Fourier integral

Let the function $f(x)$ absolutely integrable on any interval of the real axis, piecewise continuous $\forall x \in (-\infty, +\infty)$ and has for any real x one-sided derivatives.

Then, by analogy with the definition of the trigonometric Fourier series, replacing the operation of summation with integration, it can be associated with the function $\Upsilon(x)$, which is an improper integral depending on the parameter $x \in (0, +\infty)$, the species

$$Y(x) = \int_{0}^{+\infty} \big(a(u)\cos xu + b(u)\sin xu\big)du \quad , \qquad (1)$$

Where $a(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos ut\,dt$ And $b(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin ut\,dt$.

Takes place

Theorem 1. At the point x the equality will be true

$$Y(x)=\frac{f(x+0)+f(x-0)}{2}.$$

Functions $a(u)$ And $b(u)$ can be considered (by analogy with sequences $\{a_k\}$ And $\{b_k\}$, which are the discrete spectrum of a periodic function) as *continuous* spectra *non-periodic* functions $f(x)$.

And the function itself $Y(x)$, called *Fourier integral*, can be interpreted as a harmonic expansion (i.e., as a spectrum) for *non-periodic* functions.

The connection between the Fourier series and the Fourier integral can be demonstrated more clearly and naturally using the following passage to the limit in the standard Riemannian integral sum.

As we saw earlier, each absolutely integrable on the interval $[-A,A]$ functions $f(x)$ can be put into correspondence defined on $\mathbf{R}$, $2A$ -periodic function $\Phi(x)$ the species

$$\Phi(x)=\frac{a_0}{2}+\sum_{k=1}^{+\infty}\left(a_k\cos\frac{\pi kx}{A}+b_k\sin\frac{\pi kx}{A}\right),\qquad (2)$$

called *amount Fourier series*, where the coefficients a_k And b_k were determined by formulas

$$a_0=\frac{1}{A}\int_{-A}^{A}f(u)\,du,\quad a_k=\frac{1}{A}\int_{-A}^{A}f(u)\cos\frac{\pi ku}{A}\,du\quad\forall k\in\mathbf{N}$$

And

$$b_k=\frac{1}{A}\int_{-A}^{A}f(u)\sin\frac{\pi ku}{A}\,du\quad\forall k\in\mathbf{N}.$$

Then $\forall x\in(-A,A)\quad \Phi(x)=\dfrac{f(x-0)+f(x+0)}{2}$.

Let's make the following transformations based on formula (1).

$$J(x,A)=\frac{a_0}{2}+\sum_{k=1}^{+\infty}\left(a_k\cos\frac{\pi kx}{A}+b_k\sin\frac{\pi kx}{A}\right)=$$

$$=\frac{1}{2A}\int_{-A}^{A}f(t)\,dt+\sum_{k=1}^{+\infty}\left(\left(\frac{1}{A}\int_{-A}^{A}f(t)\cos\frac{\pi kt}{A}dt\right)\cos\frac{\pi kx}{A}+\left(\frac{1}{A}\int_{-A}^{A}f(t)\sin\frac{\pi kt}{A}dt\right)\sin\frac{\pi kx}{A}\right)=$$

$$=\frac{1}{2A}\int_{-A}^{A}f(t)\,dt+\sum_{k=1}^{\infty}\frac{1}{A}\int_{-A}^{A}f(t)\left(\cos\frac{\pi kt}{A}\cdot\cos\frac{\pi kx}{A}+\sin\frac{\pi kt}{A}\cdot\sin\frac{\pi kx}{A}\right)dt=$$

$$=\frac{1}{2A}\int_{-A}^{A}f(t)\,dt+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\pi}{A}\int_{-A}^{A}f(t)\cos\frac{\pi k}{A}(x-t)dt\,. \qquad (3)$$

Consider the function $\Psi(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-A}^{A}f(t)\cos\omega(t-x)dt$, defined for $\omega\in(0,A)$. Let's build it for her *Riemannian integral sum* the species $\sigma_N=\sum_{k=1}^{N}\Delta_k\Psi(\omega_k)$, in which $\Delta_k=\dfrac{A}{N}$ − fineness of the interval division $(0,A]$, A $\omega_k=\dfrac{\pi k}{A}$ belongs k -th section of the partition.

