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GENERALIZED FUNCTIONS

Definition and basic properties

In a large number of cases, the formulation of a physical law, a description of a phenomenon or process can be accomplished using the concept *functions of many variables* − rules establishing *one-to-one match* between two sets, the first of which consists of finite ordered sets of numbers - "vectors", and the second - of numbers.

However, as physics developed, it became clear that this tool may not be enough. For example, using functions it is not possible to construct a correct quantitative description $\rho_M(x,r)$ – spatial *density of a material point with center at* r *and final mass* M .

S from a physical point of view, from equality $\int\limits_{-\infty}^{+\infty} \rho_M(x,r)\, dx = M$ should *unlimited increase* $\rho_M(x,r)$ with unlimited approach x To r , which is impossible.

Physicists have known about this “theoretical trouble” since the time of Newton-Leibniz, but they were not particularly upset, since it was possible to measure *only the integral of* $\rho_M(x,r)$, and the “density function” itself $\rho_M(x,r)$ for a material point" was not required in practically important calculations.

Let us consider this problem in the one-dimensional case from a formal, mathematical point of view. In this case, we will assume that all integrals used in the notations exist, and r − arbitrary fixed real parameter.

Let us first note that if we use only the concept of “function”, then the solution to the problem is:

find function $f(x,r)$ *such that* $\int\limits_{-\infty}^{+\infty} f(x,r)\varphi(x)\,dx = \varphi(r)$, (1)

does not exist. Simply put, there is no such function $f(x,r)$, which “would be able to do” what equality (1) requires, written using an improper parametric integral.

But what then, in principle, could be the solution to this problem?

It is easy to see that equality (1) *each* valid function $\varphi(x)$ matches *the only thing* number, in this example – $\varphi(r)$.

In other words, condition (1) defines some dependence, **argument** which is the usual *function*, while, **meaning** this dependence exists *number*.

Dependencies of this type are known in mathematics. For example,

- each geometric vector in E^3 it is possible to uniquely match its length,
- each square matrix of order n can be correlated with its determinants,
- Each function continuous on a certain interval of the real axis can be assigned a unique correspondence to its definite integral.

Dependencies of this type are usually called *functionals*. They can be defined, for example, like this:

> we will say that on a certain set of mathematical objects X determined *functional*, having values in the numerical set Y, if a rule is specified according to which *to everyone* element from X compliant *the only thing* number of Y.

There is no generally accepted uniform designation for functionals. Although forms similar to the notation of ordinary functions are often used, formulas of the form $Y = \Phi(X)$. Quite appropriate, because in the case when X there is a numerical (or "vector") set, the definitions of functionals and functions coincide.

When determining functionals, requirements for the properties of a set X can vary quite widely depending on the problem at hand. Let us take advantage of this freedom when describing the properties of the domain of definition of functionals, which can be solutions to problems of the form (1).

At the same time, we will not hide our far-reaching plans here: the set of requirements formulated below will make it possible in the future to build methods for solving problems significantly more serious than problem (1).

We are talking, for example, about Cauchy problems, boundary value and mixed problems for differential equations *in second order partial derivatives.* These tasks are usually called *equations of mathematical physics*.

Let the domain of definition of the functionals we are considering, usually denoted as D, consists of functions $\varphi(x)$.

Let's formulate the requirements individually for functions $\varphi(x)$, which are functions of one real variable, and for their entire set.

1⁰. Let the functions $\varphi(x) \in D$ defined on *all* real axis and have on $\mathbf{R}$ derivative *any* order;

2⁰. $\forall \varphi(x) \in D \quad \exists a \geq 0: \quad \forall x: \ |x| \geq |a| \quad \Rightarrow \quad \varphi(x) \equiv 0$.

