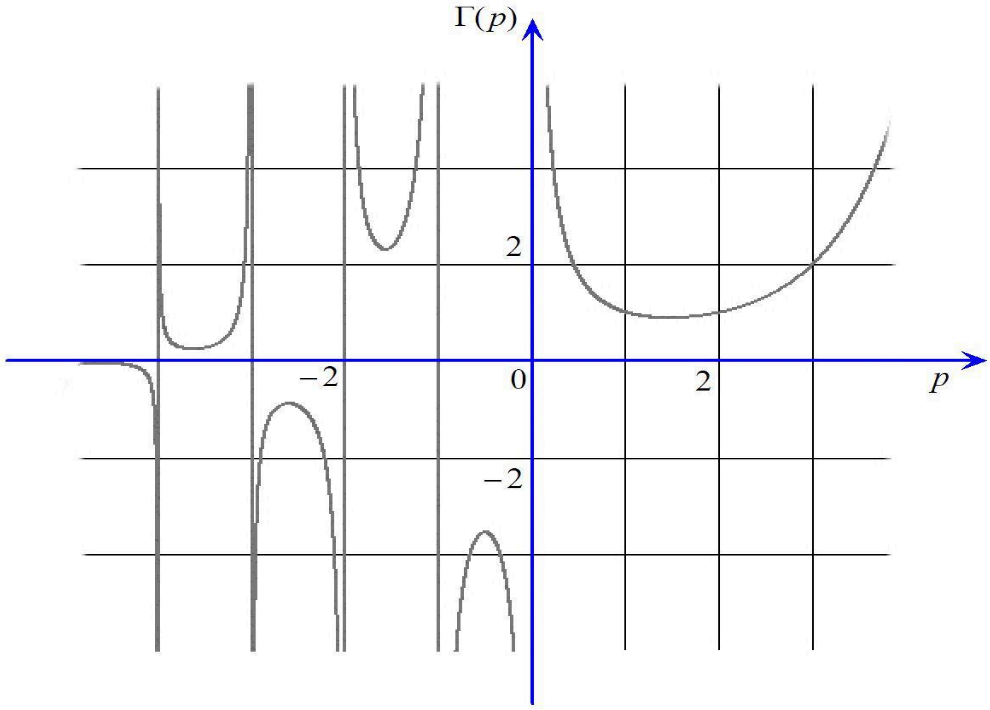
Whence it follows that

$$\Gamma(p+1) = \int_0^{+\infty} x^p e^{-x} dx$$

can be considered as a generalization of the definition *factorial* to any positive number.

Let it now p real, but not equal $0,-1,-2,-3,\ldots$, number. Then formula 4) can be taken as <u>definition</u> *gamma functions* on “almost the entire” real axis.

The graphical interpretation of the result of such a determination has the following interesting form:



Finally, as you will learn later in the TFKP course, even more interesting and useful results are obtained by expanding the scope of definition *gamma functions* to the complex plane.

Example 2. Find the value of the "untaken" Poisson integral $\int\limits_{-\infty}^{+\infty} e^{-x^2} dx$.

Solution: Let's change the integration variable:

$$u = x^2, \quad 2xdx = du, \quad dx = \frac{du}{2\sqrt{u}}$$

in the Poisson integral, we get:

$$\int\limits_0^{+\infty} e^{-x^2} dx = \frac{1}{2}\int\limits_0^{+\infty} u^{-\frac{1}{2}} e^{-u} du = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

since from 5) - properties of the “complement” we have

$$\Gamma^2\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi.$$

It should be noted (for it may be useful) that *beta function* allows other forms of its recording.

For example, by replacing

$$x = \cos^2 \varphi \quad \text{with} \quad dx = -2\cos\varphi \sin\varphi \, d\varphi ,$$

we will get in the end

$$B(\alpha,\beta) = 2\int_0^{\frac{\pi}{2}} \cos^{2\alpha-1}\varphi \cdot \sin^{2\beta-1}\varphi \, d\varphi, \quad \alpha > 0, \quad \beta > 0 .$$

And, if you use the replacement

$$x = \frac{u}{u+1}, \quad 1 - x = \frac{1}{u+1} \quad \text{And} \quad dx = \frac{du}{(1+u)^2},$$

That *beta function* can be presented when $u \in (0, +\infty)$ How

$$B(\alpha, \beta) = \int_0^{+\infty} \frac{u^{\alpha-1} du}{(1+u)^{\alpha+\beta}} . \quad \text{(A)}$$

Euler integrals so well researched, described and programmed that the problem can be considered solved if the answer is expressed in terms of *beta*- And *gamma functions*.

Example 3. Find at $\alpha > -1$ And $\beta > -1$ integral

$$I(a,b) = \int\limits_0^{\frac{\pi}{2}} \sin^\alpha x \cos^\beta x \, dx$$
.

Solution: making a replacement $u = \cos^2 x$, we get in force $du = -2\cos x \sin x \, dx$ and representations (A)

$$I(a,b) = \frac{1}{2}\int\limits_0^1 u^{\frac{a+1}{2}} (1-u)^{\frac{b+1}{2}} du = \frac{1}{2} B\left(\frac{a+1}{2}, \frac{b+1}{2}\right)$$
.

The following examples demonstrate the variety of possible uses *Euler integrals*.

Example 4. Calculate improper integral $I = \int\limits_0^{+\infty} \frac{dx}{x^3+1}$.

Solution: This integral is "taken" because it is taken from a fractional rational function.

Indeed, using decomposition into simple fractions (remember the written MA exam in the spring of the 1st year), from

$$\int \frac{dx}{x^3+1} = \frac{1}{6}\ln\frac{x^2+2x+1}{x^2-x+1} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C ,$$

we get (check!) that $I = \frac{2\pi}{3\sqrt{3}}$.

However, such boring calculations can be avoided if you notice that when replacing $u = x^3$ by virtue of formula (A) we obtain

$$I = \frac{1}{3}\int\limits_0^{+\infty} \frac{du}{(u+1)\sqrt[3]{u^2}} = \frac{1}{3}\int\limits_0^{+\infty} \frac{u^{\frac{1}{3}-1}\,du}{(u+1)^{\frac{1}{3}+\frac{2}{3}}} = \otimes .$$

But then the equalities follow from properties 3) and 5):

$$\otimes = \frac{1}{3}B\left(\frac{1}{3},\frac{2}{3}\right) = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{2\pi}{3\sqrt{3}} .$$

A similar method can be used to solve

Example 5. Calculate improper integral $I = \int\limits_0^{\frac{\pi}{2}} \operatorname{tg}^{2a-1} x\,dx$ at $a \in (0,1)$.

Solution: Let's make a replacement $\operatorname{tg} x = \sqrt{u}$ at $u > 0$. Then $dx = \frac{1}{1+u} \cdot \frac{du}{2\sqrt{u}}$ and therefore

$$I = \int\limits_0^{\frac{\pi}{2}} \operatorname{tg}^{2a-1} x\,dx = \frac{1}{2}\int\limits_0^{+\infty} u^{\frac{2a-1}{2}} \frac{1}{1+u} \cdot \frac{du}{\sqrt{u}} = \frac{1}{2}\int\limits_0^{+\infty} \frac{u^{a-1}}{(1+u)^1}\,du = \otimes$$.

Let us again apply (A), properties 3) and 4), which will give us finally

$$\otimes = \frac{1}{2} B(a, 1-a) = \frac{1}{2} \cdot \frac{\pi}{\sin \pi a}$$.