Here we have three passages to the limit: the summation of the Fourier series at $N \to \infty$, $A \to +\infty.$ and finally $-A \to -\infty.$ Let us assume that they ensure that the fineness of the partition tends to zero. Then, taking into account $\lim\limits_{A\to+\infty} \frac{1}{2A}\int\limits_{-A}^{A} |f(t)|\,dt = 0$, we get

$$\lim_{A\to+\infty} J(x,A) = \lim_{A\to+\infty} \frac{1}{\pi}\sum_{k=1}^{N} \Delta_k \Psi(\omega_k) = \frac{1}{\pi}\int\limits_{0}^{+\infty} d\omega \int\limits_{-\infty}^{+\infty} f(t)\cos\omega(x-t)dt =$$

$$= \int\limits_{0}^{+\infty} \big(a(\omega)\cos\omega x + b(\omega)\sin\omega x\big)d\omega = Y(x)\,, \qquad (4)$$

Where $a(\omega) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} f(t)\cos\omega t\,dt$ And $b(\omega) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} f(t)\sin\omega t\,dt$.

In these discussions, it is important to take into account that in these formulas three limit passages were combined into one:

- transition from an integral sum to a definite integral due to the tendency of the fineness of the partition to zero;
- transition from a definite integral to an improper integral at a singular point $+\infty$;
- transition from a definite integral to an improper integral at a singular point $-\infty$.

In this case, the joint execution of limit passages of the first with the second, as well as the first with the third, is quite correct. But the joint execution of the second and third transitions clearly violates the definition of the existence of an improper integral with several singular points (it requires the existence of an integral in *everyone* singular points at *independent* limit passages to each of them). We will consider this issue later.

To illustrate this interpretation, consider

Example 6. Represent by Fourier integral for $\tau = 10$ function

$$f(x) = \begin{cases} 1, \text{при } |x| \le \tau, \\ 0, \text{при } |x| > \tau. \end{cases}$$

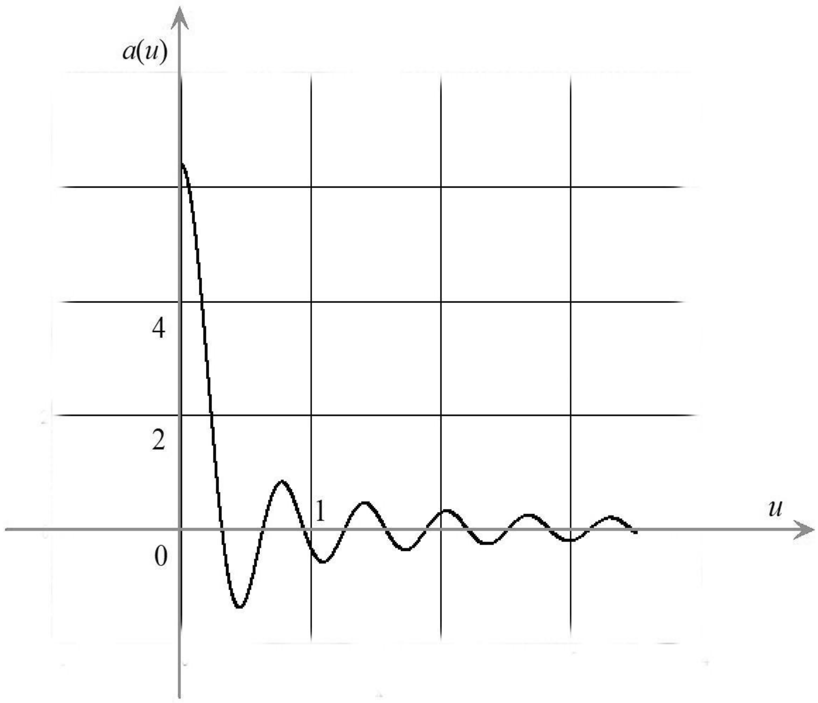
Solution: Let fixed $\tau > 0.$ Since the function $f(x)$ even, then it is obvious that $b(u) \equiv 0$. For $a(u)$ we have

$$a(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\cos ut\,dt = \frac{2}{\pi}\int_{0}^{+\infty} f(u)\cos ut\,dt = \frac{2}{\pi}\int_{0}^{\tau}\cos ut\,dt = \frac{2\sin \tau u}{u}.$$