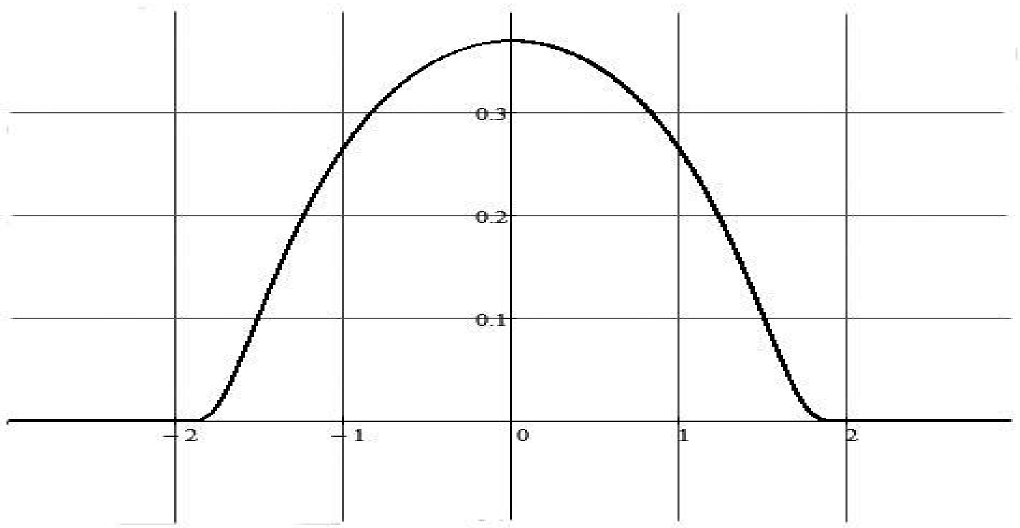
Functions that satisfy the 2⁰ condition are usually called *finite*, and, simultaneously 1⁰ and 2⁰, – *main*.

It is easy to check that the set D is *linear space* with the standard operations of adding and multiplying a number by an element.

Example 1: The main one is the "cap" function, specified by the formula

$$\varphi(x)=\begin{cases} e^{-\frac{4}{4-x^2}}, \text{ при } |x|<2, \\ 0, \quad \text{при } |x|\geq 2, \end{cases}$$

the graph of which is shown in Fig. 1.



Rice. 1. Example of the main function.

Let us now define *convergence* sequences $\{\varphi_{(k)}(x)\}$ elements in D. Symbol $\lim\limits_{k\to\infty}\varphi_{(k)}(x)=\varphi^*(x)$ will mean that on the set $\mathbf{R}$ takes place *uniform* by x convergence of the sequence itself and for sequences of derivatives *any* order $\varphi_{(k)}^{(n)}(x)\Rightarrow\varphi^{*(n)}\quad\forall n=0,1,2,\ldots$ . .

It should be noted that this definition of convergence cannot be reduced to the use of any norm in D (see, for example, Petrovich A.Yu., Ch.3 Str. 292-293).

Example 2: sequence of basic functions of the form $\varphi_{(k)}(x)=\dfrac{\varphi^*(x)}{k}$, Where $\varphi^*(x)$ – “hat” function, converges in D at $k\to\infty$ to a function identically equal to zero $\forall x\in\mathbf{R}$.

Let us now move on to the definition of functionals in space D − basic functions.

Let's call *functional in space* D the rule according to which *each* main function $\varphi(x)$ is put in accordance *singular real* number.

Note also that the functionals defined on D, can be added and multiplied by a number. As a result, new functionality will be obtained.

Using properties of the space of basic functions D, among all possible types of functionals we can distinguish a special class, the elements of which we will call *linear* And *continuous* functionals. Let us give its definition.

Definition 1: Functional $\mathsf{F}(\varphi)$ called *linear* V D, If $\forall \lambda_1, \lambda_2 \in \mathbf{R}$ And $\forall \varphi_1, \varphi_2 \in D$ equality holds

$$\mathsf{F}(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 \mathsf{F}(\varphi_1) + \lambda_2 \mathsf{F}(\varphi_2) .$$

Definition 2: Functional $\mathsf{F}(\varphi)$ called *continuous* V D, If $\forall \{\varphi_{(k)}\}$ converging to $\varphi^*(x)$ V D, takes place $\lim\limits_{k\to\infty} \mathsf{F}(\varphi_{(k)}(x)) = \mathsf{F}(\varphi^*(x))$.

Now we will consider *only* linear and continuous functionals in D, which we will call following the established tradition. *generalized functions*.

It's easy to see that *in its entirety* with the operations of adding functionals and multiplying a number by a functional, generalized functions (as well as basic functions) form a linear space. It is usually denoted D'.

What functionals are called *functions*, we will have to “swallow.” This is a tradition of translation into Russian, apparently not very successful. In English for elements D' the (also very ambiguous in meaning) term is used *distribution*.

Here's an adjective *generalized* leads to a completely legitimate question: what, in fact, do the functionals we are considering generalize?