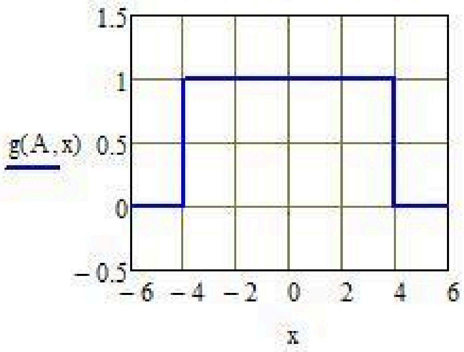
Therefore, the required representation will be

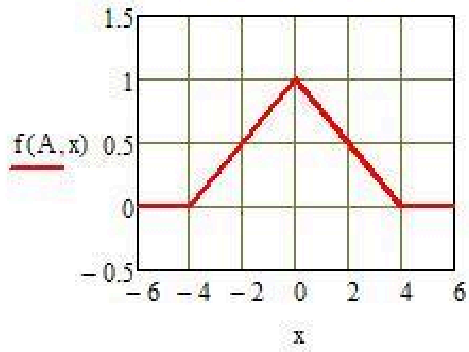
$$f(x) = \frac{2}{\pi}\int_{0}^{+\infty}\frac{\sin \tau u \cdot \cos xu}{u}\,du.$$

Continuous Spectral Function Graph $a(u)$ for $\tau = 10$ has a look.

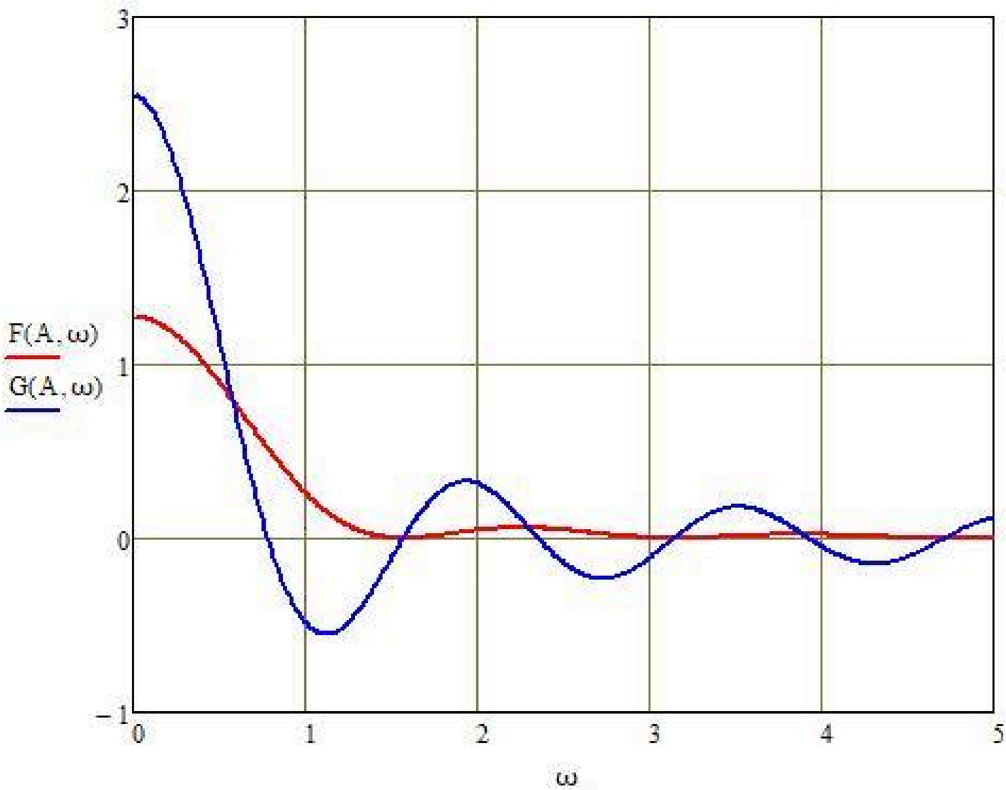


$$g(A,x) := \begin{cases} 1 & \text{if } |x| \le A \\ 0 & \text{otherwise} \end{cases} \qquad f(A,x) := \begin{cases} 1 - \dfrac{|x|}{A} & \text{if } |x| \le A \\ 0 & \text{otherwise} \end{cases}$$





$$G(A,\omega) := \frac{2}{\pi} \cdot \frac{\sin(A \cdot \omega)}{\omega} \qquad F(A,\omega) := \frac{2}{\pi A} \cdot \frac{1 - \cos(A \cdot \omega)}{\omega^2}$$