Returning to problems of type (1), we can give the following explanation.

Let the function $f(x)$ absolutely integrable on any interval of the real axis. Then (this is a theorem!) an integral of the form

$$\mathsf{G}(\varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx \tag{2}$$

there is a linear and continuous functional on D . This means the functionality $\mathsf{G}(\varphi) \in D'$.

That is, some of the functionalities from D' can be generated according to formula (2) by ordinary, absolutely integrable *functions* $f(x)$. We will call such generalized functions *regular*. All others − *singular*.

This gives rise to the use of the term *generalized* to denote a set D' − as a set of regular and singular functionals.

So, regular generalized functions are defined using (2).

The methods for describing singular generalized functions can be very diverse, even in verse.

Let's consider

Example 3: Let the functional $\mathsf{F}(\varphi(x))$ matches the main function $\varphi(x)$ number $\varphi(a)$ − its value at the point $a \in \mathbf{R}$. Show what the functionality is $\mathsf{F}(\varphi(x)) \in D'$.

Solution:

1) What $\mathsf{F}(\varphi(x)) = \varphi(a) \in \mathbf{R}$ there is functionality on D, - obviously. Let us show that it is linear and continuous.

2) It's obvious that

$$\mathsf{F}(\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x)) = \lambda_1\varphi_1(a) + \lambda_2\varphi_2(a) = \lambda_1 \mathsf{F}(\varphi_1(x)) + \lambda_2 \mathsf{F}(\varphi_2(x)).$$

This proves linearity.

3) If we have $\lim\limits_{k\to\infty}\varphi_{(k)}(x) = \varphi^*(x)$, then by definition of convergence in D $\varphi_{(k)}^{(n)}(x) \rightrightarrows \varphi^{*(n)} \quad \forall n = 0,1,2,\ldots$ But this means that (including) there is also $\varphi_{(k)}(a) \to \varphi^*(a)$. Whence it follows that

$$\lim_{k\to\infty} \mathsf{F}(\varphi_{(k)}(x)) = \lim_{k\to\infty}\varphi_{(k)}(a) = \varphi^*(a) = \mathsf{F}(\varphi^*(x)).$$

Thus, continuity is also proven.

The generalized function considered in Example 3 has a special name. They call her − *Dirac delta function* or, simply, *delta function*. Its standard designation $\delta_a(x)$.

An important way of relating between regular and singular generalized functions is *passage to the limit*, i.e., when a singular generalized function can be represented as the limit of a sequence of regular functionals in D'.

Example 4: Let us show that in D' for the Dirac delta function $\lim\limits_{\varepsilon\to+0}\dfrac{\varepsilon}{x^2+\varepsilon^2}=\pi\,\delta_0(x)$.

Solution: For any positive ε function $f_\varepsilon(x)=\dfrac{\varepsilon}{x^2+\varepsilon^2}$ will determine $\mathsf{F}_\varepsilon(\varphi)$ − regular generalized function in D'. Let us assume that, due to finiteness, $\exists\; 0\le A<+\infty$ such that everything $\varphi(x)$ the main functions are equal to zero outside the segment $[-A,A]$. Then the equalities will be valid

$$\mathsf{F}_\varepsilon(\varphi(x))=\int\limits_{-\infty}^{+\infty}f_\varepsilon(x)\varphi(x)\,dx=\int\limits_{-A}^{+A}f_\varepsilon(x)\varphi(x)\,dx=\int\limits_{-A}^{+A}\frac{\varepsilon}{x^2+\varepsilon^2}\varphi(x)\,dx=$$

$$=\int\limits_{-A}^{+A}\frac{\varepsilon}{x^2+\varepsilon^2}\big(\varphi(0)+\varphi(x)-\varphi(0)\big)\,dx=\varepsilon\,\varphi(0)\int\limits_{-A}^{+A}\frac{dx}{x^2+\varepsilon^2}+\varepsilon\int\limits_{-A}^{+A}\frac{\varphi(x)-\varphi(0)}{x^2+\varepsilon^2}\,dx\,.$$

For the first term we have:

$$\varepsilon\,\varphi(0)\int\limits_{-A}^{+A}\frac{dx}{x^2+\varepsilon^2}=2\varepsilon\,\varphi(0)\int\limits_{0}^{+A}\frac{dx}{x^2+\varepsilon^2}=\frac{2\varepsilon\,\varphi(0)}{\varepsilon}\operatorname{arctg}\frac{A}{\varepsilon}=2\varphi(0)\operatorname{arctg}\frac{A}{\varepsilon}\to\pi\,\varphi(0)$$

.

For the second term, by virtue of the estimate following from Lagrange's theorem:

$$|\varphi(x)-\varphi(0)|\le|x\varphi'(\xi)|\le|x|\max_{x\in[-A,+A]}|\varphi'(x)|=C|x|$$

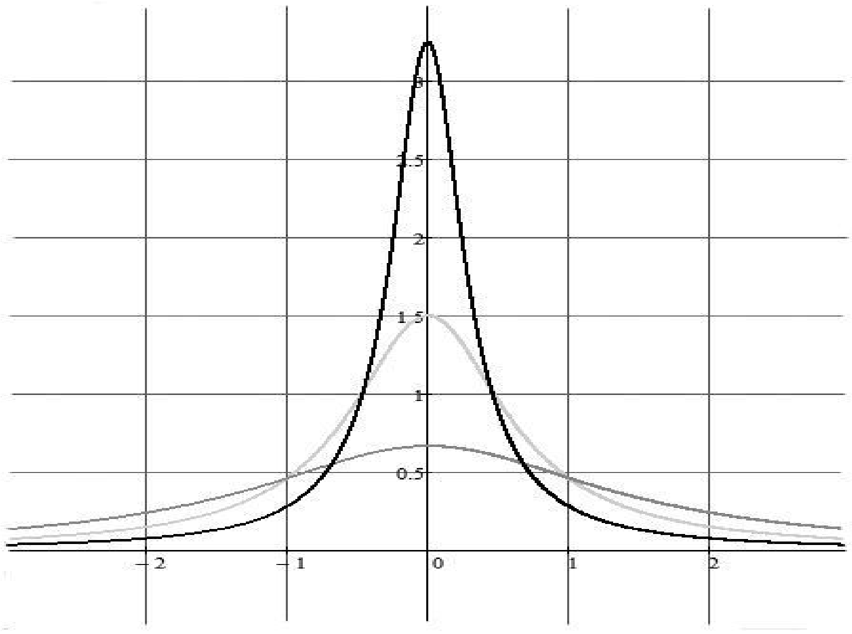
get

$$\left|\varepsilon\int\limits_{-A}^{+A}\frac{\varphi(x)-\varphi(0)}{x^2+\varepsilon^2}\,dx\right|\le\varepsilon\int\limits_{-A}^{+A}\frac{|\varphi(x)-\varphi(0)|}{x^2+\varepsilon^2}\,dx\le C\varepsilon\int\limits_{-A}^{+A}\frac{|x|\,dx}{x^2+\varepsilon^2}=$$

$$=C\varepsilon\int\limits_{0}^{+A}\frac{2x\,dx}{x^2+\varepsilon^2}=C\varepsilon\ln\left(x^2+\varepsilon^2\right)\Big|_0^A=C\varepsilon\ln\left(A^2+\varepsilon^2\right)-2C\varepsilon\ln\varepsilon\underset{\varepsilon\to+0}{\to}0\,.$$

Where do we get it from? $\lim\limits_{\varepsilon\to+0}\dfrac{\varepsilon}{x^2+\varepsilon^2}=\pi\,\delta_0(x)$.

Figure 2 shows the graphs $f_\varepsilon(x)$ for different values ε .



Now let's discuss how to write generalized functions. Note that the right-hand side of formula (2) in the regular case can be considered as a representation of the scalar product in E − some Euclidean space, that is, we can use the notation

$$\mathsf{F}(\varphi)=\int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx=(f,\varphi)\,.$$

Here the left bracket and the functional identifier seem to be “castled”.

This equality, if extended to singular cases, allows us to symbolically denote *any* generic functions by symbol (f,φ), Where f − functional identifier, and φ denotes its argument - the main function.

Note that the reverse redesignation is not always mathematically correct. For example, from the symbolically correct $(\delta_a(x),\varphi)=\varphi(a)$ shouldn't $(f,\varphi)=\int_{-\infty}^{+\infty}\delta_a(x)\varphi(x)\,dx=\varphi(a)$, because $\delta_a(x)$ not a function and has no specific meaning at a point x .

Nevertheless, such “formulas” can sometimes be found in information resources that use such notations for simplicity when explaining “on the fingers”.