Consider the following Cauchy problem for the harmonic equation

$$x''+\omega_0^2 x = A\cos\omega t$$

with initial conditions: $x(0) = x'(0) = 0$, Where $\omega_0 > 0, \omega > 0, A \neq 0$ − some constants.
In the theory of oscillations, a similar problem arises when studying an external harmonic influence with a frequency ω and amplitude *A* to a linear system, the frequency of natural oscillations of which is equal to ω_0 .

Note that for this problem the characteristic equation $\lambda^2 + \omega_0^2 = 0$ has roots $\pm i\,\omega_0$, and the right side is the sum of quasipolynomials of zero order $\frac{1}{2}e^{i\varpi t} + \left(-\frac{1}{2}\right)e^{-i\varpi t}$.

It is known that the solution to this Cauchy problem for $\omega \neq \omega_0$ there will be a non-resonant function $x(t) = \frac{A}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right)$.

While in the resonant case, with $\omega = \omega_0$, the solution has the form $x(t) = \frac{A}{2\omega_0^2}\, t\sin\omega_0 t$.

It follows that in the non-resonant case the amplitude of oscillations of the solution is constant, and in the case of resonance it is proportional to the independent variable *t*.

Finally, it becomes obvious that when exposed to a continuous spectrum, there will be a harmonic that has a resonant frequency.

Comparison of the properties of the Fourier series and the Fourier integral

Let us compare the definitions and properties of the Fourier series and integral for an absolutely integrable function $f(x)$.

Hopefully: function $f(x)$ defined and absolutely integrable on the interval $[-A, A]$, $A > 0$.	Hopefully: function $f(x)$ definite, absolutely integrable $\forall x \in (-\infty, +\infty)$, is piecewise continuous on any segment of the real axis and has one-sided derivatives for each real x.
It is associated with the function $\Phi(x) = \frac{a_0}{2} + \sum_{k=1}^{+\infty}\left(a_k \cos\frac{\pi k x}{A} + b_k \sin\frac{\pi k x}{A}\right)$,	It is associated with the function $Y(x) = \int_0^{+\infty}\big(a(\omega)\cos\omega x + b(u)\sin\omega x\big)d\omega$
Where $a_0 = \frac{1}{A}\int_{-A}^{A} f(t)\, dt$, $a_k = \frac{1}{A}\int_{-A}^{A} f(t)\cos\frac{\pi k t}{A}\,dt \quad \forall k \in \mathbf{N}$ $b_k = \frac{1}{A}\int_{-A}^{A} f(t)\sin\frac{\pi k t}{A}\,dt \quad \forall k \in \mathbf{N}$	Where $a(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos\omega t\,dt$ $b(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\omega t\,dt$
Equality is fair $\lim_{x\to x_0}\Phi(x) = \frac{f(x_0-0)+f(x_0+0)}{2} \quad \forall x_0 \in (-A, A)$	Equality is fair $\lim_{x\to x_0}Y(x) = \frac{f(x_0-0)+f(x_0+0)}{2} \quad \forall x_0 \in (-\infty, +\infty)$
Function $\Phi(x)$ determined on $x \in (-\infty, +\infty)$, $2A$ *-periodic*	Function $\mathrm{Y}(x)$, set at $x \in (-\infty, +\infty)$, generally speaking, *non-periodic*
Sequences $\{a_k\}$ And $\{b_k\}$, $k \in \mathbf{N}$, are called *discrete spectro*m $f(x)$	Functions $a(\omega)$ And $b(\omega)$, $\omega \in (0, +\infty)$, are called *continuous spectrum* functions $f(x)$.